MICHIGAN STATE UNIVERSITY

PHY 422/820: Classical Mechanics

Subject/Final Exam, December 2025

Formula Sheet

- **Coordinate Systems**

$$x = \rho\cos\phi = r\sin\theta\cos\phi\,, \quad y = \rho\sin\phi = r\sin\theta\sin\phi\,, \quad z = r\cos\theta$$

$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z = \rho\vec{e}_\rho + z\vec{e}_z = r\vec{e}_r$$

$$\begin{aligned}\dot{\vec{r}} &= \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + \dot{z}\vec{e}_z \\ &= \dot{\rho}\vec{e}_\rho + \rho\dot{\vec{e}}_\rho + \dot{z}\vec{e}_z = \dot{\rho}\vec{e}_\rho + \rho\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z \\ &= \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\sin\theta\;\dot{\phi}\;\vec{e}_\phi\end{aligned}$$

$$\dot{\vec{e}}_\alpha = \vec{\omega}\times\vec{e}_\alpha$$

$$dV = \rho\,d\rho\,d\phi\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi = r^2\,dr\,d\Omega$$

$$\vec{\nabla} = \vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z} = \vec{e}_\rho\frac{\partial}{\partial\rho} + \vec{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \vec{e}_z\frac{\partial}{\partial z} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\vec{\nabla}f(r) = f'(r)\frac{\vec{r}}{r}\,, \quad r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

- **D'Alembert's Principle**

$$\sum_{i=1}^{N}\left(\vec{F}_i^{(a)} - m_i\ddot{\vec{r}}_i\right)\cdot\delta\vec{r}_i = 0$$

- **Lagrangian Dynamics**

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$S = \int_{t_1}^{t_2} dt\,L(q,\dot{q},t)\,, \quad \delta S = 0$$

$$\widetilde{L}(q,\dot{q},t,\lambda) = L(q,\dot{q},t) + \sum_{\alpha=1}^{n_c}\lambda_\alpha f_\alpha(q,t)$$

$$\text{(Dissipative)}\quad \frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i^D = -\frac{\partial D}{\partial\dot{q}_i}\,, \qquad Q_i^D = \sum_i^N \vec{F}_i^D\cdot\frac{\partial\vec{r}_i}{\partial q_i}$$

- **Jacobi Integral**

$$h(q,\dot{q}) = \sum_i^n\left(\dot{q}_i\frac{\partial L}{\partial\dot{q}_i}\right) - L$$

- **Noether's Theorem**

$$q_i'(t') = q_i(t) + \epsilon\eta_i(t)\,, \quad t' = t + \epsilon\tau(t)\,, \quad L' = L + \epsilon\frac{dF}{dt}$$

$$J = \sum_i \frac{\partial L}{\partial \dot{q}_i}\left(\dot{q}_i\tau - \eta_i\right) - L\tau + F = \text{const.}$$

- **Central Forces**

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r)\,, \qquad \vec{A} = \frac{\vec{p}\times\vec{l}}{m\kappa} - \frac{\vec{r}}{r}$$

$$\phi - \phi_0 = \pm\int_{r(\phi_0))}^{r(\phi)} dr'\,\frac{l}{r'^2\sqrt{2m\left(E - V_{\text{eff}}(r')\right)}}$$

$$r(\phi) = \frac{\alpha}{1 + \epsilon\cos(\phi - \phi_0)}\,, \quad \alpha = \frac{l^2}{m\kappa}\,, \quad \epsilon = |\vec{A}|$$

- **Rotations**

$$\vec{r}' = \vec{r}\cos\phi + \vec{n}(\vec{n}\cdot\vec{r})(1 - \cos\phi) + (\vec{n}\times\vec{r})\sin\phi$$

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}\,, \quad R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}\,, \quad R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = BCD = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{pmatrix}$$

- **Rigid Body Motion**

$$\vec{R} = \frac{1}{M}\int_M dm\vec{r} = \frac{1}{M}\int_V dV\vec{r}\rho(\vec{r})$$

$$I_{ij} = \int_V dV\rho(\vec{r})\left(\vec{r}^2\delta_{ij} - r_ir_j\right)$$

$$I_{ij} = I_{ij}^{\text{cm}} + M\left(\vec{R}^2\delta_{ij} - R_iR_j\right)$$

$$T_{\text{rot}} = \frac{1}{2}\underline{\omega}^T\,\boldsymbol{I}\,\underline{\omega}\,, \quad \underline{L} = \boldsymbol{I}\underline{\omega}$$

$$\underline{\omega} = \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

$$\left(\frac{d}{dt}\right)_\Sigma = \left(\frac{d}{dt}\right)_{\Sigma'} + \vec{\omega}\times$$

$$
\begin{aligned}
A\dot{\omega}_{x'} + (C-B)\omega_{y'}\omega_{z'} &= N_{x'} \\
B\dot{\omega}_{y'} + (A-C)\omega_{z'}\omega_{x'} &= N_{y'} \\
C\dot{\omega}_{z'} + (B-A)\omega_{x'}\omega_{y'} &= N_{z'}
\end{aligned}
$$

- **Small Oscillations**

$$
L = \frac{1}{2}\underline{\dot{\eta}}^T\,\boldsymbol{T}\,\underline{\dot{\eta}} - \frac{1}{2}\underline{\eta}^T\,\boldsymbol{V}\,\underline{\eta}\,, \qquad \left(\boldsymbol{V} - \omega^2\boldsymbol{T}\right)\underline{\rho} = \underline{0}\,, \qquad \det\left(\boldsymbol{V} - \omega^2\boldsymbol{T}\right) = 0
$$

$$
\boldsymbol{A} = \begin{pmatrix}\underline{\rho}^{(1)} & \cdots & \underline{\rho}^{(n)}\end{pmatrix}, \qquad \underline{\eta} = \boldsymbol{A}\underline{\zeta}\,, \qquad \underline{\zeta} = \boldsymbol{A}^T\boldsymbol{T}\underline{\eta}\,, \qquad \boldsymbol{A}^T\boldsymbol{T}\boldsymbol{A} = \mathbb{1}
$$

$$
\left(\vec{a},\vec{b}\right) = \underline{a}^T\,\boldsymbol{T}\,\underline{b} = \sum_{kl} a_k T_{kl} b_l
$$

- **Hamiltonian Dynamics**

$$
p_j = \frac{\partial L}{\partial \dot{q}_j}\,, \quad H(q,p,t) = \sum_k p_k\dot{q}_k(q,p,t) - \widetilde{L}(q,p,t)
$$

$$
\dot{q}_j = \frac{\partial H}{\partial p_j}\,, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}
$$

$$
\{f,g\} = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial p_k}\right), \quad \frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t}
$$

$$
\{fg,h\} = \{f,h\}g + f\{g,h\}\,, \quad \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0
$$

$$
\{q_j,q_k\} = \{p_j,p_k\} = 0\,, \quad \{q_j,p_k\} = \delta_{jk}
$$

$$
\dot{q}_i = \{q_i,H\}\,, \quad \dot{p}_i = \{p_i,H\}
$$

$$
p_i = \partial F_\alpha/\partial q_i\,, \quad P_i = -\partial F_\alpha/\partial Q_i\,, \quad q_i = -\partial F_\alpha/\partial p_i\,, \quad Q_i = \partial F_\alpha/\partial P_i
$$

$$
K = H + \frac{\partial F_\alpha}{\partial t}
$$

- **Taylor Expansions**

$$
\frac{1}{(1+x)^n} = 1 - nx + \frac{1}{2}n(n+1)x^2 - \frac{1}{6}n(n+1)(n+2)x^3 + O(x^4)
$$

$$
\sqrt{1-x} = 1 - \frac{x}{2} + O(x^2)\,, \quad x \ll 1
$$

$$
\sin x = x - \frac{1}{6}x^3 + O(x^5)\,, \qquad \cos x = 1 - \frac{1}{2}x^2 + O(x^4)
$$

$$
\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)
$$

$$
e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5)\,, \quad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right), \quad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)
$$

- **Linear Algebra**

$$
\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{pmatrix}
$$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \boldsymbol{A} = \det \boldsymbol{A}^T \,, \quad \det \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \,, \quad \det \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_1 \cdot \ldots \cdot a_n$$

$$\vec{a} \times \vec{b} \leftrightarrow \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

- **Trigonometric Identities**

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(2x) = 2 \sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\int_0^{2\pi} dx \, \sin x = \int_0^{2\pi} dx \, \cos x = 0 \qquad \int_0^{2\pi} dx \, \sin^2 x = \int_0^{2\pi} dx \, \cos^2 x = \pi$$

$$\frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}$$

- **Miscellaneous**

$$ax^2 + bx + c = 0 \quad \Rightarrow \qquad x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$