PHY 831: Final exam

Welcome to your statistical physics exam! A few things to note:

- Make sure to put your name on both sides of every page (to help keep track of who's who once I scan everything).
- The exam consists of five problems, each having a few parts.
- You have three hours for the exam. I'd recommend first taking a quick look at all problems to see which may require more or less time.
- Try to keep your answer to each part within the box provided — that'll make the grading much more amenable to Gradescope. That said, it is of course no problem at all if you need extra space. I have blank sheets for you to take — just keep each extra page next to its associated problem when turning the exam in.
- I might provide you with a big box even for a short part of a question, but I won't provide you with a small box for a long part — use that as a hint.
- Feel free to ask any questions you may have. I might not be willing to answer, but there's no harm in asking.

Good luck!

Don't write anything on this page — it won't be scanned with the rest.

Name:

1) Consider a non-interacting *classical* gas in three dimensions, subject to a quadratic potential in the $\hat{x}$ and $\hat{y}$ directions but a linear potential in the $\hat{z}$ direction:

$$H = \sum_{j=1}^{N} \left(\frac{|\vec{p}_j|^2}{2m} + \frac{k(x_j^2 + y_j^2)}{2} + \lambda z_j \right). \tag{1}$$

Assume that x_j and y_j can take any value in $(-\infty, \infty)$ but z_j can only take values in $(0, \infty)$.

a) [8 pts] Calculate the free energy as a function of temperature. Include the appropriate factors of $N!$ and $2\pi\hbar$.

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b) [6 pts] Calculate the average energy as a function of temperature.

c) [6 pts] Determine the entropy as a function of *energy*.

d) [2 pts] What is the number of (many-body) states having energy E? (It's fine to leave your answer in terms of the results to previous parts.)

Name:

2) We'll again consider the van der Waals gas, for which the entropy is (assuming constant N throughout)

$$S(E,V) = N \log \frac{V - bN}{N} + \frac{3N}{2} \log \left(\frac{E}{N} + \frac{aN}{V} \right). \tag{2}$$

a) [10 pts] Determine the energy-temperature and volume-pressure relationships.

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b) [8 pts] Different gases can have different parameters a and b. Consider two such gases, the first of which has $a = 0$ and $b = b_1$, and the second of which has $a = 0$ and $b = b_2$. Suppose these two gases have the same number N of particles (which again is fixed for each), but they can exchange energy and volume.

If the total energy is E and the total volume is V, and the two gases are in equilibrium with each other, how much energy and volume does gas 1 have?

Hint: When systems are in equilibrium with each other, which quantities are equal among them?

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3) Consider a gas of bosons whose energy as a function of momentum is

$$\epsilon(\vec{p}) = c|\vec{p}|. \tag{3}$$

We'll study these bosons in a general dimension d (not necessarily 3D).

a) [8 pts] Calculate the density of states $g(\epsilon)$. (Recall that we've been denoting the surface area of a d-dimensional unit sphere by B_d — feel free to use the same notation in your answer.)

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b) [6 pts] For which values of d does a BEC form at finite temperature?

c) [6 pts] How does the critical temperature for BEC formation scale with density (again in general d)? In other words, T_c is proportional to n^α for some exponent α — find α in terms of d.

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4) Consider a system of *spin-1*s (meaning that each spin s_j takes values in $\{-1, 0, +1\}$), with Hamiltonian

$$H_0 = -\sum_{j=1}^{N} \left(h s_j + \Delta s_j^2\right). \tag{4}$$

a) [7 pts] Calculate the free energy.

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b) [7 pts] Calculate $\langle s_j \rangle$ and $\langle s_j^2 \rangle$.

c) [3 pts] Now suppose that, in addition to the terms above, the spins interact ferromagnetically. In other words, the Hamiltonian has additional terms

$$H_{\text{int}} = -J \sum_{\langle jk \rangle} s_j s_k, \tag{5}$$

and the full Hamiltonian is $H_0 + H_{\text{int}}$.

Within mean-field theory, what equation should you solve to determine the magnetization $m \equiv \langle s_j \rangle$? Assume that each spin interacts with z others.

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5) Here we'll consider a random walk with drift: during a time interval τ, not only does the particle move a distance l in a random direction, but in addition it moves a set amount $a\hat{x}$. Thus the location of the particle after k steps (with $t \equiv k\tau$) is

$$\vec{r}(t) = \left(a\hat{x} + l\hat{n}_1\right) + \left(a\hat{x} + l\hat{n}_2\right) + \cdots + \left(a\hat{x} + l\hat{n}_k\right), \tag{6}$$

where each $\hat{n}_j$ is an uncorrelated random unit vector as usual.

a) [3 pts] Calculate $\langle \vec{r}(t) \rangle$.

b) [6 pts] Calculate the variance in $\vec{r}(t)$, i.e., $\langle |\vec{r}(t)|^2 \rangle - |\langle \vec{r}(t) \rangle|^2$.

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c) [4 pts] Putting aside random walks for the moment, now let's turn to how we might account for drift in the diffusion equation. We add an additional term as follows (we'll focus on the 1D equation for simplicity):

$$\frac{\partial f(x,t)}{\partial t} + v\frac{\partial f(x,t)}{\partial x} = D\frac{\partial^2 f(x,t)}{\partial x^2}. \tag{7}$$

What is the equation of motion for the (spatial) Fourier transform of $f(x,t)$, denoted $\widetilde{f}(k,t)$, and what is the solution of that equation (in terms of the initial condition $\widetilde{f}(k,0)$)?

Hint: What do Fourier transforms do to derivatives? You can use this fact freely — no need to derive it.

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d) [6 pts] Suppose that at time 0, the density profile is $f(x,0) = f_0\delta(x)$. Show that the density profile at a later time t is

$$f(x,t) = \frac{f_0}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x-vt)^2}{4Dt}\right]. \tag{8}$$

Hint: Make use of your answer to part c.

e) [4 pts] The random walk (with drift) has parameters l, τ, and a. Meanwhile, the diffusion equation (with drift) has parameters D and v. Comparing Eq. (8) to your results for parts a and b — which apply in any dimension, including 1D — how should D and v be related to l, τ, and a (not worrying about numerical prefactors)?