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QUANTUM MECHANICS

Subject Exam

Total $4 \times 20 + 5 \times 4 = 100$ points

April 29, 2019

PROBLEM 1. A state of a particle of mass m is described by the wave function

$$\psi(\mathbf{r}) = f(\mathbf{r})e^{i(\mathbf{k}\cdot\mathbf{r})}, \tag{1}$$

where $f(\mathbf{r})$ is a real square integrable function, and $\mathbf{k}$ is a real vector. Determine the probability density in the coordinate space and in the momentum space, probability current $\mathbf{j}(\mathbf{r})$ and the expectation value of the momentum $\langle \mathbf{p} \rangle$.

PROBLEM 2. Using the Born approximation, calculate the differential cross section for scattering of a fast particle by the Yukawa potential,

$$U(r) = \frac{g}{r}\, e^{-\mu r}. \tag{11}$$

Explain the limit $\mu \to 0$. Formulate the condition of validity of the Born approximation. Is the total cross section finite?

PROBLEM 3. Consider a system of two different particles with spin 1/2 each. The spins interact through the Hamiltonian

$$\hat{H}_0 = A(\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2). \tag{16}$$

a. Find the stationary states of the system and their energies.

b. In addition, the static magnetic field $\vec{\mathcal{B}}$ is applied to the system. Find the stationary states of the system and their energies if the spin gyromagnetic ratios of the particles are equal to g_1 and g_2.

PROBLEM 4. Show that the solutions of the Dirac equation with negative energy can be reinterpreted as solutions with positive energy for the antiparticle.

PROBLEM 5a. In the sodium-23 atom (nuclear charge $Z = 11$), a split spectral line is observed corresponding to the transition of the valence electron $p_{1/2} \rightarrow s_{1/2}$. The level $s_{1/2}$ has a hyperfine splitting into two components. (Argue that it is possible to neglect the splitting of the $p_{1/2}$ level.) The ratio of intensities (not energies!) of these two lines is 5:3. Assuming that the hyperfine structure levels are populated *statistically* (proportionally to the number of available magnetic substates), determine the ground spin of the nucleus ^{23}Na.

PROBLEM 5b. A wave function of the particle depends on the azimuthal angle φ with respect to the polar axis z as

$$\psi(\varphi) = \text{const} \cdot \cos^2 \varphi. \tag{29}$$

Determine the probabilities of various values of the orbital momentum projection $\ell_z = m$.

PROBLEM 5c. Estimate the lifetime and the width of the $2p$ state in the hydrogen-like atom where the electron is substituted by the negative muon heavier than the electron by a factor 207 (neglect spin effects and use the long wavelength approximation). Compare this time with the lifetime 2.2 microseconds of the free muon (the muon decays into electron and two neutrinos).

PROBLEM 5d. For a stationary bound one-dimensional state of a particle find the expectation value of the operator prodict $\langle \hat{x}\hat{p} \rangle$.

PROBLEM 5e. Four identical atoms of spin $s = 1$ are placed in a potential trap on the ground state level with orbital momentum $\ell = 0$. Find possible values of the angular momentum J of the whole system. Check the full number of states.