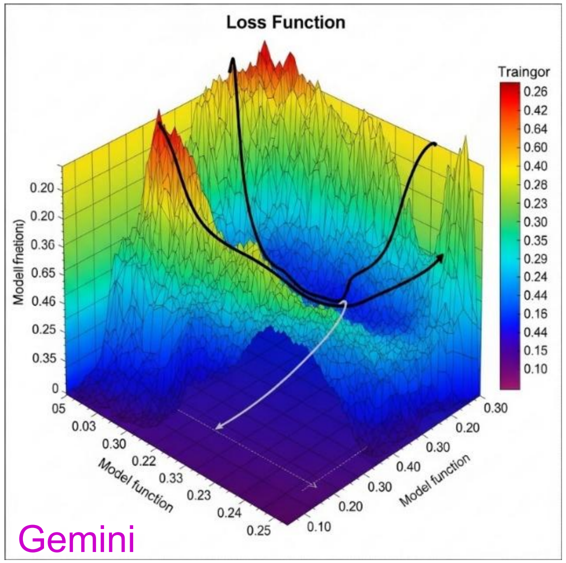
AI for Theory Data



Lance Dixon (SLAC)

T. Cai, K. Cranmer, F. Charton, LD, G. Merz, N. Nolte, M. Wilhelm,
2405.06107 [cs.LG], 2501.05743 [hep-th], and in progress

Michigan State University
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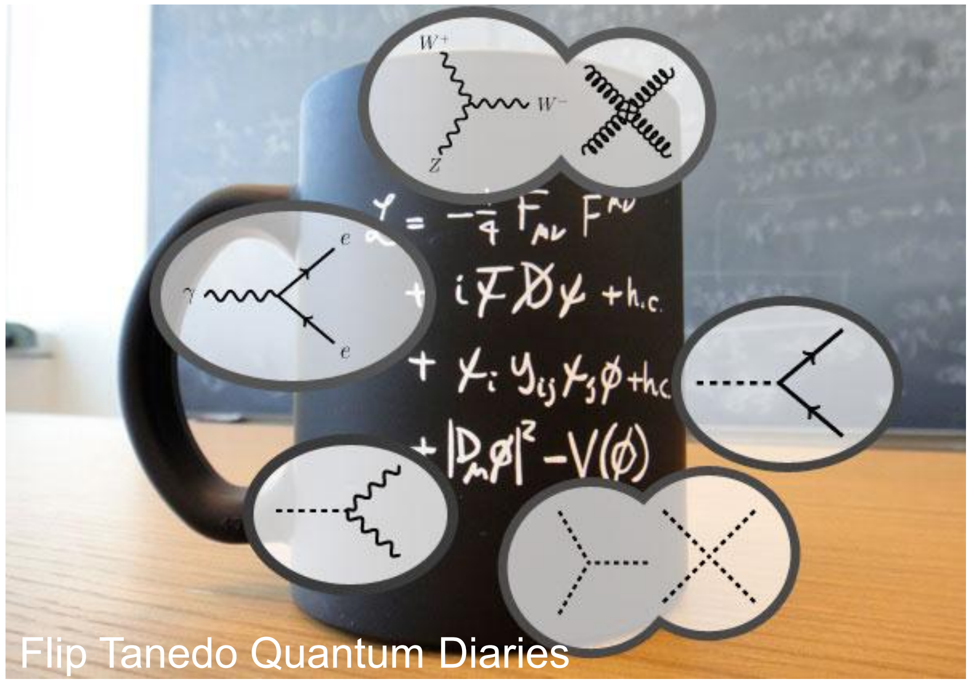
SLAC
NATIONAL ACCELERATOR LABORATORY

Introduction

- Theoretical particle physics is reductionist: organizing all phenomena of elementary particles & forces between them into a minimal set of rules.
- Electricity + magnetism → electromagnetism
- Special relativity + quantum mechanics + electromagnetism → QED
- Weak interactions + QED
  → $\mathrm{SU}(2) \times \mathrm{U}(1)$ electroweak theory
- All strong interaction phenomena → $\mathrm{SU}(3)$ QCD
- $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ + … = Standard Model

Standard Model



Flip Tanedo Quantum Diaries

Special relativity + quantum mechanics
+ gauge symmetry [$SU(3) \times SU(2) \times U(1)$]
+ minimal way to spontaneously break $SU(2)$ [Higgs]
→ Explains essentially all terrestrial experimental results

SM Unsatisfactory

- No neutrino masses (however, can easily accommodate)
- No particle for dark matter, let alone dark energy, inflation
- Gravity can be added but minimal approach not renormalizable (→ string theory?)
- Many "arbitrary" parameters, especially in flavor sector of how fermions talk to Higgs boson
- Electroweak scale "fine tuned"
- **BIG PROBLEM**: No "smoking gun" (solid experimental deviation from SM) telling us where to look next

What can theorists do?

?

propose new frameworks and models leading directly to (new) experimental tests

compute consequences of SM at higher precision to aid in stringent experimental tests

search for new principles by studying structural aspects of observables in SM and related (simpler) theories

How will AI assist?

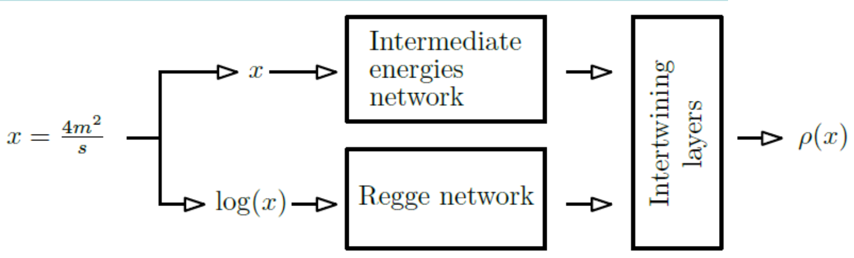
How will AI assist?

- Optimize computational problems
- Find new patterns in “theory data” – symbolic regression
- “Agentically”, LLMs assisting with more and more complex tasks, e.g.
- M. Schwartz [& Claude], 2601.02484
- Agrawal, Craig, Madden, Lombera, FERMIACC, 2603.22538
- → “thinking like physicists”
- Commercial LLMs *vs.* custom architectures for specific (sets of) tasks

“AI” for optimizing trial functions

For example:

- For S-matrix bootstrap



Analyticity, Crossing, Unitarity

– e.g. Atkinson-Mandelstam approach to 2→2 scattering with non-linear unitarity constraints, beyond semi-definite linear programming

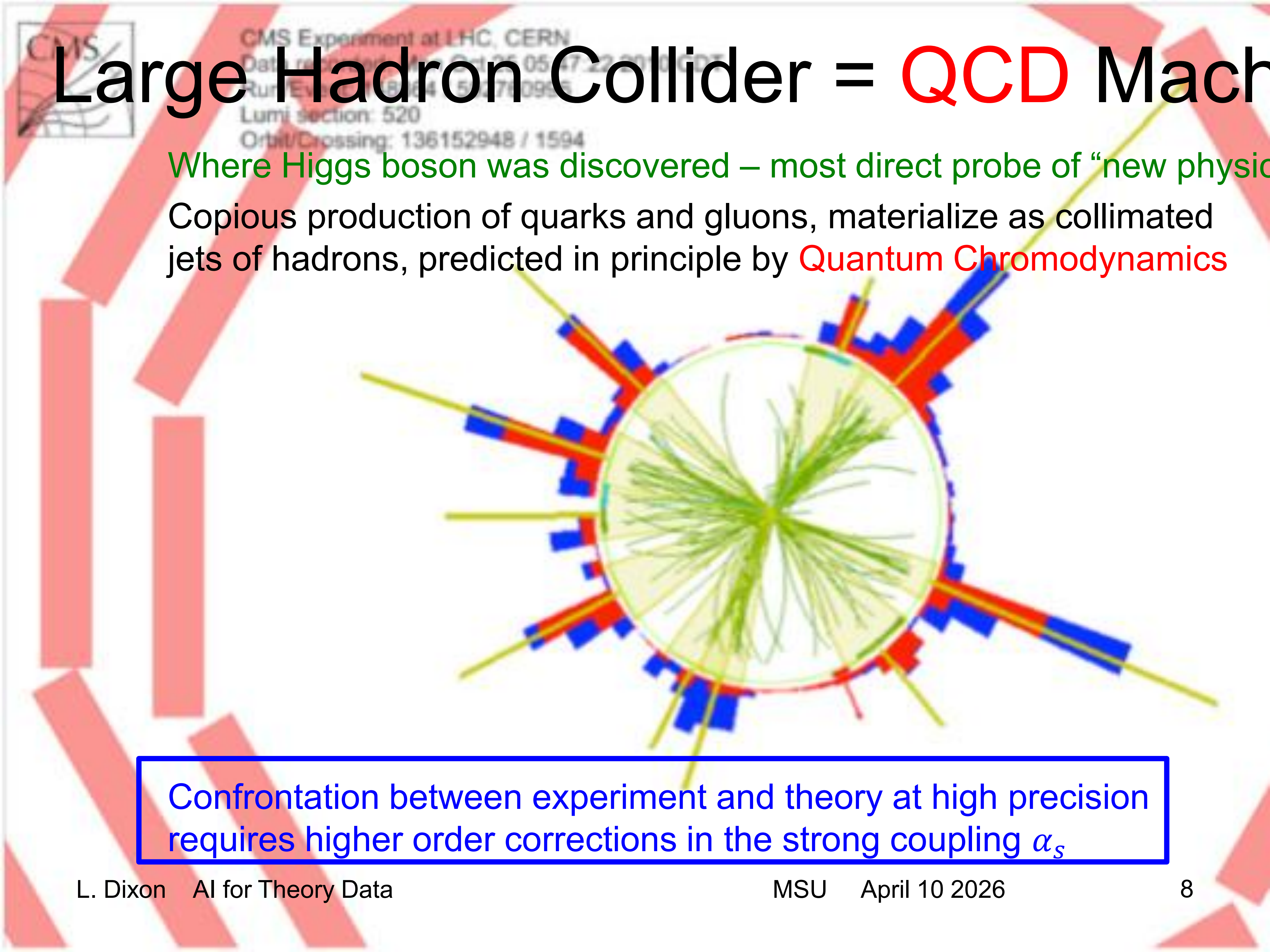
Gumus, Leflot, Tourkine, Zhiboedov, 2412.09610, 2601.22145

- For CFT bootstrap

ACU similar, also OPE expansion into known conformal blocks -- Reinforcement learning for cheap high-dimensional searches in the space of scaling dimensions and OPE-squared coefficients

Kantor, Niarkos, Papageorgakis, 2108.09330

Large Hadron Collider = QCD Mach



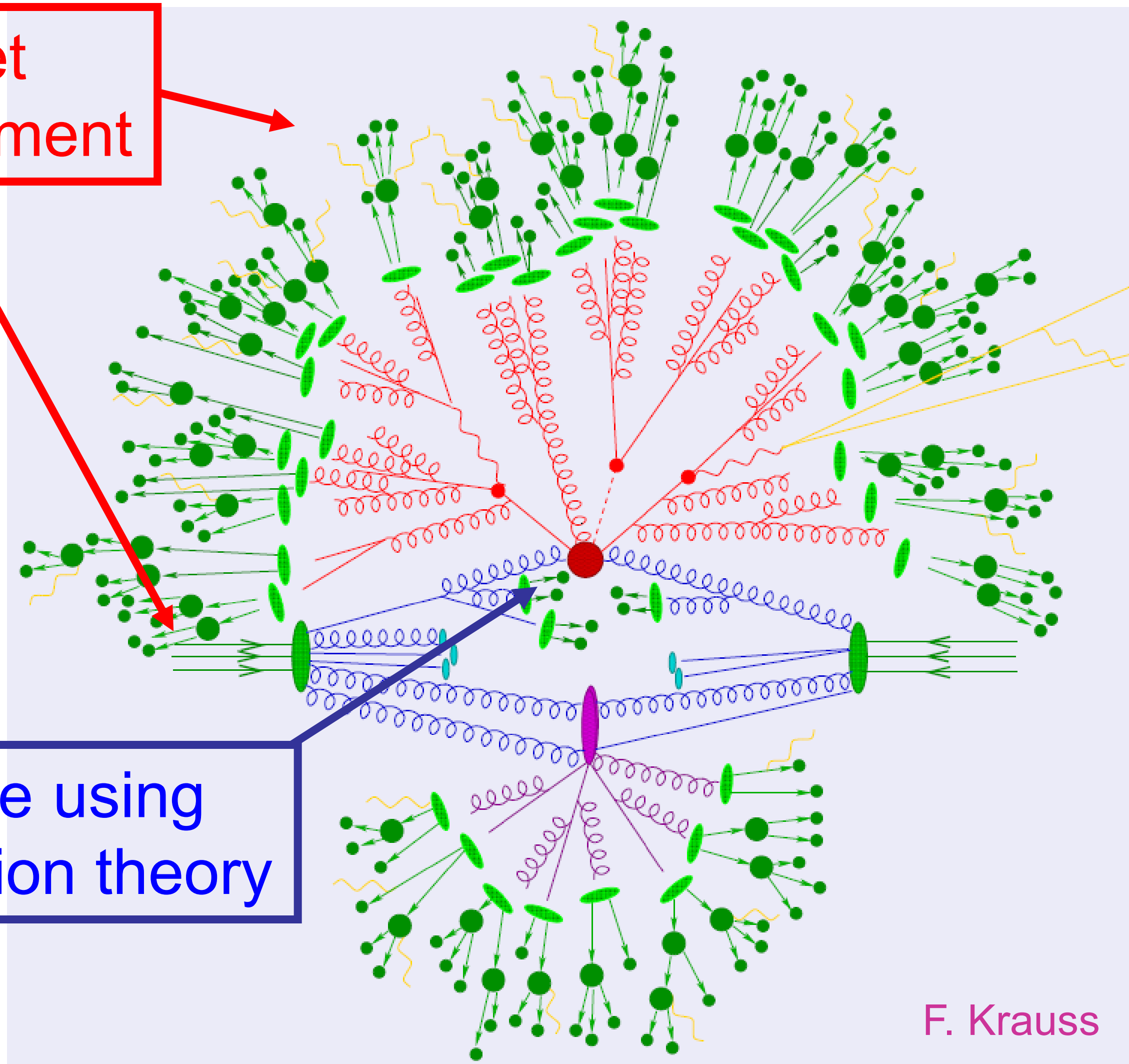
Where Higgs boson was discovered – most direct probe of “new physic

Copious production of quarks and gluons, materialize as collimated jets of hadrons, predicted in principle by Quantum Chromodynamics

Confrontation between experiment and theory at high precision requires higher order corrections in the strong coupling α_s

Typical LHC Collision

Model or get from experiment



Calculable using perturbation theory

Perturbative Short-Distance Cross Section

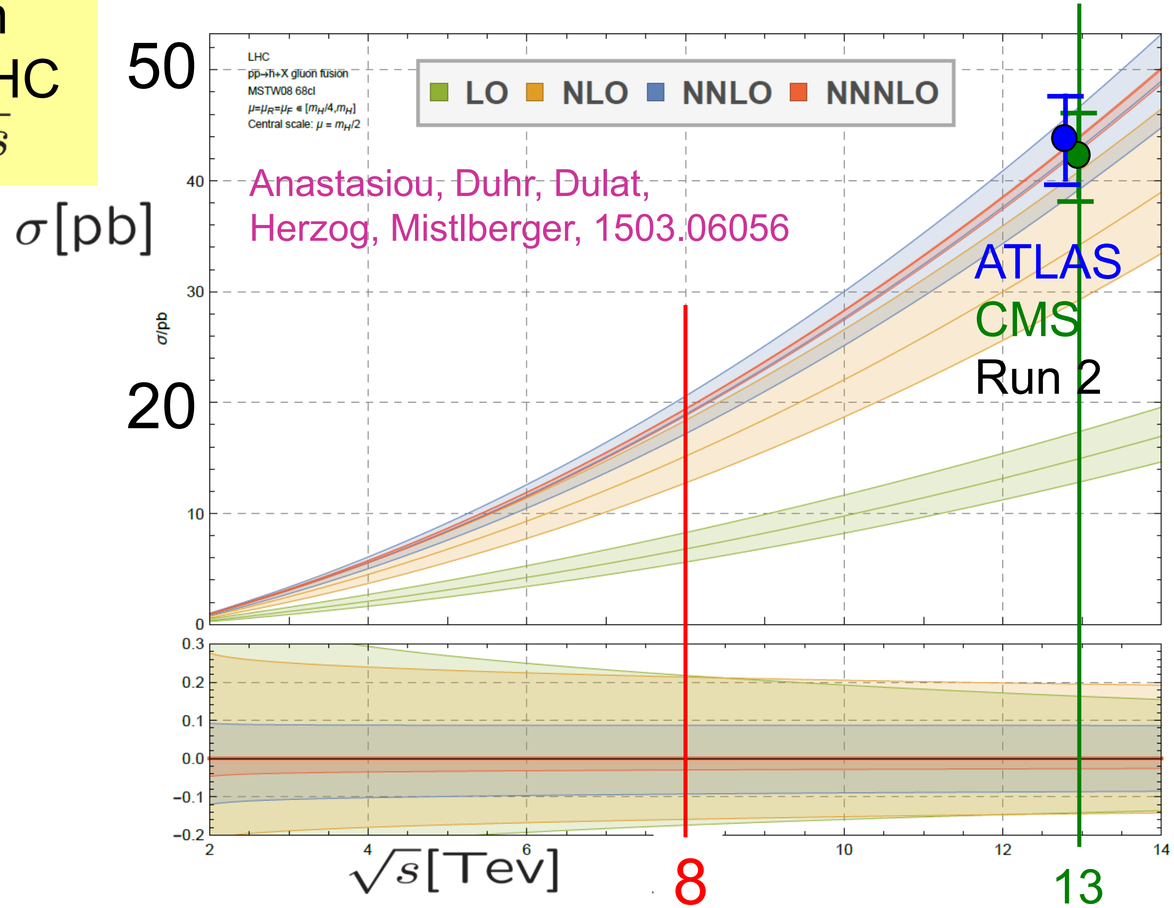
$$\hat{\sigma}(\alpha_s, \mu_F, \mu_R) \;=\; [\alpha_s(\mu_R)]^{n_\alpha}\left[\hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi}\hat{\sigma}^{(1)}(\mu_F, \mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}(\mu_F, \mu_R) + \cdots\right]$$

LO NLO NNLO

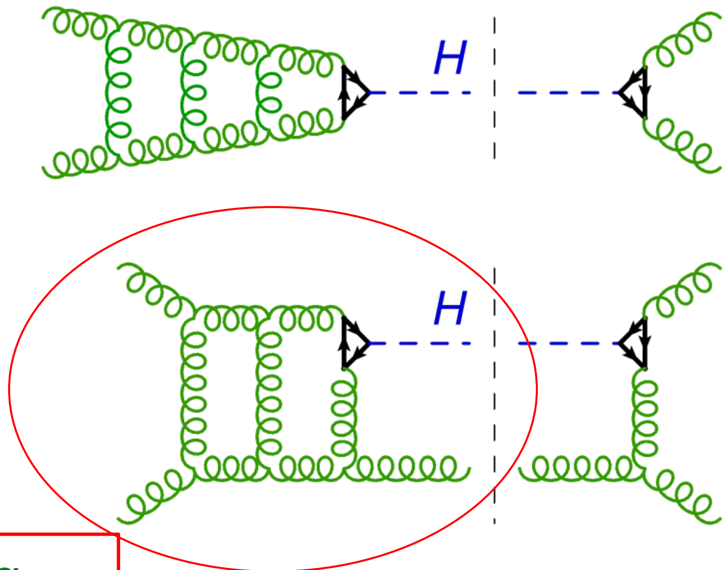
Higgs gluon fusion cross section at LHC vs. CM energy $\sqrt{s}$

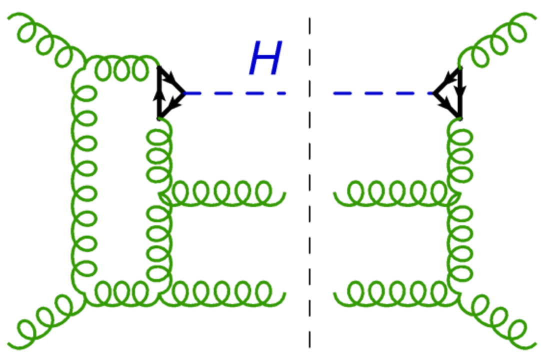
LO terrible!
LO → NNNLO
→ factor of 2 or 3 increase!

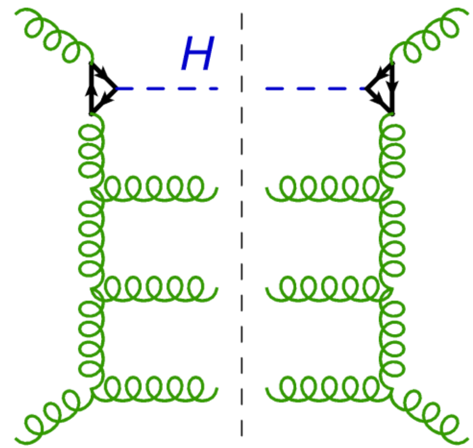
Poor convergence of expansion in $\alpha_s(\mu)$ necessitates high orders!



Very few of the NNNLO QCD diagrams



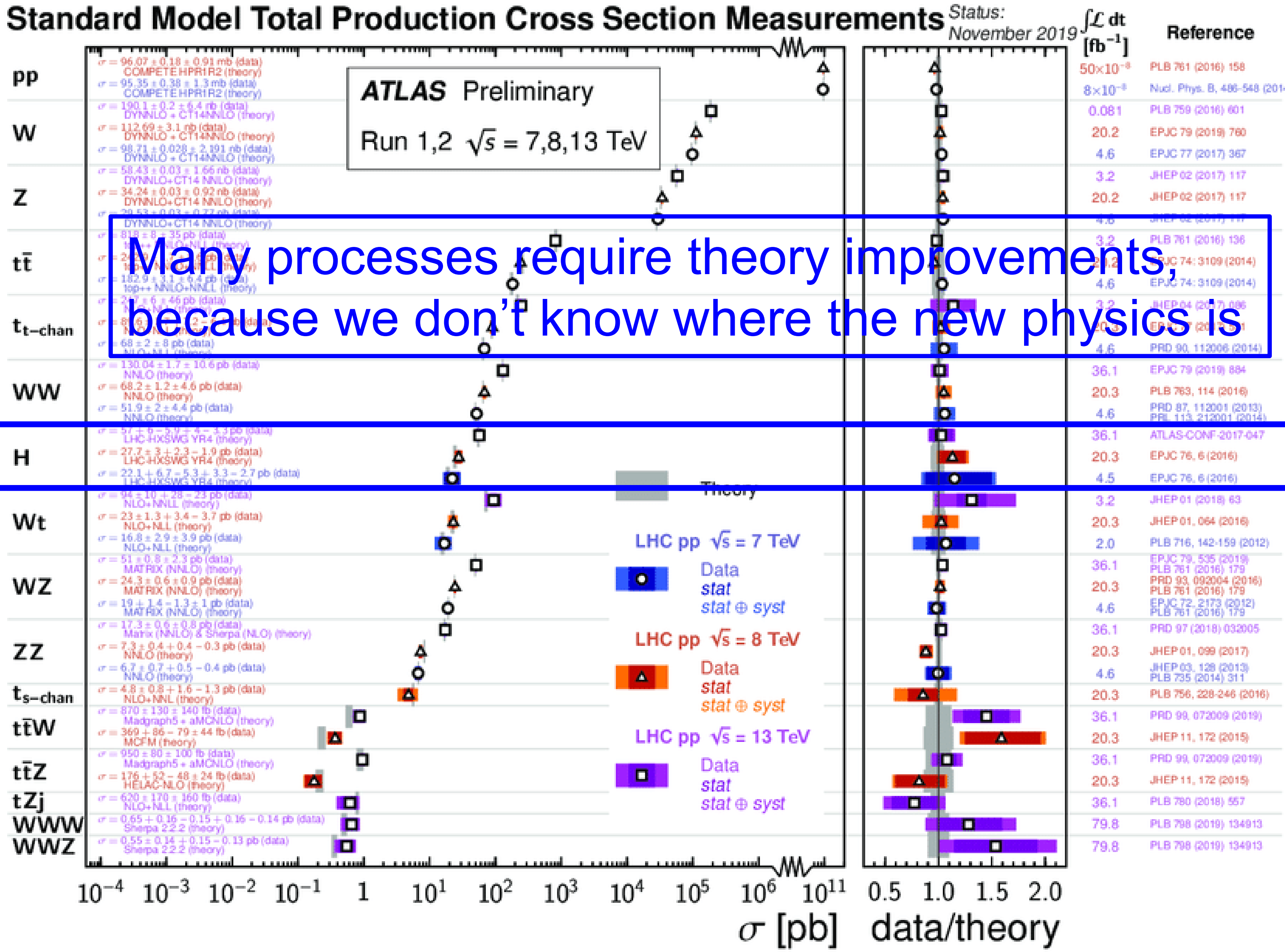




gg → *Hg*
@ 2 loops,
state of art
in QCD until
this year

+ …
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions

Scattering amplitudes are underlying building blocks



Many processes require theory improvements, because we don't know where the new physics is

Different routes to perturbative amplitudes

Draw all Feynman graphs G_i

↓

Evaluate all Feynman rules: I_i

↓

Perform all loop integrations: A_i

↓

$$A = \sum_i A_i$$

Evaluate all unitarity cuts C_α

↓

Construct **local**: integrand I

↓

Perform all loop integrations: A_α

↓

$$A = \sum_\alpha A_\alpha$$

Bootstrap: Guess $A = \sum_m c_m F_m$

F_m known functions

$c_m \in \mathbb{Q}$ unknown constants

↓

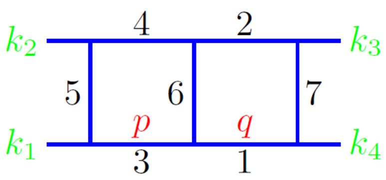
Solve constraints, linear equations for $c_m \to r_m$

↓

$$A = \sum_m r_m F_m$$

Performing Loop Integration → Linear Algebra

E.g. for planar double box integral,



$$\mathcal{I}[P] \equiv \int d^D p \, d^D q \, \frac{P(p^\mu, q^\nu; k_i)}{p_1^2 \, p_2^2 \cdots p_7^2}$$

Consider the $2 \times 5 = 10$ equations

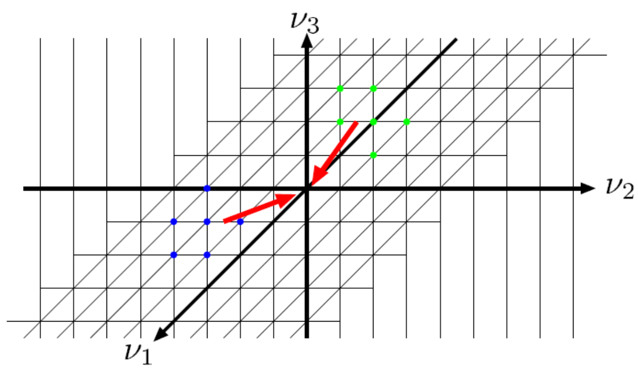
$$0 = \int d^D p \, d^D q \, \frac{\partial}{\partial \ell^\mu} \frac{b^\mu}{(p_1^2)^{\nu_1} (p_2^2)^{\nu_2} \cdots (p_7^2)^{\nu_7}} \, ; \quad \ell^\mu = p^\mu, q^\mu, \quad b^\mu = p^\mu, q^\mu, k_i^\mu$$

— also 3 from Lorentz invariance

Gehrmann, Remiddi

⇒ 13 linear equations for each point

$$0 = \sum_{\{\nu_i\}} c_M(\nu_1, \ldots, \nu_7) \mathcal{I}(\nu_1, \ldots, \nu_7), \quad M = 1, 2, \ldots, 13$$



Computational Bottleneck

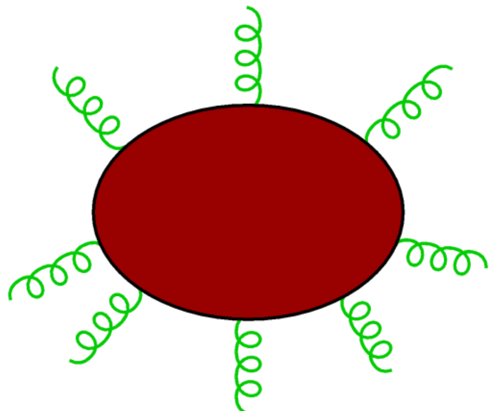
- Multi-variate problem, but generally must fix to rational kinematics, so equations are over $\mathbb{Q}$
- Then map $\mathbb{Q} \to \mathbb{Z}_p$ for some prime p, solve linear system by Gaussian elimination, and reconstruct rational answer (often using multiple primes).
- Repeat for many kinematic points until full analytic behavior is reconstructed.
- This is where most of the computation is, in multi-loop amplitudes for precision theory. (Also very expensive Monte Carlo integrations of trees over phase space.)

Can AI optimize computation time?

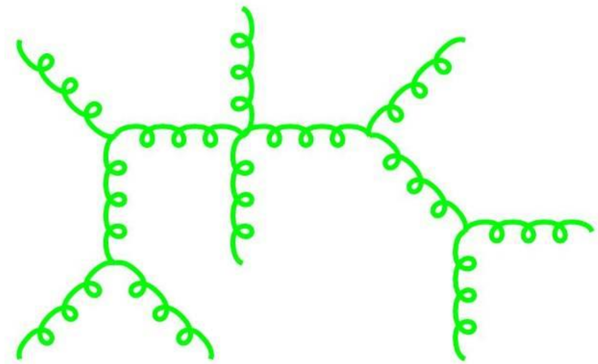
- Over-constrained linear system, even an infinite set of equations. How to pick the best finite set, computationally?
- Optimize "seeds" using genetic algorithms
  von Hippel, Wilhelm, 2502.05121
- Using FunSearch Song, Yang, Cao, Luo, Zhu, 2502.09444
- Using reinforcement learning or simulated annealing
  Zeng, 2504.14065
- Still early days

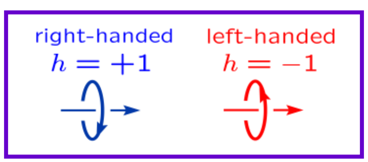
Scattering Amplitudes Story: Whole more than sum of its parts

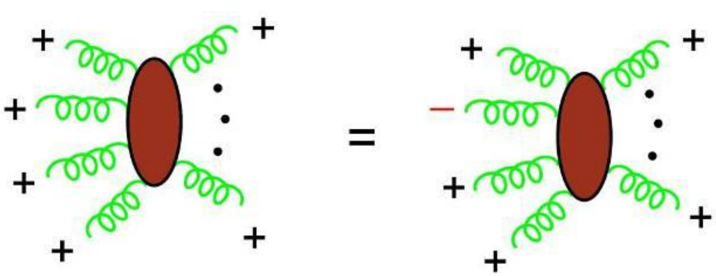
Simplicity often hidden from individual Feynman diagrams



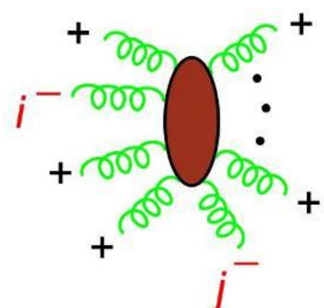
$$= \sum_{\text{trees}}$$







0*



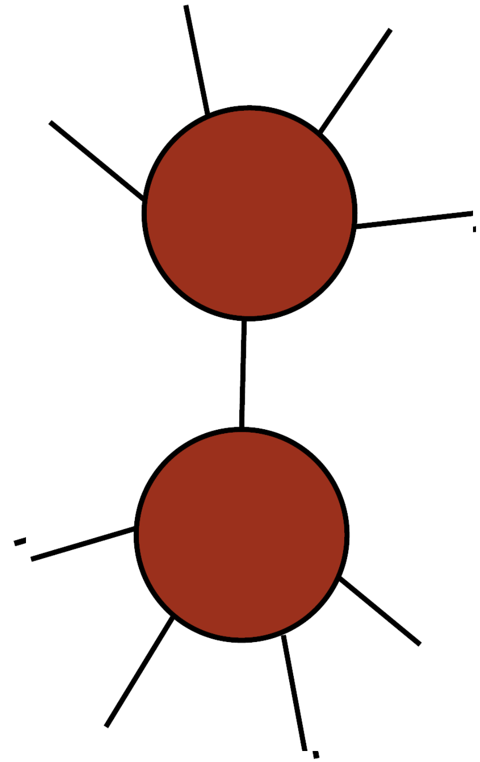
$$= \frac{< ij >^4}{< 12 >< 23 > \cdots < n1 >}$$

*for real momenta; see 2602.12176

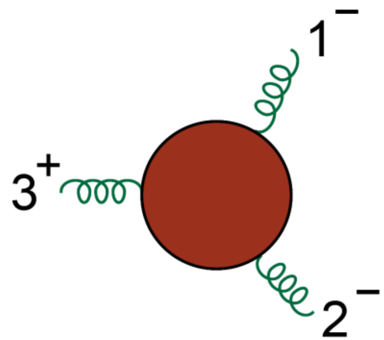
Parke-Taylor (1986)
Mangano-Parke-Xu (1988)

Scattering Amplitudes Are Smooth, Analytic Functions of Kinematics

They fall apart – factorize –
into simpler amplitudes in special limits
→ On-shell BCFW recursion relations



All Gluon Tree Amplitudes Built From:



$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}$$

In contrast to Feynman vertices, it's on-shell, completely physical

- On-shell recursion → compact analytic formulae for trees
- Can do same sort of thing at loop level for integrands: unitarity method

→ all loop BCFW recursion

Toward New formulations

- All-loop BCFW integrand, described in Grassmannian variable, led to Amplituhedron Arkani-Hamed, Trnka (2013)
- reformulation of (planar N=4 super) Yang-Mills theory in terms of geometry instead of locality (Lagrangians)
- Can AI help find new other reformulations that explain YM physics from different perspectives, and point beyond SM?

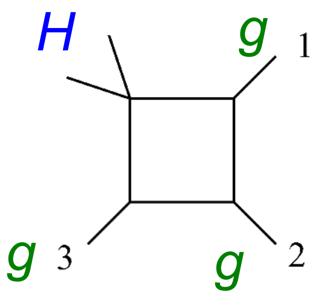
"AI for Analytic Amplitudes"

T. Cai, F. Charton, K. Cranmer, L. Dixon, G. Merz, J. Pages, M. Wilhelm

- There is a set of problems in HEP where the answers can be phrased as language
- In some cases, we have billions of words/sentences worth of theoretical data
- We know the answer is the solution to a large system of linear equations: hard to generate for a big ansatz, but easy to evaluate on a candidate solution → verifiability
- Transformer models can predict many properties of the answer, when part of it is provided.
- Can we use similar models to get previously unknown answers (loop orders)?
- Can we find new patterns in the data?

Loop amplitudes contain transcendental functions from performing Feynman integrals

For example, one-loop box integral for *gg* → *Hg* in $m_t \to \infty$ limit



$$I = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p-k_1)^2 (p-k_1-k_2)^2 (p-k_1-k_2-k_3)^2}$$

$$I \propto \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

$$= \mathrm{Li}_2\left(1 - \frac{1}{u}\right) + \mathrm{Li}_2\left(1 - \frac{1}{v}\right) + \frac{1}{2}\ln^2\left(\frac{u}{v}\right) + \cdots$$

where $u = \frac{s_{12}}{s_{123}}$ and $v = \frac{s_{23}}{s_{123}}$ are the only 2 dimensionless variables ($w = \frac{s_{13}}{s_{123}} = 1 - u - v$)

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x \frac{dt}{t} \int_0^t \frac{dt'}{1-t'}$$

weight 2 iterated integral

Multi-loop transcendental structure

- At L loops, instead of just Li_2's, get special functions with up to $2L$ integrations = weight $2L$ "iterated integrals"
- **Best case**: multiple polylogarithms, defined iteratively:

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

- Still very intricate multi-variate functions, but can be understood very systematically
- Have to proceed process by process
- Much computation required at state-of-art, e.g.
- 2→ 2 with 1 external mass (W,Z,H) at 3 loops,
- all massless 2 → 4 at 2 loops,
- 2 → 3 at 2 loops with 1 or 2 external masses (W,Z,H)

Multiple polylogarithms

Chen, Goncharov, Brown,…

- Iterated integrals, e.g.

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

one weight lower

- Or define differentially: $dF = \sum_{s_k \in \mathcal{S}} F^{s_k} \, d \ln s_k$

- s_k **are letters in the symbol alphabet** $\mathcal{S}$

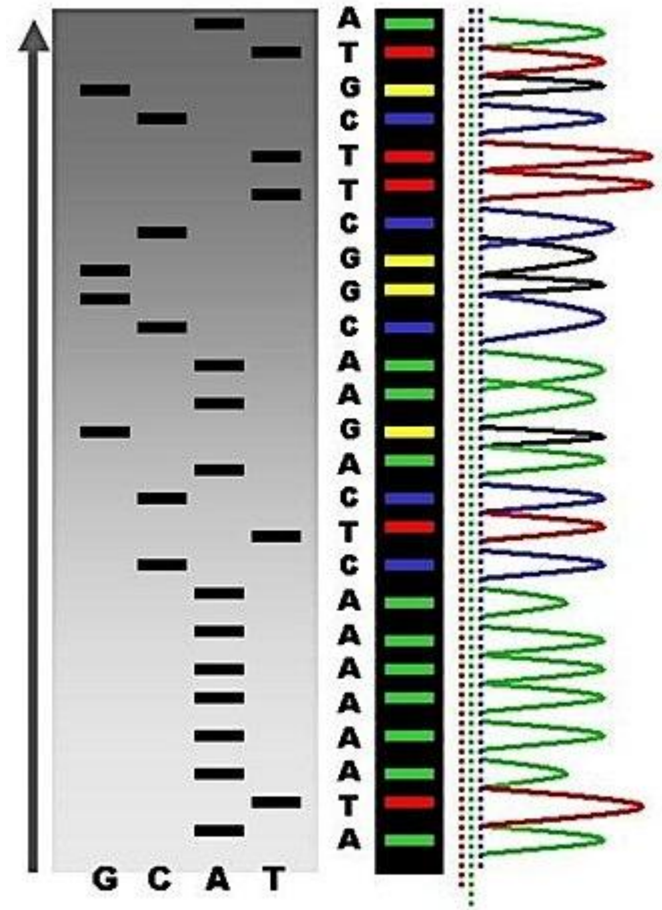
Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Building symbol like sequencing DNA

Taking derivative
of function exposes
last letter of string

Iterate $2L$ times to
obtain symbol of
generalized polylogarithm
of weight $2L$

chemical reaction cleaves
A,G,T,C off end of string



The symbol

- Iterating the proceduree n times for weight n function, gives symbol $\mathcal{S}[F]$,

$$\mathcal{S}[F] \equiv \Delta_{1,1,\ldots,1}\,[F] = \sum_{s_{i_1},\ldots,s_{i_n}\in\mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}}\; s_{i_1} \otimes \ldots \otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}} \in \mathbb{Q}$ and it's conventional to drop the $\ln$'s

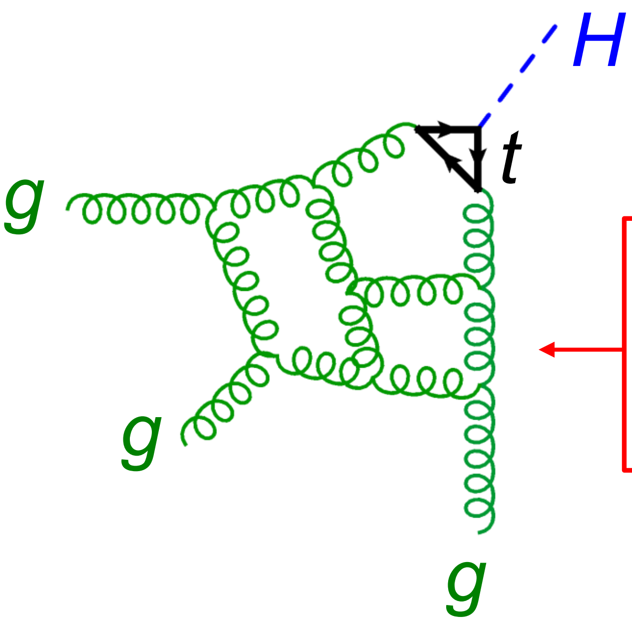
Goncharov, Spradlin, Vergu, Volovich, 1006.5703

- In our examples, $F^{s_{i_1},\ldots,s_{i_n}} \in \mathbb{Z}$ (!)
- All elements easily tokenized for study by a (transformer) language model

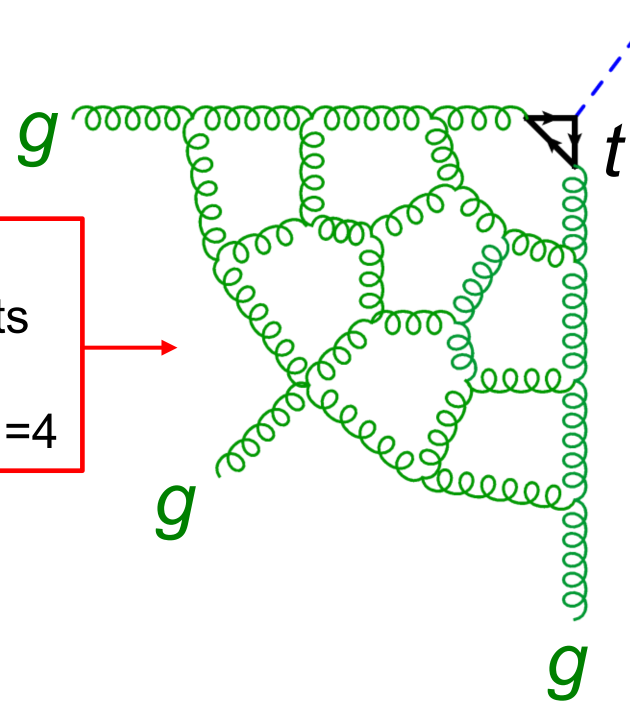
"Goldilocks Process" ~ $gg \to Hg$

QCD state of art is now three loops @ large N_c (not counting top quark loop)
Chen, Guan, Mistlberger, 2504.06490

Can get to **eight** loops – in analog process: 3-point form factor of $\mathrm{tr}\phi^2$ operator in planar N=4 SYM
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm 2012.12286, 2112.06243, 2204.11901



Leading weight $2L$ transcendental parts match between QCD ←→ planar N=4



Alphabet for planar N=4 *gg* → *Hg*

Let $u = \frac{s_{12}}{s_{123}}$, $v = \frac{s_{23}}{s_{123}}$, $w = \frac{s_{31}}{s_{123}} = 1 - u - v$

Then $\mathcal{L} = \{a, b, c, d, e, f\}$

where $a = \sqrt{\frac{u}{vw}},\ b = \sqrt{\frac{v}{wu}},\ c = \sqrt{\frac{w}{uv}},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w}$

- Symbols of *gg*→*Hg* amplitude $F_3^{(L)}$ simplify (remarkably) at $L = 1$ and 2 loops, to just 6 and 12 terms:

$$\mathcal{S}\left[F_3^{(1)}\right] = (-2)[\, b\otimes d + c\otimes e + a\otimes f + b\otimes f + c\otimes d + a\otimes e]$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 8[\, b\otimes d\otimes d\otimes d + c\otimes e\otimes e\otimes e + a\otimes f\otimes f\otimes f + b\otimes f\otimes f\otimes f + c\otimes d\otimes d\otimes d + a\otimes e\otimes e\otimes e] \\ +16\,[b\otimes b\otimes b\otimes d + c\otimes c\otimes c\otimes e + a\otimes a\otimes a\otimes f + b\otimes b\otimes b\otimes f + c\otimes c\otimes c\otimes d + a\otimes a\otimes a\otimes e]$$

(really only 1 and 2 terms, plus images under dihedral symmetry)

N=4 *gg*→*Hg* symbol terms per loop

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

Not advisable to look at full symbol directly without "eye protection"

But a lot of data to be mined from it!!

- Word length $= 2L$ (uniform weight for N=4)
- All coefficients integers!
- Many, many correlations/patterns

Examples of patterns

- Every term in symbol starts with a, b, c; never d, e, f
- Physical reason for start related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- Many patterns still not understood physically
- For example, *gg*→*Hg* symbol turns out to be the **same** as symbol of 6-gluon *gg*→*gggg* MHV amplitude written backwards (antipodal duality)!! 2112.06423, 2212.02410
- What other surprises lurk within this data?
- Can they be used to find new principles, and/or predict the next loop order?

Host of Linear Relations

- E.g. 3 pair relations from integrability, $d^2F = 0$:

$$F^{a,b} - F^{b,a} + F^{a,c} - F^{c,a} = 0$$

$$F^{c,a} - F^{a,c} + F^{c,b} - F^{b,c} = 0$$

$$F^{d,b} - F^{b,d} + F^{e,c} - F^{c,e} + F^{f,a} - F^{a,f}$$
$$+ F^{c,d} - F^{d,c} + F^{a,e} - F^{e,a} + F^{b,f} - F^{f,b} + 2(F^{c,b} - F^{b,c}) = 0$$

- They hold in every pair of slots, e.g. first one means:

$$F^{l_1,\ldots,l_{i-1},a,b,l_{i+2},\ldots,l_{2L}} - F^{l_1,\ldots,l_{i-1},b,a,l_{i+2},\ldots,l_{2L}} + F^{l_1,\ldots,l_{i-1},a,c,l_{i+2},\ldots,l_{2L}}$$
$$- F^{l_1,\ldots,l_{i-1},c,a,l_{i+2},\ldots,l_{2L}} = 0$$

for all $l_1, \ldots, l_{i-1}, l_{i+2}, \ldots, l_{2L} \in \mathcal{L}$

- Inhomogeneous relations that involve taking (near-)collinear limits of symbol and matching to known behavior, etc.
- All told, we have enough relations to validate any symbol an AI model might propose! Challenge to the AI community.

Our Transformer Models

Cai, Cranmer, Charton, LD, Merz, Nolte, Wilhelm, 2405.06107 [cs.LG]

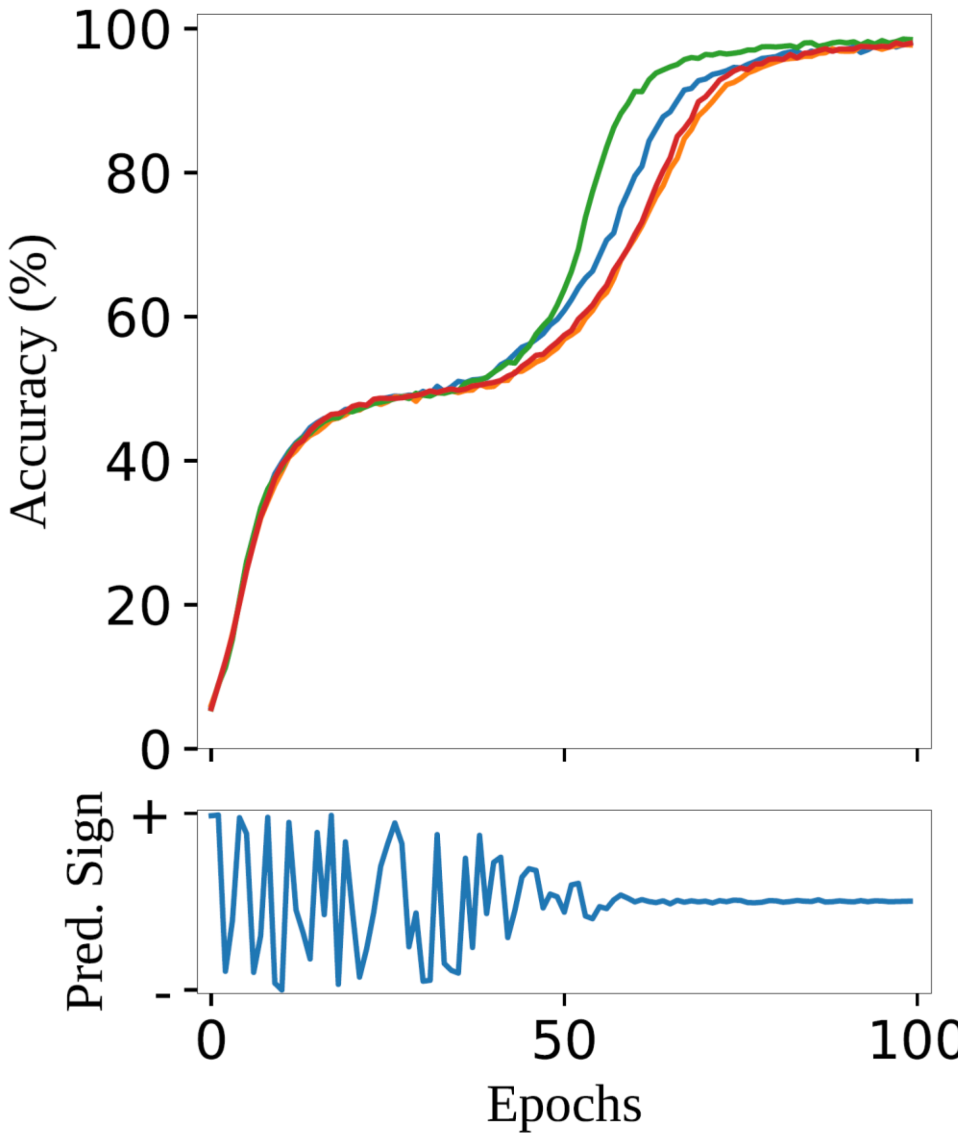
- Encoder-decoder, with bidirectional encoder and autoregressive decoder linked by cross-attention.
- Encoder & decoder have same number of layers (≤ 8), same number of attention heads (8 or 16), and same dimension ($d = 256, 512, \text{or } 1024$).
- Between 4.5 and 245 million trainable parameters ($\ll$ LLMs with 100s of billions)
- Tokenize integer coefficients as:

  $\pm \times [\text{magnitude in base } 1000]$

First “experiment”: coefficient-from-word

- Train model to “learn” symbol of $F_3^{(L)}$, using a subset of the 5 million terms at $L = 6$ loops
- Task: predict the integer coefficient, given the letter sequence for the term
- Minimize **“cross-entropy” loss function** between model’s current probability distribution for coefficients and the training set
- Accuracy predicting coefficients in test set is >98% after 200 epochs

- Two learning phases:
  - First the magnitudes
  - Then the signs



“Seed” sequences

- Certain sequences can be inferred to all loop orders, given 8 loops worth of “data” (so far mainly by humans).
- For example, $\mathcal{S}\left[F_3^{(L)}\right] \supset c_L(\mathbf{afff}\,\ldots)\; a \otimes f \otimes f \otimes \ldots \otimes f$

through 6 loops has $c_L = -2,\ 8,\ -96,\ 1920,\ -53760,\ 1936360,\ \ldots$

which is clearly $c_L = (-2)^L \frac{[2(L-1)]!}{(L-1)!}$

as confirmed by 7,8 loops

- Can infer hundreds of similar sequences where first 8 letters are arbitrary, but followed by $(2L-8)\ \mathbf{f}'s$ Cai et al., 2501.05743

$$c_L(\mathbf{X}_8\mathbf{f}\ldots\mathbf{f}) = p_L(\mathbf{X}_8\mathbf{f}\ldots\mathbf{f}) \times (-1)^L 2^{2L-8}(2L-9)!!$$

where $p_L(\mathbf{X}_8\mathbf{f}\ldots\mathbf{f})$ is a cubic polynomial. (Fit the 4 parameters in the cubic to the data for $L = 5{,}6{,}7{,}8$.)

"Seed" sequences (cont.)

- Inspired by certain "strike-out" AI experiments, we recast some of the all-L sequences into recursive form, e.g.

$$c_L(\mathbf{abbbdd\ldots}) = (36 - 8\mathrm{L})c_{L-1}(\mathbf{abbbdd\ldots}) - 16\, c_{L-1}(\mathbf{abbddd\ldots})$$

$$c_L(\mathbf{aaabbdd\ldots}) = (28 - 8\mathrm{L})c_{L-1}(\mathbf{aaabbdd\ldots}) + 40c_{L-1}(\mathbf{aabbddd\ldots}) + 12\, c_{L-1}(\mathbf{abbddd\ldots})$$

- We'd like to find many more such sequences to "seed" the AI model

- We (Matthias W.) used "pySR" M. Cranmer, 2305.01582

to determine some sequences that "HumInt" failed to find:

$$c_L(\mathbf{a\ldots aeee}) = (-1)^L[-2^{2L}(L-6) - 8L]\,\frac{[2(L-2)]!}{6\,(L-2)!}$$

$$c_L(\mathbf{a\ldots aeae}) = (-1)^{L+1}[2^{2L}(2L-3) - 8L]\,\frac{[2(L-2)]!}{12\,(L-2)!}$$

- More sequences under study with various LLMs…

Next phase

- We don’t know most of the symbol at the next loop order
- But we do know all the linear relations
- We are now training models on the relations instead of on individual coefficients, using compressed forms
- Will also include the “seed” data
- And we will look for more seeds
- And are studying more efficient ways to solve the (too many) linear equations

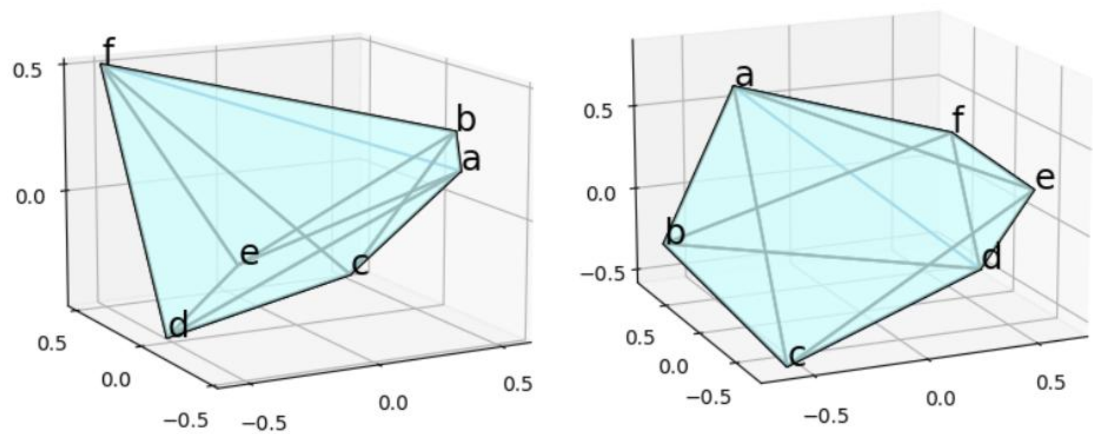
Other future “AIAA” directions

- Lots of other planar N=4 data to play with, e.g.
  - full MHV 6-gluon amplitude to 8 loops 2308.18199
  - NMHV 6-gluon to 7 loops 1903.10890 + to appear
  - “ tr ϕ^3 “ 3-point form factor to 8 loops
    Basso, LD, Tumanov, 2410.22402; LD, Z. Li, to appear
  - 7-point amplitudes to 5 loops 1612.08976, 1812.04640, 2511.09669 [S. He et al.]
- Will this work?
- How about for QCD? → Many challenges with understanding rational prefactors too!

Extra Slides

Dihedral Symmetry Learned

- Tokens mapped into vector space in "embedding layer" of the transformer.
- Plot leading 3 PCA (principal component analysis) components for positions of $\{a, b, c, d, e, f\}$ in the embedding space:



$L = 5\ (64\%\ of\ variance\ in\ 3D)$ $L = 6\ (82\%\ of\ variance\ in\ 3D)$

- In full space, dihedral symmetry holds for both $L = 5$ and $L = 6$: all angles close to $60°$. For $L = 5$, not apparent in projection to 3D

More compressed "data" still works

- Present AI model with symbol in a compressed format, because:

  A) we don't want it to "cheat" on "easy" rules, e.g. dihedral symmetry
  B) at 7 and 8 loops, the full symbol is too big to train cheaply
- For example, there are not 6^4 independent quadruples of the 4 final entries; many linear relations → only 24 quads are independent.
- The 24 form 8 3-plets under dihedral symmetry
- So we ask the AI model to predict coefficients for the $2L-4$ letter sequences associated with just 8 quads.
- It can still do so very successfully, although larger models are required

“Strike-out” experiment: $L-1 \leftrightarrow L$

- If we strike out 2 letters from a string of length $2L$, it becomes a string of length $2(L-1)$ which appears in the $(L-1)$ loop amplitude.
- There are $L(2L-1)$ ways of removing 2 letters (``parent words’’)
- Q: Can a model find hidden relations between coefficients for parent words from the $(L-1)$ loop amplitude and that for the term in the L loop amplitude, and use them to predict the latter?
- A: Yes, when provided with enough $(L-1, L)$ correlated pairs

(k is maximum distance between the two letters struck out):

	Accuracy	Magnitude accuracy	Sign accuracy
Strike-two, all parents	98.1%	98.4%	99.6%
Strike-two, $k=5$	98.3%	98.6%	99.7%
Strike-two, $k=3$	98.4%	98.7%	99.7%
Strike-two, $k=2$	98.1%	98.3%	99.5%
Strike-two, $k=1$	94.3%	95.2%	98.5%

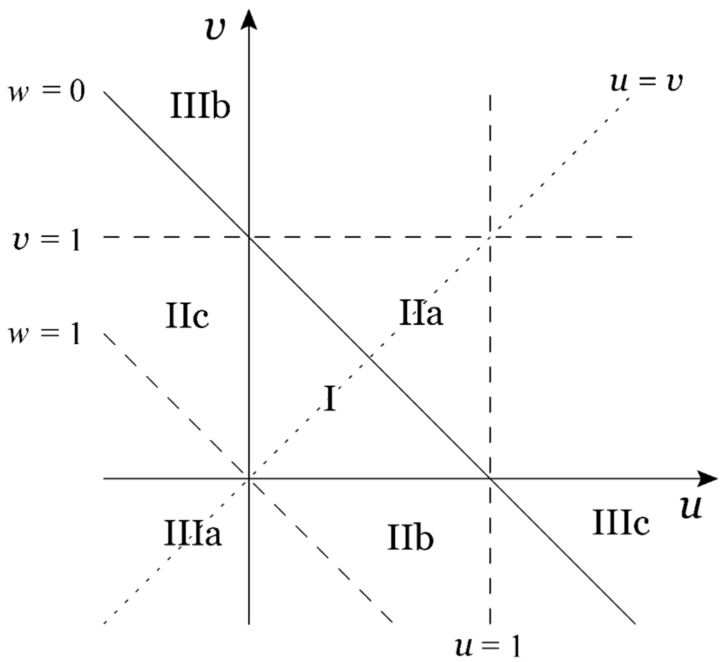
$Hggg$ kinematics is two-dimensional

$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$



$$u + v + w = 1$$

I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or

$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

Many linear relations

dihedral cycle: $a \to b \to c \to a, \quad d \to e \to f \to d$
dihedral flip: $a \leftrightarrow b, \quad d \leftrightarrow \mathrm{e}$

- Implies equality of many 3-plets and 6-plets of integer coefficients, e.g. all nonzero ones at 1 loop:

$$\mathcal{S}\left[F_3^{(1)}\right] = (-2)[\, b \otimes d + c \otimes e + a \otimes f + b \otimes f + c \otimes d + a \otimes e]$$

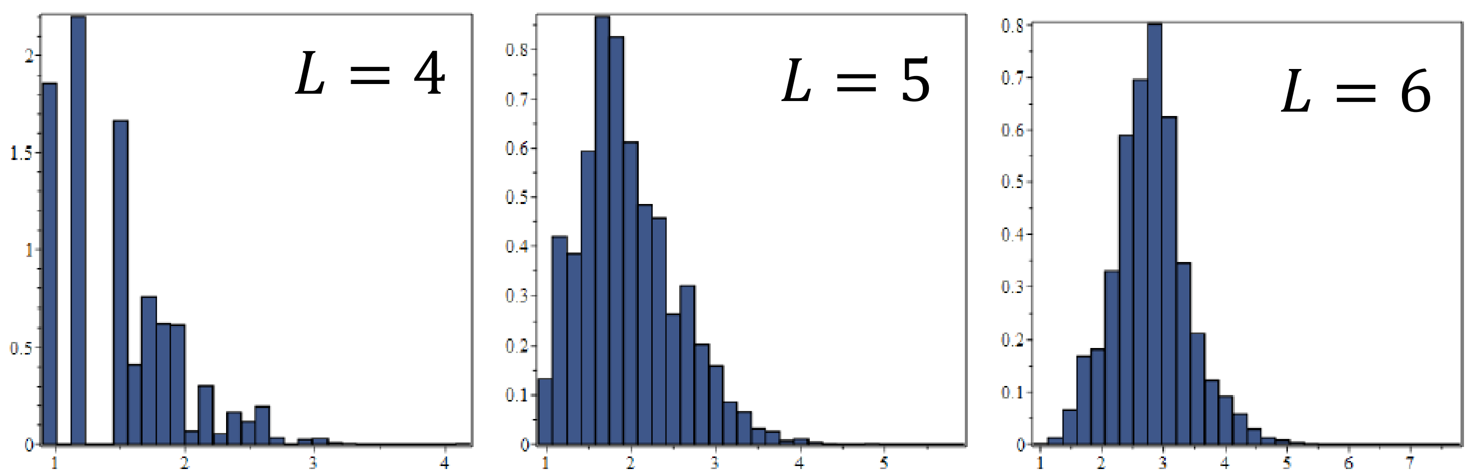
- Integrability, or $d^2 F = 0$, leads to 9 antisymmetric “pair” constraints on adjacent pairs of letters.
- 6 of the 9 are subsumed by some “adjacency conditions”

Adjacency rules → “trivial zeros”

- But still half of the remaining terms are zero:

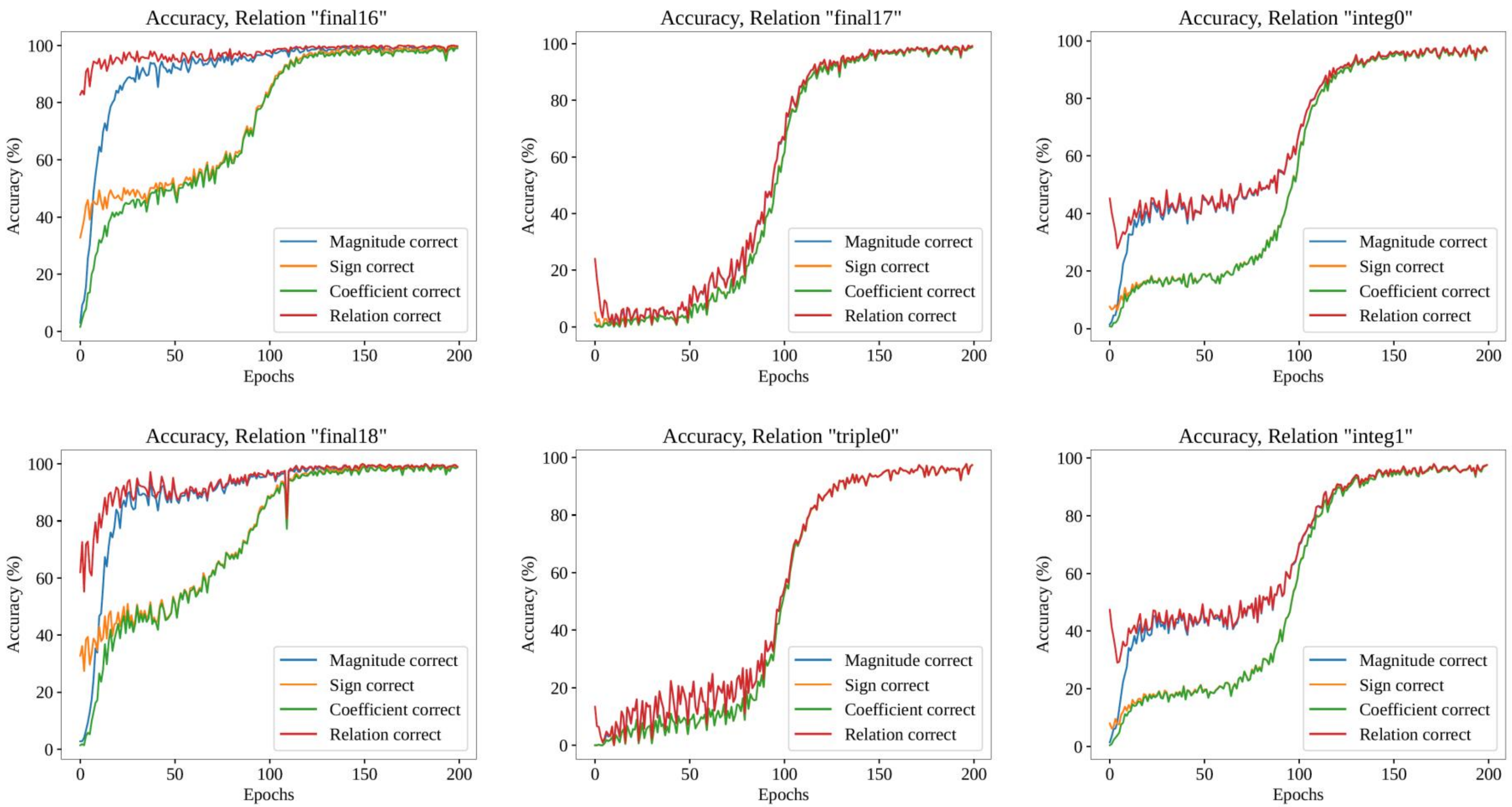
Loop	1	2	3	4	5	6	7	8
Total (6^{2L})	36	1,296	46,656	$1.7 \cdot 10^6$	$6.1 \cdot 10^7$	$2.2 \cdot 10^9$	$7.8 \cdot 10^{10}$	$2.8 \cdot 10^{12}$
W/O trivial zeros	6	102	1,830	32,838	589,254	$1.1 \cdot 10^7$	$1.9 \cdot 10^8$	$3.4 \cdot 10^9$
Total nonzero	6	12	636	11,208	263,880	$4.9 \cdot 10^6$	$9.3 \cdot 10^7$	$1.7 \cdot 10^9$

- Nonzero coefficients’ magnitudes grow with loop order:



$\log_{10}|coeff.|$

Linear relations learned at different rates



2 term, same sign   2-3 terms, opposite signs   4 terms, mixed signs

Hopf algebra "coacts"

- Splits functions into simpler pieces
- Total differential of polylogarithmic function F (with respect to underlying coordinates x_i)

$$dF = \sum_{s_k \in \mathcal{L}} F^{s_k} \, d \ln s_k \quad \Leftrightarrow \quad \triangle_{n-1,1} F = \sum_{s_k \in \mathcal{L}} F^{s_k} \otimes \ln s_k$$

part of Hopf algebra coaction

- Iterate: Define F^{s_j, s_k} via derivatives of F^{s_k}:

$$dF^{s_k} \equiv \sum_{s_j \in \mathcal{S}} F^{s_j, s_k} \, d \ln s_j \quad \Leftrightarrow \quad \triangle_{n-2,1,1} F = \sum_{s_j, s_k \in \mathcal{L}} F^{s_j, s_k} \otimes \ln s_j \otimes \ln s_k$$

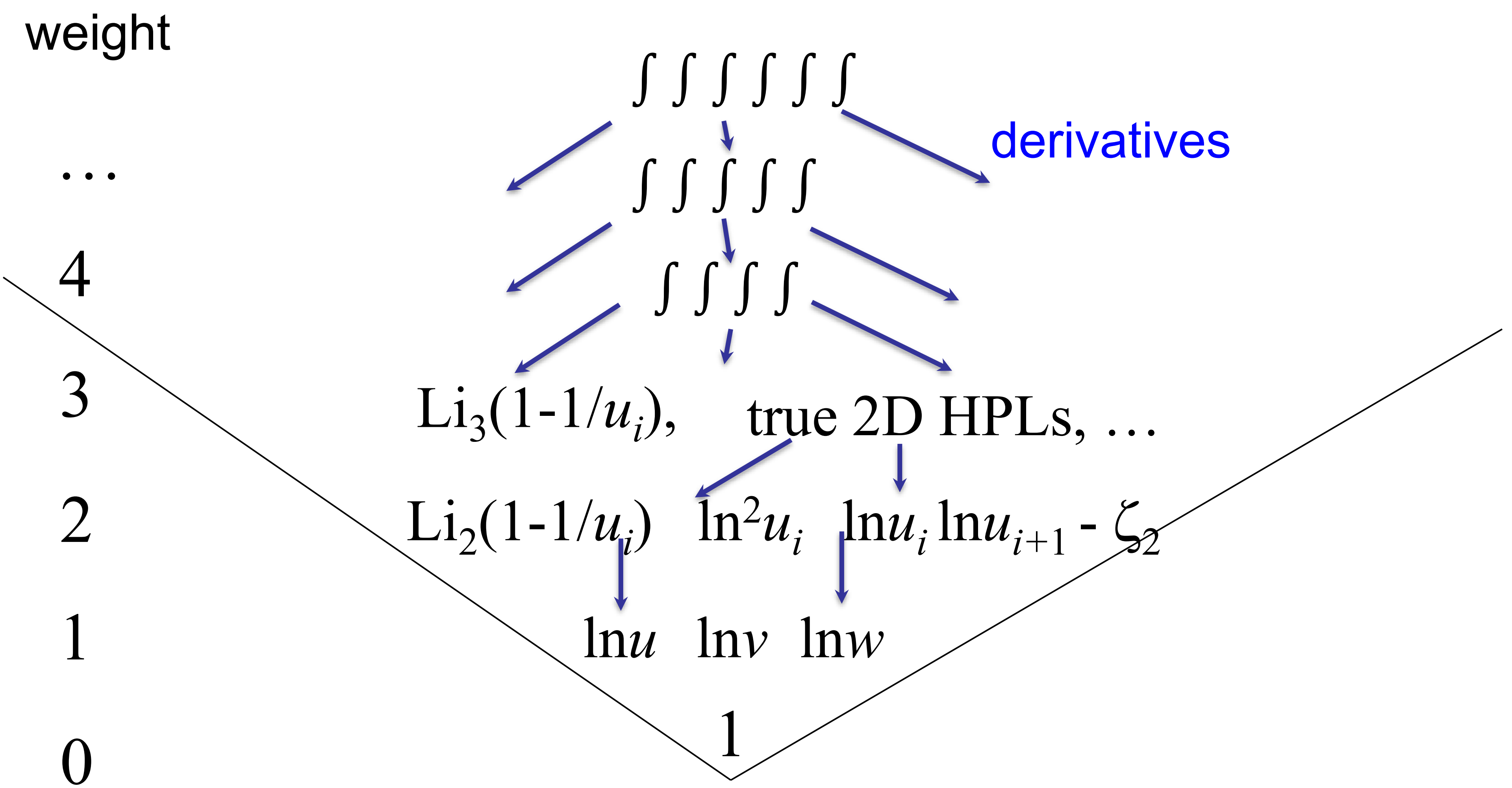
another part of Hopf algebra coaction

Number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	3	1														
$L=2$	1	3	6	3	1												
$L=3$	1	3	9	12	6	3	1										
$L=4$	1	3	9	21	24	12	6	3	1								
$L=5$	1	3	9	21	46	45	24	12	6	3	1						
$L=6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L=7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L=8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- Properly normalized L loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right

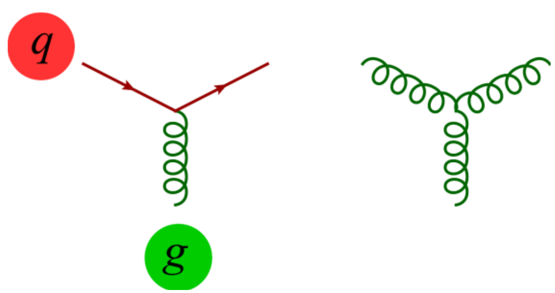
Heuristic view of function space



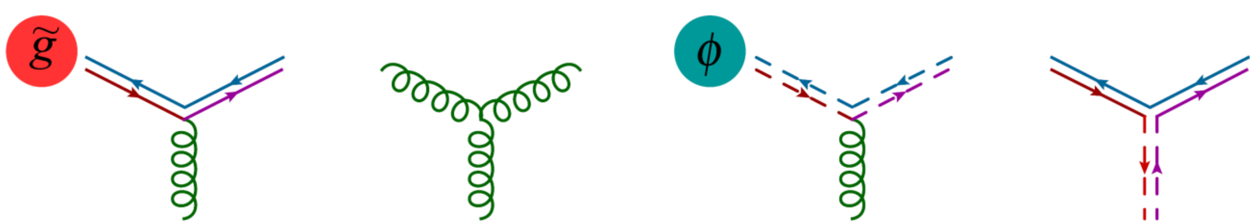
QCD vs. N=4 SYM

- QCD: gluons, plus quarks in fundamental rep. of $SU(N_c)$
- N=4: Replace quarks with 4 copies of fermions in adjoint rep. (gluinos) and add 6 real adjoint rep. scalars
- All in same supermultiplet (like one particle)
- Feynman vertices:

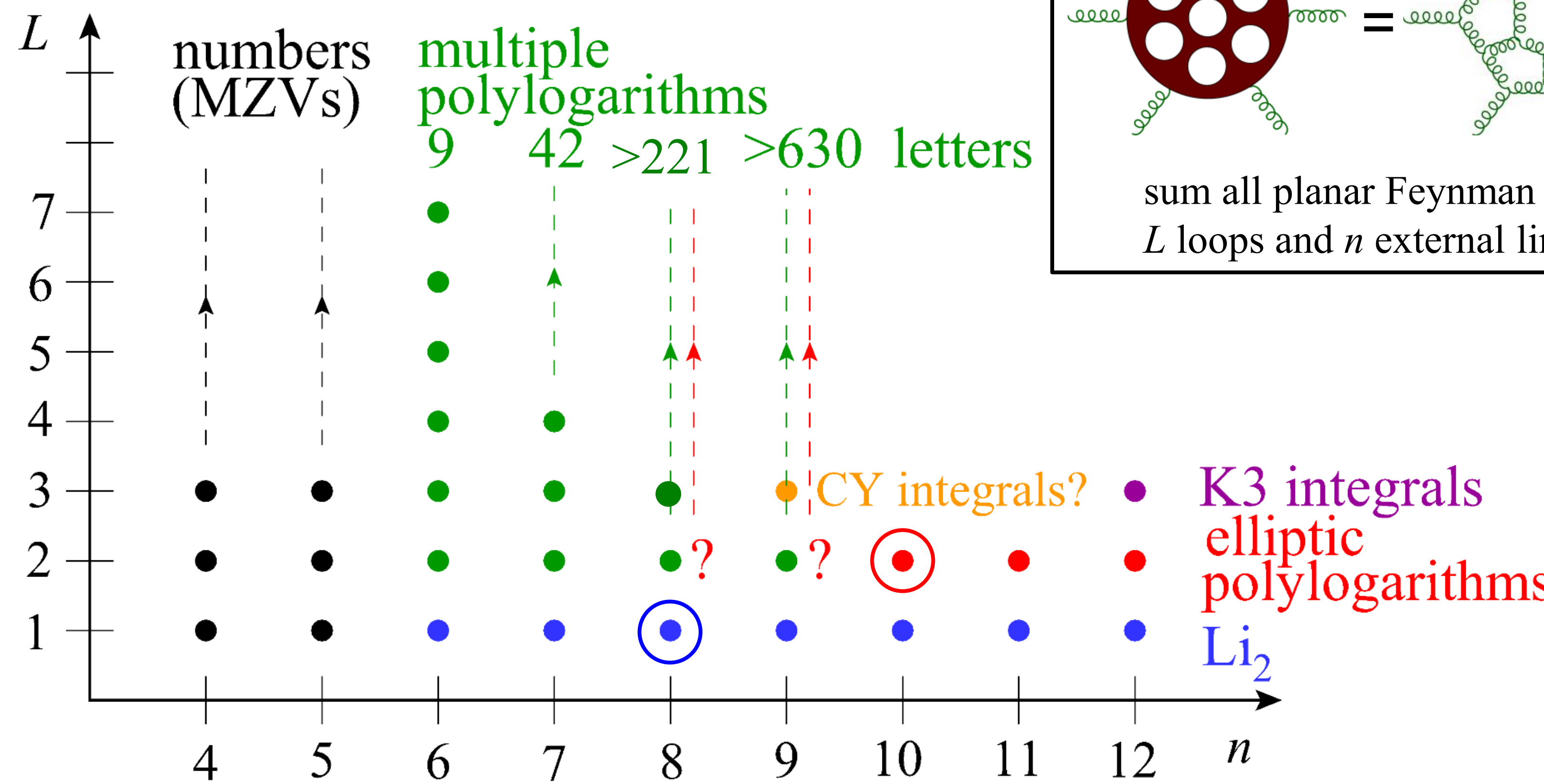
QCD

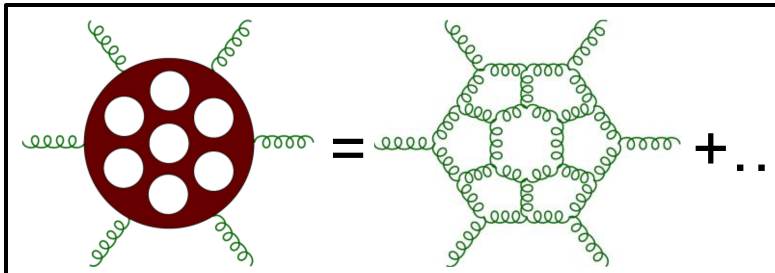


N=4 SYM



Known Planar N=4 SYM n-Point Amplitudes





sum all planar Feynman graphs with
L loops and n external lines