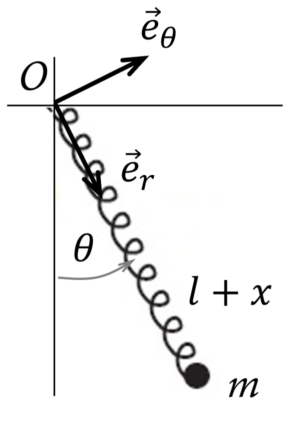
MICHIGAN STATE UNIVERSITY

Final Exam / Subject Exam

(December 11, 2024)

Problem F1 – Spring pendulum

[10 points = 2 + 2 + 2 + 4] A massless spring has an unstretched length l and a spring constant k. It is fixed to the wall with a nail at one end and has a mass m attached to the other end (see figure). The pendulum is assumed to move in a vertical plane.



1. Using the generalized coordinates θ and x, express the instantaneous velocity, $\dot{\vec{r}}$, and the instantaneous acceleration, $\ddot{\vec{r}}$, of the mass in the polar basis $\{\vec{e}_r, \vec{e}_\theta\}$.

2. Write the kinetic and the potential energies of the system in terms of the coordinates θ and x. Deduce the Lagrangian and indicate by inspection whether any of the coordinates is cyclic (do not calculate Lagrange equations).

The equations of motion of the system can be cast in the form

$$\ddot{x} - (l + x)\dot{\theta}^2 + \frac{k}{m}x = g\cos\theta \tag{1}$$

$$(l + x)\ddot{\theta} + 2\dot{\theta}\dot{x} + g\sin\theta = 0\,. \tag{2}$$

3. Express the conditions of equilibrium of the mass and find the solutions (θ, x) at the equilibrium positions.

Consider the approximation of small oscillations around an equilibrium position with $\theta \ll 1$. The deviations from equilibrium are respectively denoted by α and η for the θ and x coordinates. It is furthermore assumed that $\dot{\alpha}$ and $\ddot{\alpha}$ are smaller or of the same order than α and that $\dot{\eta}$ and $\ddot{\eta}$ are smaller or of the same order as η.

4. Under these approximations, deduce the equations of motion to leading order in α and η by justifying each step, and find the frequencies of oscillation.

Problem F2 – Central Force and Stability of Orbits

[10 points = 1 + 3 + 2 + 2 + 2] A mass m moves in a central force field defined by

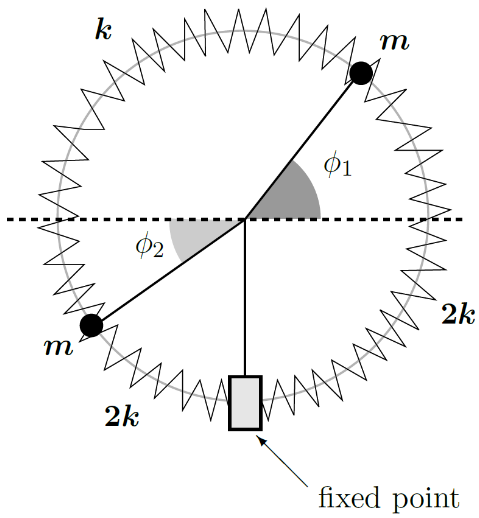
$$\vec{F}(\vec{r}) = -\frac{\alpha}{r}\vec{e}_r \,, \tag{3}$$

where $\alpha > 0$ is a constant.

1. Determine (within an arbitrary integration constant V_0) the potential, $V(r)$, associated to this force field.
2. Given the magnitude, l, of the mass angular momentum in this central field, determine the radii R of circular orbits and the energies E_R of such orbits as a function of the parameters of the problem.
3. Check whether the circular orbits are stable or unstable.
4. State the equation which enables to determine the turning points for orbits with $E > E_R$ (do not solve the equation) and indicate whether the equation has always solutions or not.
5. Justify shortly what type of orbits (bound, unbound, open, closed) result for $E < E_R$ and for $E > E_R$.

Problem F3 – Two Masses on a Circular Track

[10 points = 3 + 2 + 4 + 1] Consider two identical masses m that slide without friction on a circular horizontal track of radius R (see figure). Each mass is connected to a fixed point by a spring with spring constant $2k$, and it is coupled to the other mass by a spring with spring constant k. In the figure, ϕ_1 and ϕ_2, designate the counterclockwise angular displacements of the masses from their equilibrium positions, indicated by the dotted lines.



1. Construct the Lagrangian of the system in terms of the angular displacements ϕ_1 and ϕ_2.

2. Express the Lagrangian in the quadratic form

$$L = \frac{1}{2}\dot{\underline{\phi}}^T \boldsymbol{T}\, \dot{\underline{\phi}} - \frac{1}{2}\underline{\phi}^T \boldsymbol{V}\, \underline{\phi}\,, \tag{4}$$

where

$$\underline{\phi} = (\phi_1, \phi_2)^T\,, \tag{5}$$

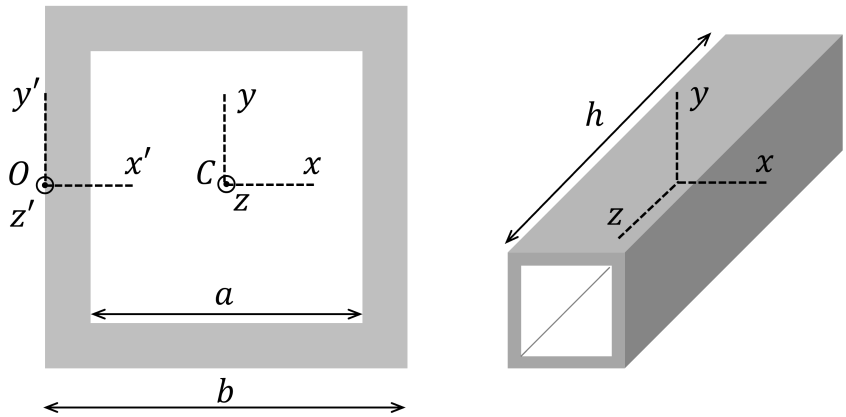
and identify the matrices $\boldsymbol{T}$ and $\boldsymbol{V}$.

3. Determine the characteristic frequencies and the associated eigenvectors of the normal modes of oscillation (the vectors do not need to be normalized).

4. Interpret the solutions and make a sketch for each normal mode of oscillation.

Problem F4 – Rotating Beam with Square Section

[10 Points = 5 + 1 + 2 + 2] A commonly used profile in civil construction is that of a pipe with a square cross section (see figure). The mass of the pipe is M and the dimensions are given on the figure.



1. Construct the complete moment of inertia tensor of the pipe relative to the system of axes (Cx, Cy, Cz) where the point C is at the geometrical center of the pipe. Express the moment of inertia tensor as a function of the mass M of the pipe.

   Notes: 1) Use symmetry properties and properties of the inertia tensor to limit the number of moments of inertia to be calculated, clearly mentioning the property used. 2) Moments of inertia are additive. A possible approach is to calculate the moments of inertia for homogeneous rectangular cuboids.

For the following questions, the diagonal moments of inertia of the pipe can be denoted by A, B and C.

2. Calculate the kinetic energy, T, of the pipe when it rotates with an angular velocity $\vec{\omega}_1 = \omega_0 \vec{e}_z$, around the Cz axis.

3. Calculate the kinetic energy T' when the body rotates with an angular velocity $\vec{\omega}_2 = \omega_0 \vec{e}'_z$ around the axis Oz' parallel to Cz.

4. In the pipe-fixed frame (Cx, Cy, Cz), determine the torque which is necessary to maintain the pipe rotating with an angular velocity $\vec{\omega}_3 = \omega_0(\vec{e}_x + \vec{e}_y + \vec{e}_z)$.

Problem F5 – Building a One Dimensional Crystal

[10 Points = 2 + 3 + 3 + 2] A building block for a one-dimensional crystal is composed by a system of three identical particles of mass m in which the masses only interact with their neighbors. The dynamics of the elementary system is described by the Lagrangian

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V_0 e^{a(x_3 - x_2)} - V_0 e^{a(x_2 - x_1)}\,, \tag{6}$$

where the x_i are the Cartesian coordinates of the masses in the one-dimensional lattice and V_0 and a are constant parameters.

1. Determine the canonical momenta, p_1, p_2 and p_3, and use them to construct the Hamiltonian of the system.
2. Derive Hamilton's equations for the system.
3. Calculate the Poisson bracket $\{p_1 + p_2 + p_3, H\}$ and give the physical interpretation and the explanation of the result.
4. Argue or demonstrate through an explicit calculation whether the total energy of the system is conserved.