SECRET STUDENT NUMBER: STUDNUMBER

DO NOT WRITE YOUR NAME OR STUDENT NUMBER ON ANY SHEET!

$$
\begin{aligned}
\vec{a}\times(\vec{b}\times\vec{c}) &= \vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b}),\\
\vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b}),\\
(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}),\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) &= 0,\\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\vec{a}) &= 0,\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\vec{a}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\vec{a})-\nabla^2\vec{a},\\
\boldsymbol{\nabla}\cdot(\psi\vec{a}) &= \vec{a}\cdot\boldsymbol{\nabla}\psi+\psi\boldsymbol{\nabla}\cdot\vec{a},\\
\boldsymbol{\nabla}\times(\psi\vec{a}) &= \boldsymbol{\nabla}\psi\times\vec{a}+\psi\boldsymbol{\nabla}\times\vec{a},\\
\boldsymbol{\nabla}(\vec{a}\cdot\vec{b}) &= (\vec{a}\cdot\boldsymbol{\nabla})\vec{b}+(\vec{b}\cdot\boldsymbol{\nabla})\vec{a}+\vec{a}\times(\boldsymbol{\nabla}\times\vec{b})+\vec{b}\times(\boldsymbol{\nabla}\times\vec{a}),\\
\boldsymbol{\nabla}\cdot(\vec{a}\times\vec{b}) &= \vec{b}\cdot(\boldsymbol{\nabla}\times\vec{a})-\vec{a}\cdot(\boldsymbol{\nabla}\times\vec{b}),\\
\boldsymbol{\nabla}\times(\vec{a}\times\vec{b}) &= \vec{a}(\boldsymbol{\nabla}\times\vec{b})-\vec{b}(\boldsymbol{\nabla}\times\vec{a})+(\vec{b}\cdot\boldsymbol{\nabla})\vec{a}-(\vec{a}\cdot\boldsymbol{\nabla})\vec{b},\\
\boldsymbol{\nabla}\cdot\vec{r} &= 3,\\
\boldsymbol{\nabla}\times\vec{r} &= 0,\\
\boldsymbol{\nabla}\cdot\hat{r} &= 2/r,\\
\boldsymbol{\nabla}\times\hat{r} &= 0,\\
(\vec{a}\cdot\nabla)\hat{r} &= \frac{1}{r}[\vec{a}-\hat{r}(\vec{a}\cdot\hat{r})]=\frac{\vec{a}_\perp}{r}.
\end{aligned}
$$

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\begin{aligned}
\int_V d^3r\ \boldsymbol{\nabla}\cdot\vec{A} &= \int_S d\vec{S}\cdot\vec{A},\\
\int_V d^3r\ \boldsymbol{\nabla}\psi &= \int_S \psi\ d\vec{S},\\
\int_V d^3r\ \boldsymbol{\nabla}\times\vec{A} &= \int_S d\vec{S}\times\vec{A},\\
\int_V d^3r\ (\phi\nabla^2\psi+\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi) &= \int_S \phi\ d\vec{S}\cdot\boldsymbol{\nabla}\psi,\\
\int_V d^3r\ (\phi\nabla^2\psi-\psi\nabla^2\phi) &= \int_S (\phi\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\phi)\cdot d\vec{S},\\
\int_S (\boldsymbol{\nabla}\times\vec{A})\cdot d\vec{S} &= \oint d\vec{\ell}\cdot\vec{A},\\
\int_S d\vec{S}\times\boldsymbol{\nabla}\psi &= \oint_C d\vec{\ell}\ \psi.
\end{aligned}
$$

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\begin{aligned}
\nabla^2 &= \partial_r^2+\frac{2}{r}\partial_r-\frac{\ell(\ell+1)}{r^2},\\
\nabla^2 &= \partial_\rho^2+\frac{1}{\rho}\partial_\rho-\frac{m^2}{r^2},\\
\nabla^2\left(\frac{1}{r}\right) &= -4\pi\delta(\vec{r}).
\end{aligned}
$$

$$L^{\alpha}{}_{\beta} = \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{aligned}
F_{\alpha\beta} &= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \\
F^{\alpha\beta} &= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
m\frac{d}{d\tau}u^{\alpha} &= eF^{\alpha\beta}u_{\beta}, \\
\frac{d\vec{p}}{dt} &= e\vec{E} + e\vec{v}\times\vec{B}, \\
\omega_c &= \frac{eB}{\gamma m}, \\
\boldsymbol{\nabla}\cdot\vec{E} &= 4\pi J^0, \\
(\boldsymbol{\nabla}\times\vec{B}) - \partial_t\vec{E} &= 4\pi\vec{J}, \\
\boldsymbol{\nabla}\cdot\vec{B} &= 0, \\
\partial_t\vec{B} + \boldsymbol{\nabla}\times\vec{E} &= 0, \\
\partial_{\alpha}F^{\alpha\beta} &= 4\pi J^{\beta}, \\
\partial_{\alpha}\tilde{F}^{\alpha\beta} &= 0, \\
e^2 &= \frac{\hbar c}{137.036}, \\
T^{\alpha\beta} &= \pi^{\alpha}\partial^{\beta}\phi - g^{\alpha\beta}\mathcal{L}, \\
\pi^{\alpha} &\equiv \frac{\partial\mathcal{L}}{\partial(\partial_{\alpha}\phi)}, \\
T^{00} &= \frac{1}{8\pi}(E^2 + B^2), \\
T^{0i} &= \frac{1}{4\pi}\epsilon_{ijk}E_jB_k, \\
T^{ij} = -T^{i}{}_{j} &= \frac{1}{8\pi}(\delta_{ij}(E^2 + B^2) - 2E_iE_j - 2B_iB_j), \\
\vec{E} &= -\boldsymbol{\nabla}A_0 - \partial_t\vec{A} = -\boldsymbol{\nabla}\Phi - \partial_t\vec{A}, \quad \vec{B} = \boldsymbol{\nabla}\times\vec{A}.
\end{aligned}$$

$$\begin{aligned}
P_{\ell}(\cos\theta) &= \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell m=0}(\theta), \\
Y_{0,0} &= \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \\
Y_{1,\pm1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi}, \quad Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \\
Y_{2,\pm1} &= \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi}, \quad Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}, \\
Y_{\ell -m}(\theta,\phi) &= (-1)^m Y^*_{\ell m}(\theta,\phi), \\
\delta_{\ell\ell'}\delta_{mm'} &= \int d\Omega\; Y_{\ell,m}(\theta,\phi)Y_{\ell',m'}(\theta,\phi), \\
P_0(x) &= 1, \quad P_1(x) = x, \\
P_2(x) &= \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \\
P_{\ell}(x=1) &= 1, \quad \int_{-1}^{1} dx\; P_{\ell}(x)P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}, \\
(3D)\quad \Phi &= \sum_{\ell m}\left(A_{\ell m}r^{\ell} + B_{\ell m}r^{-\ell-1}\right)Y_{\ell m}(\theta,\phi)e^{im\phi}, \\
(2D)\quad \Phi &= A_0\ln(\rho) + \sum_m e^{im\phi}\left(A_m\rho^m + B_m\rho^{-m}\right), \\
\Phi &= A_0J_0 = \sum_m e^{im\phi}\left(A_mJ_m(k\rho) + B_mN_m(k\rho)\right)e^{\pm kz}, \\
\Phi &= \frac{q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \sum_m \frac{4\pi}{5r^3}q_{2m}(r)Y_{2m}(\theta,\phi), \\
\vec{E} &= -\frac{1}{r^3}\vec{p} + 3\frac{\vec{p}\cdot\vec{r}}{r^5}\vec{r} + \cdots, \\
\Phi(r,\theta,\phi) &= \sum_{\ell m}\frac{4\pi}{(2\ell+1)r^{\ell+1}}q_{\ell m}(r)Y_{\ell m}(\theta,\phi), \\
q_{22} &= \sqrt{\frac{15}{32\pi}}\int d^3r\;\rho(\vec{r})(x-iy)^2 = \sqrt{\frac{15}{288\pi}}(Q_{11} - 2iQ_{12} - Q_{22}), \\
q_{21} &= -\sqrt{\frac{15}{8\pi}}\int d^3r\;\rho(\vec{r})(x-iy)z = -\sqrt{\frac{15}{72\pi}}(Q_{13} - iQ_{23}), \\
q_{20} &= \sqrt{\frac{5}{16\pi}}\int d^3r\;\rho(\vec{r})(3z^2 - r^2) = \sqrt{\frac{5}{16\pi}}Q_{33}, \\
Q_{ij} &\equiv \int d^3r\;(3r_ir_j - r^2\delta_{ij})\rho(\vec{r}), \\
U &= q\Phi_0 - \vec{p}\cdot\vec{E} - \frac{1}{6}Q_{ij}\partial_iE_j,
\end{aligned}$$

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\begin{aligned}
\nabla^2 A^\alpha &= -4\pi J^\alpha, \\
\vec{m} &= \frac{1}{2}\int d^3r\ \vec{r}\times\vec{J} = \frac{I}{2}\int\ \vec{r}\times d\vec{\ell}, \\
\vec{B} &= -\frac{\vec{m}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{m}\cdot\vec{r}), \\
\mu_e &= g_e\frac{e\hbar}{2m_e}, \\
U &= \frac{(\vec{\mu}_N\cdot\vec{\mu}_e)}{r^3} - \frac{3(\vec{\mu}_N\cdot\vec{r})(\vec{\mu}_e\cdot\vec{r})}{r^5} - \frac{8\pi}{3}(\vec{\mu}_N\cdot\vec{\mu}_e)\delta^3(\vec{r}) \\
&\quad -e\frac{(\vec{\mu}_N\cdot\vec{L})}{mr^3}, \\
T_{00} &= \frac{1}{8\pi}\left(|\vec{E}|^2+|\vec{B}|^2\right) \\
&= \frac{a_i^2+b_i^2}{8\pi} = \frac{|\vec{a}|^2}{4\pi}\cos^2(\vec{k}\cdot\vec{r}-\omega t), \\
T_{0i} &= \epsilon_{ijk}\frac{E_jB_k}{4\pi} \\
&= \hat{k}_i\frac{|\vec{a}|^2}{4\pi}\cos^2(\vec{k}\cdot\vec{r}-\omega t), \\
T^{ij} = -T^i{}_j &= \frac{1}{8\pi}(\delta_{ij}(E^2+B^2) - 2E_iE_j - 2B_iB_j). \\
&= \frac{1}{4\pi}\left\{|\vec{a}|^2\delta_{ij} - a_ia_j - b_ib_j\right\}\cos^2(\vec{k}\cdot\vec{r}-\omega t), \\
\omega_s &= \omega\sqrt{\frac{1-v}{1+v}}, \\
(TM)\quad E_z &= \psi(x,y)e^{-i\omega t+ik_zz}, \\
\nabla_t^2\psi &= -(\omega^2-k_z^2)\psi, \quad \Psi|_S = 0, \\
\vec{E}_t(x,y) &= \frac{ik_z}{(\omega^2-k_z^2)}e^{-i\omega t+ik_zz}\boldsymbol{\nabla}_t\psi(x,y), \\
\vec{B}_t(x,y) &= \left(\frac{k_z}{\omega}\right)\hat{z}\times\vec{E}_t, \\
(TE)\quad B_z &= \psi(x,y)e^{-i\omega t+ik_zz}, \\
(\hat{n}\cdot\boldsymbol{\nabla}_t)\psi(x,y)|_S &= 0, \\
\vec{B}_t(x,y) &= \frac{ik_z}{(\omega^2-k_z^2)}e^{-i\omega t+ik_zz}\boldsymbol{\nabla}_t\psi(x,y), \\
\vec{E}_t(x,y) &= -\left(\frac{k_z}{\omega}\right)\hat{z}\times\vec{B}_t.
\end{aligned}
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\begin{aligned}
A^\alpha(x) &= \int d^4x'\ \frac{1}{|\vec{x}-\vec{x}'|}J^\alpha(x')\delta(x_0-x_0'-|\vec{x}-\vec{x}'|), \\
\vec{E} &= e\left\{\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3|\vec{x}|}\right\}, \\
\vec{B} &= \hat{n}\times\vec{E}.
\end{aligned}
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\begin{aligned}
P &= \frac{2e^2}{3c}|\dot{\vec{\beta}}|^2 \quad \text{(Non.Rel.)}, \\
\frac{dP}{d\Omega} &= \frac{e^2}{4\pi(1-\vec{\beta}\cdot\hat{n})^6}|(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}})|^2, \\
P &= \frac{2}{3c}e^2\gamma^6\left[\dot{\beta}^2 - |\vec{\beta}\times\dot{\vec{\beta}}|^2\right], \\
\frac{dP}{d\Omega} &= \frac{e^2}{4\pi(1-\beta\cos\theta)^5}|\dot{\vec{\beta}}|^2\sin^2\theta \quad \text{(linear)}, \\
P &= \frac{2e^2\dot{\beta}^2}{3c}\gamma^6 \quad \text{(linear)}, \\
\frac{dP}{d\Omega} &= \frac{e^2}{4\pi(1-\beta n_\beta)^5}|\dot{\vec{\beta}}|^2\left((1-\beta n_\beta)^2 - (1-\beta^2)n_r^2\right) \quad \text{(circular)}, \\
P &= \frac{2}{3c}e^2\dot{\beta}^2\gamma^4 \quad \text{(circular)}.
\end{aligned}
$$

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\begin{aligned}
\frac{dP}{d\Omega} &= \frac{1}{8\pi}\omega^4|\hat{n}\times\vec{p}|^2, \\
P &= \frac{\omega^4}{3}|\vec{p}|^2,
\end{aligned}
$$

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\begin{aligned}
\text{(Thomson)}\quad \sigma &= \frac{8\pi e^4}{3m^2}, \\
\frac{\Delta\lambda}{\lambda} &= \frac{\hbar\omega}{m}(1-\cos\theta_s).
\end{aligned}
$$

electron	$-2.00231930436182 \pm 0.00000000000052$
muon	$-2.0023318418 \pm 0.0000000013$
proton	$5.585694702 \pm 0.000000017$
neutron	$-3.82608545 \pm 0.00000090$

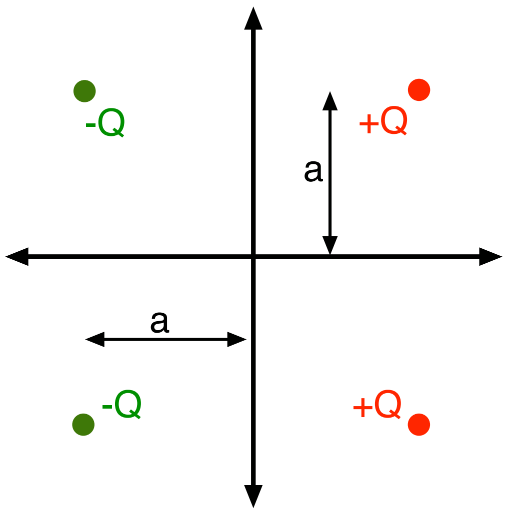
LONG ANSWER SECTION

1. (15 pts) Consider an infinitely long thin cylindrical shell of radius R oriented along the z axis. The shell has a surface charge density,

$$\sigma = \sigma_0 \cos\phi.$$

Find the electric potential at all positions as a function of the transverse radius $r = \sqrt{x^2 + y^2}$ and the azimuthal angle$\phi = \tan^{-1}(y/x)$.

Extra workspace for #1



2. Consider a set of four charges: $+Q$ at $x = a, y = a, z = 0$, $+Q$ at $x = a, y = -a, z = 0$, $-Q$ at $x = -a, y = -a, z = 0$, $-Q$ at $x = -a, y = a, z = 0$.

   (a) (5 pts) For large distances r, the electric potential can be written as $\Phi(\vec{r}) = F(\theta, \phi)/r^n$. What is n?

   (b) (10 pts) Find $F(\theta, \phi)$, where θ and ϕ are spherical coordinates (defined around the z axis).

Extra workspace for #2

3. An antenna is designed by circulating a current around a circular loop of radius R with its axis along the z direction. The current has the form

$$I(\phi, t) = I_0 \cos(\omega t - \phi),$$

where ϕ denotes is the azimuthal angle of a point on the loop.

(a) (5 pts) Find the charge per unit length, $\lambda(\phi, t)$.

(b) (10 pts) Find the radiated power. (Use dipole approximation)

Extra work space for #3

4. A rectangular wave guide has transverse dimensions, $0 < x < a$ and $0 < y < a$. For a transverse electric (TE) wave moving along the z axis with wave number k_z.

   (a) (5 pts) Find the frequency of the propagating wave. Choose the solutions with the fewest nodes in the transverse wave function.

   (b) (10 pts) Find the magnetic field $\vec{B}(x, y, z, t)$ for this solution.

   (c) (5 pts) What is the group velocity of the wave?

Extra work space for #4

SECRET STUDENT NUMBER: STUDNUMBER

SHORT ANSWER SECTION

5. (3 pts each) Light is emitted from a distant source from early in the universe. Choose $>$, $<$ or $=$ for each answer

   (a) The initial frequency of the source is ___________ than the frequency of the light measured by a present-day observer.

   (b) If an observer moves toward the source, the observed frequency will be ___________than the frequency measured by a static observer.

6. (4 pts) In terms of M_p/m_e (mass of proton to mass of electron) calculate the ratio of the radiative powers P_e/P_p emitted for a very high-energy circular accelerator of a given radius R that features either electron or proton beams of the same energy and same currents. Note: Magnetic fields would be quite different to hold particles to the same energy and radius.

7. (3 pts each) Sally Slowpoke measures two events that both occur right in front of her nose separated by a time $\Delta\tau = 1.0$ second. Roberto Rapido travels by in his space ship at some speed $\vec{v}$. (Circle the correct answers)

   (a) The difference in the times of the two events Roberto measures, $\Delta t'$, will
      - always be positive
      - may be positive or negative depending on $\vec{v}$.

   (b) The distance between the two events measured by Robert will be
      - always $< c|\Delta\tau|$
      - greater or less than $c|\Delta\tau|$ depending on $\vec{v}$.

8. You wish to solve the following problem using the method of images:
   "A point charge Q is placed far outside a grounded conducting spherical shell of radius R. The position of the charge is $x = 0, y = 0, z = A >> R$." You are solving for the potential outside the shell.
   True or false: (4 pts)

   (a) The image charge is inside the sphere. ______________
   (b) The potential inside the sphere is constant. _____________
   (c) The magnitude of the image charge must be less than $|Q|$. _____________

9. (5 pts) Which of the following are odd under parity? Circle the answers.

   (a) $\vec{A}$ (the vector potential)
   (b) A_0 (the electric potential)
   (c) $\vec{E}$ (the electric field)
   (d) $\vec{B}$ (the magnetic field)
   (e) $\vec{E} \times \vec{B}$
   (f) $|\vec{B}|^2 - |\vec{E}|^2$
   (g) $|\vec{E}|^2 + |\vec{B}|^2$
   (h) $\vec{E} \cdot \vec{B}$
   (i) $J \cdot A$ (J is the electric current density)

10. Consider a system with non-zero electric and magnetic fields,

$$\vec{B} = B_z\hat{z}, \quad \vec{E} = E_x\hat{x},$$
$$E_x > 0, B_z > 0, \quad E_x > B_z.$$

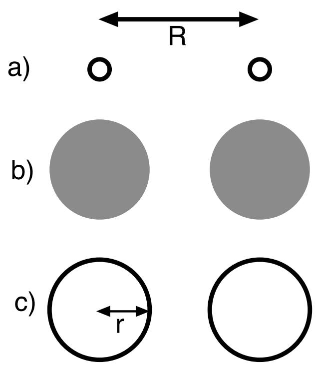
A charged particle is placed at the origin, initially with zero momentum. For each question answer True or False. (6 pts)

(a) There exists a reference frame where $\vec{B} = 0$. _______________

(b) At large times the coordinate $x \to \pm\infty$. _______________

(c) At large times the coordinate $y \to \pm\infty$. _______________

(d) At some parts of the trajectory the particle's momentum p_x would be negative. _______________



11. (4 pts) Consider the three charge configurations shown above:
(a) two point charges $+Q$ separated by R
(b) two spheres of radius $r < R/2$, each with charge $+Q$ uniformly spread throughout the volume, with the centers separated by R
(c) two spherical shells of radius $r < R/2$, each with charge $+Q$ uniformly spread throughout the surface, with the centers separated by R.

The work required to move the spheres (or points) from infinity to the separation R is labeled W_a, W_b and W_c for each configuration. Label each statement as true or false.

(a) $W_a > W_b$ ___________

(b) $W_a > W_c$ ___________

(c) $W_b > W_c$ ___________