Student Number:________________________

Electricity & Magnetism **Subject Exam** **January 14, 2015**

Please read all of the following before starting the exam:

- Before starting the exam, write your student number on each page of the exam. If you require extra paper, write your student number and the relevant problem number on the extra page(s).
- All problems are assumed to be in Gaussian units. If you choose to convert to SI units, please state so. You will be responsible for the correct conversion factors.
- You may use a simple calculator, but no external notes, books, etc.
- Show all work as neatly and logically as possible to maximize your credit. Circle or otherwise indicate your final answers.
- This test has 5 problems for a total of 100 points. Please make sure that you have all of the pages.
- Good luck!

VECTOR CALCULUS

Gradient vector $= \vec{\nabla} = (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$
$\mathrm{div}\,\mathbf{v} \equiv \vec{\nabla}\cdot\mathbf{v}$
$\mathrm{curl}\,\mathbf{v} \equiv \vec{\nabla}\times\mathbf{v}$

Cylindrical coordinates (ρ, ϕ, z):

$$\nabla S = \frac{\partial S}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial S}{\partial \phi}\mathbf{e}_\phi + \frac{\partial S}{\partial z}\mathbf{e}_z$$

$$\vec{\nabla}\cdot\mathbf{v} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\begin{aligned}\vec{\nabla}\times\mathbf{v} &= \left[\frac{1}{\rho}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\mathbf{e}_\rho \\ &\quad + \left[\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho}\right]\mathbf{e}_\phi \\ &\quad + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial \phi}\right]\mathbf{e}_z\end{aligned}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical polar coordinates (r, θ, ϕ):

$$\nabla S = \frac{\partial S}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\mathbf{e}_\phi$$

$$\vec{\nabla}\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned}\vec{\nabla}\times\mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\mathbf{e}_r \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\mathbf{e}_\theta \\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\mathbf{e}_\phi\end{aligned}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Spherical Harmonics:

$$l = 0: \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l = 1: \qquad Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$l = 2: \qquad Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

$$Y_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

Legendre Polynomials:

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l,0}(\theta,\phi)\ .$$