MICHIGAN STATE UNIVERSITY

PHY 422/820: Classical Mechanics

Final Exam / Subject Exam

(December 13, 2023)

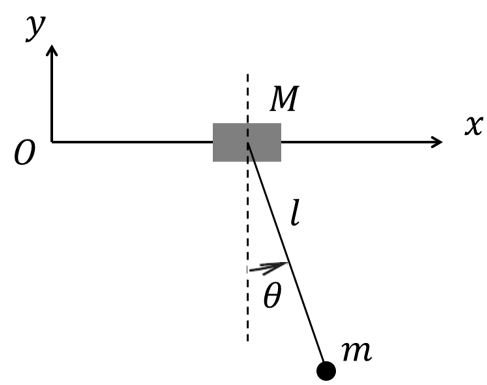
Exam copy number............

Points					
F1	F2	F3	F4	F5	total
10	10	10	10	10	50

- **Please keep this exam closed until instructed to begin.**
- **Do not write your name or your student ID on any page of this copy.**
- **Students in PHY422 need to complete 3 out of the 5 problems.**
- **This is a closed-notes exam. The use of text books, course material, personal notes, etc. is NOT allowed. The use of electronic devices is NOT allowed. Only the use of the distributed formula sheet is allowed.**
- **Do not hesitate to ask questions if something is unclear.**

Problem F1 – Sliding mass and pendulum

[10 points] A mass M slides without friction on a horizontal line and a pendulum of length l and mass m hangs from M, as shown in the figure.



1. Show, justifying all steps in the derivation, that the Lagrangian of the system is given by

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right) + mgl\cos\theta\,, \tag{1}$$

   where x is the position of mass M on the line relative to the origin O, and θ is the angle between the pendulum and the vertical direction.

2. Derive the Lagrange equations for the system.

3. Express the condition of equilibrium of the mass m relative to the mass M and solve the equations of motion to find the equilibrium positions for θ.

4. Make a sketch of the equilibrium positions and comment shortly, without doing any calculation, about the stability of these positions.

Problem F2 – Central Forces

[10 Points] A mass m moves in a central potential defined by

$$V(r) = -\frac{V_0}{r^4}\,, \tag{2}$$

where V_0 is a constant which can be either positive or negative.

1. Give the expression of the central force vector, $\vec{F}(r)$, and indicate for which sign of V_0, the force is attractive (i.e. toward the potential center) or repulsive (away from the potential center).

2. Express with words and in maths the condition on the effective potential for the existence of circular orbits. Calculate the radius R for such orbit and indicate for which sign of V_0 circular orbits exist.

In the following we consider the sign of V_0 which is consistent with the existence of circular orbits.

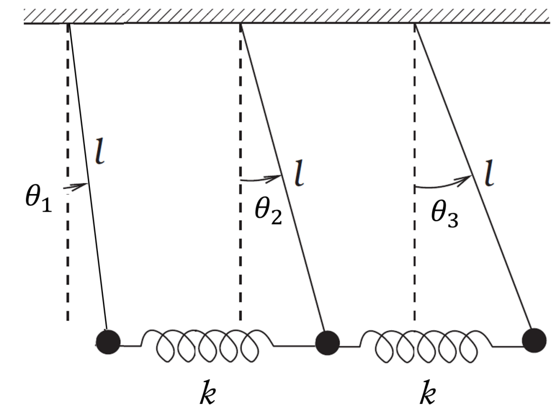
3. Express the condition on the effective potential for the existence of stable orbits and determine whether the solution for the circular orbit found above is stable or unstable.

4. Determine $\lim_{r\to 0} V_{\text{eff}}(r)$; $\lim_{r\to\infty} V_{\text{eff}}(r)$; and $V_{\text{eff}}(R)$ using the available information.

5. Deduce, when applicable, the range of values for the energy E of the mass, for which bounded orbits are possible.

Problem F3 – Small Oscillations and Normal Modes

[10 Points] Three pendula with equal masses, m, and equal lengths, l, are coupled with two springs of equal spring constant, k, as shown in the figure. In the limit of small oscillations, the motion of the masses can be approximated to be in one dimension, so that the potential energy due to gravity can be neglected and the horizontal position of each mass out of equilibrium is $x_i \approx l\theta_i$.



1. Construct the Lagrangian of the system in terms of the angular coordinates $\theta_1, \theta_2, \theta_3$ of the masses relative to their equilibrium positions. Justify each step in the derivation.

2. Express the Lagrangian in the quadratic form

$$L = \frac{1}{2}\dot{\vec{\theta}} \cdot \boldsymbol{T} \cdot \dot{\vec{\theta}} - \frac{1}{2}\vec{\theta} \cdot \boldsymbol{V} \cdot \vec{\theta} \tag{3}$$

and show that the matrices $\boldsymbol{T}$ and $\boldsymbol{V}$ are given by

$$\boldsymbol{T} = ml^2\mathbf{1}\,, \quad \boldsymbol{V} = kl^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\,. \tag{4}$$

3. Determine the normal modes of the system, i.e. the proper frequencies and the associated characteristic vectors. (The characteristic vectors do not need to be normalized).

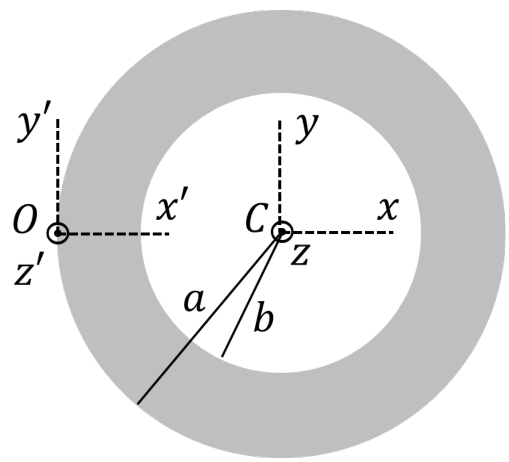
4. For each normal mode, interpret in few words or with a sketch the relative motions of the masses.

Problem F4 – Rotating Ring

[10 Points] The mass density of a flat ring, with inner radius b, and outer radius a, can be parameterized as

$$\rho(r,\phi,z) = \frac{M}{\pi(a^2-b^2)}\Theta(a-r)\Theta(r-b)\delta(z)\,, \tag{5}$$

where M is the total mass and where the polar coordinates (r,ϕ,z) are defined relative to the axes (Cx, Cy, Cz), shown in the figure.



1. Compute the volume integral of $\rho(r,\phi,z)$ to show that it indeed yields the total mass of the ring.

2. Construct the complete moment of inertia tensor of the ring relative to the system of axes (Cx, Cy, Cz). **Notes:** 1) Use the symmetry properties of the ring and the properties of the inertia tensor relative to the principal axes to limit the number of elements of the inertia tensor to be calculated; 2) Moments of inertia are additive. Consider first the calculation of the moments of inertia for a full disk to deduce those of the ring.

3. Determine the moment of inertia tensor relative to the system of axes (Ox', Oy', Oz'), where $\overrightarrow{CO} = -a\vec{e}_x$.

4. Calculate the kinetic energy, T, of the ring when it rotates with an angular velocity $\vec{\omega} = \omega_0\vec{e}_z$, around the axis Cz. Calculate also the kinetic energy T' when it rotates with an angular velocity $\vec{\omega}' = \omega_0\vec{e}'_z$ around the axis Oz'.

Problem F5 – Hamilton Dynamics in One Dimension

[10 Points] Consider the motion of a mass m, described by the Lagrangian of a single generalized coordinate q,

$$L = \frac{1}{2}m\dot{q}^2 + \frac{\alpha}{q^2}\,, \tag{6}$$

where α is a real constant.

1. Determine the canonical momentum, p, and use it to construct the Hamiltonian of the system.

2. Derive Hamilton's equations and deduce the equation of motion for q.

3. Consider the quantity

$$F(q,p,t) = \frac{qp}{2} - Ht\,. \tag{7}$$

   Express the time evolution, dF/dt, in terms of Poisson brackets and perform its calculation. Conclude whether F is a constant of motion or not.

4. Consider the coordinate transformation from (q,p) to (Q,P), defined by

$$Q = \arctan\frac{\beta q}{p}\,, \quad P = \frac{\beta q^2}{2} + \frac{p^2}{\beta}\,. \tag{8}$$

   Check whether this transformation is canonical by computing the fundamental Poisson bracket $\{Q,P\}_{(q,p)}$.