Final exam

1) Consider a classical non-interacting gas (in 3D) in a linear confining potential, i.e., with Hamiltonian

$$H = \sum_{j=1}^{N} \frac{|\vec{p}_j|^2}{2m} + \lambda \sum_{j=1}^{N} \Big(|x_j| + |y_j| + |z_j| \Big). \tag{1}$$

a) [10 pts] Calculate the free energy as a function of temperature (including factors of $N!$ and $2\pi\hbar$).

b) [6 pts] Calculate the average energy as a function of temperature.

c) [4 pts] What is the average *kinetic* energy as a function temperature?

2) Suppose the entropy of some gas is given by

$$S(E, V, N) = 2\sqrt{NE} + 2\sqrt{NV}. \tag{2}$$

a) [6 pts] Determine the energy-temperature and volume-pressure relationships.

b) [9 pts] Determine the Gibbs free energy G (which recall is a function of T and P).

c) [5 pts] Now suppose that your answers to the previous parts only apply for a certain phase of the gas — call it the "weird" phase. The gas can also be in a much more "normal" phase, where the Gibbs free energy is $G = -T \log T$ (up to constants and prefactors that we won't worry about).

Holding the pressure fixed, which phase will the gas be in at sufficiently high temperatures?

3) Consider a gas of *relativistic* fermions (in 3D), meaning that the energy-momentum relationship is

$$\epsilon(\vec{p}) = c|\vec{p}|. \tag{3}$$

The spin-statistics theorem says that fermions cannot be spinless, so assume that these fermions are spin-S (which means that there are $2S + 1$ spin states per momentum).

a) [4 pts] Without doing any math, do you expect the Fermi energy (at fixed density) to increase or decrease as the value of S increases? Why?

b) [9 pts] Now let's check. Calculate the density of states $g(\epsilon)$ and the Fermi energy ϵ_F (denoting the particle density by n).

c) [7 pts] What is the total energy of the gas at zero temperature? (It's fine to leave the answer in terms of ϵ_F.)

4) Let's return to our usual setting of non-interacting spins in a magnetic field, i.e.,

$$H = -h \sum_j \sigma_j, \tag{4}$$

but now suppose the spins are spin-3/2 — this means that σ_j can take the values $\{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$.

a) [8 pts] Calculate the average magnetization $\sum_j \sigma_j$ as a function of temperature.

b) [7 pts] What is the probability of a spin having $\sigma_j > 0$?

c) [5 pts] Describe in words how you would calculate the number of configurations $\Omega(M)$ for which the magnetization is M. (No need to actually calculate it — the math is ugly, but that's what computers are for.)

5) Here we'll consider a particle undergoing a 1D random walk. Recall that this means the position of the particle at time t is

$$x(t) = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_k, \tag{5}$$

where $k \equiv t/\tau$ (with τ a constant) and each Δx_j is a random variable.

a) [9 pts] Suppose that the particle has a chance of staying put at each step, so that each Δx_j is given by

$$\Delta x_j = \begin{cases} -l \text{ with probability } a/2 \\ 0 \text{ with probability } 1-a \\ l \text{ with probability } a/2 \end{cases} . \tag{6}$$

Determine the diffusion constant D for this random walk.

b) [6 pts] Let $p(x,t)$ denote the probability that the particle is at position x at time t. What is the equation giving $p(x,t)$ in terms of the probabilities at time $t-\tau$?

c) [5 pts] Taylor-expand the equation from part b (for small l and τ) to derive the diffusion equation, and confirm that you obtain the same diffusion constant D as in part a.