SECRET STUDENT NUMBER: ____________________

Classical Mechanics Subject Exam   August 28, 2017

DO NOT WRITE YOUR NAME ON ANY SHEET!

SECRET STUDENT NUMBER: ________________

1. [10 pts] The position of a point particle is governed by the equation of motion

$$\ddot{x} + 4x = \cos(\omega t)$$

(a) Solve this differential equation to find x as a function of time t, given that the particle starts from rest at the point $x = 0$ at $t = 0$.

(b) Take the limit $\omega \to 2$ in your answer to part (a) and describe the solution in that limit.

SECRET STUDENT NUMBER: ______________

2. [10 pts] A particle of mass M slides without friction inside the surface of a frictionless paraboloid of revolution $z = A(x^2 + y^2)$ where $A > 0$ is a constant. The symmetry axis z of the paraboloid is vertical, so there is a gravitational force in the $-\hat{z}$ direction.

   (a) Find the Lagrangian, using the polar coordinates r and ϕ as generalized coordinates, where $x = r\cos\phi$ and $y = r\sin\phi$.

   (b) There are two obvious constants of the motion. Use those constants to find $\dot{r}$ as a function of r.

SECRET STUDENT NUMBER: ________________

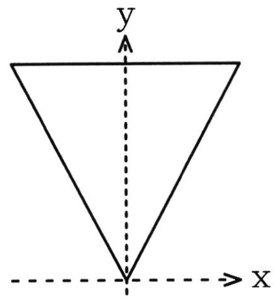
3. [10 pts] Find the path $y(x)$ from $(0,0)$ to $(1,1)$ in the (x,y) plane that makes the integral

$$I = \int_0^1 \left[\left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} + 4y^2 \right] dx$$

a minimum. This is a calculus of variations problem: you should write the Euler-Lagrange equation, solve it, and then impose the end-point conditions.

4. [10 pts] Consider the Lagrangian $L = \frac{1}{2} M \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + K\, x\, \dot{y}$ where x and y are generalized coordinates and K and M are constants.

   Change variables in this Lagrangian to cylindrical coordinates $x = r \cos\phi$, $y = r \sin\phi$, $z = z$. Then find the Hamiltonian as a function of the new coordinates r, ϕ, z and their appropriate canonical momenta.

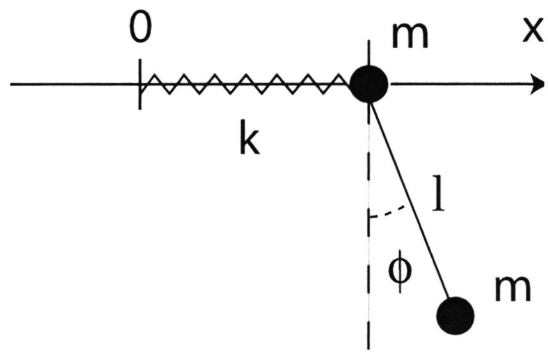


5. [5 pts] An infinitely thin flat uniform sheet of metal with mass M has the shape of a symmetric triangle: the width and height (the x and y directions in the picture) are both equal to B.

Find its three principal moments of inertia for rotations about the point $x = y = 0$.

SECRET STUDENT NUMBER: ________________

6. [10 pts] A bead of mass m slides without friction on a thin horizontal rod along the x axis. The mass is attached to one end of a spring with spring constant k and unstretched length b. The other end of the spring is fixed at $x = 0$. A second bead also of mass m hangs from the first bead by a massless thread of length ℓ.



Find the frequencies of small oscillations for this system, for motions in which the second bead moves only in the plane spanned by the rod and the vertical direction; so $x - b$ and ϕ are the only variables needed to describe the motion.

You can leave your answer in the form of a quadratic equation for ω^2 — you do not need to solve the equation.