

SCATTERING AMPLITUDES FOR  $ZZ$  PRODUCTION AT THE LHC AND  
TOP-QUARK MASS EFFECTS

by

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## ABSTRACT

### SCATTERING AMPLITUDES FOR $ZZ$ PRODUCTION AT THE LHC AND TOP-QUARK MASS EFFECTS

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With the Large Hadron Collider providing experimental data with unprecedented precision, theoretical predictions must improve similarly as well to keep up. Among a plethora of processes being studied at the LHC, the production of a pair of vector bosons is of particular importance. Consequently, precise theoretical predictions for these processes are necessary. This thesis discusses primarily the calculation of  $ZZ$  production through gluon fusion at 2-loops with full top-quark mass dependence as well as the technological improvements required to successfully perform the calculation. Also discussed briefly is the quark initiated production of  $\gamma\gamma + \text{jet}$  at 2-loops where some of these technologies allowed to overcome prior bottlenecks in the calculation of the helicity amplitudes.

The 2-loop corrections for  $ZZ$  production through massless quarks had been known; in this work, the 2-loop corrections through the massive top quark are calculated. To achieve this, a new algorithm to systematically construct linear combinations of integrals with a convergent parametric representation is developed. This algorithm finds for linear combinations of general integrals with numerators, dots, and dimension shifts as well as integrals from subsectors.

To express the amplitudes in terms of these integrals, Integration-By-Parts (IBP) reduction is performed making use of syzygies and finite field based methods. A new algorithm is employed to construct these syzygies using linear algebra. The IBP reductions for  $gg \rightarrow ZZ$  are successfully performed using these techniques. Further improvements, including pre-determining the structure of the coefficients in IBP reductions, are used to successfully perform the reductions for  $\gamma\gamma + \text{jet}$ . Multivariate partial fractioning is used to simplify the final expressions to more manageable forms and render them suitable for fast numerical evaluation.

In the case of  $gg \rightarrow ZZ$ , due to the presence of structures beyond polylogarithms, sector decomposition is employed to numerically evaluate the finite master integrals. Evaluating the amplitudes, agreement is found with previously calculated expansions specifically in the limit of large and small top mass. Improved results are presented for scattering at intermediate energies and/or for non-central scattering angles. With this calculation, the last building block required for the calculation of full NLO cross-section for  $gg \rightarrow ZZ$  is known.

*Dedicated to my parents.*

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# TABLE OF CONTENTS

<b>LIST OF TABLES</b>	<b>ix</b>
<b>LIST OF FIGURES</b>	<b>x</b>
<b>LIST OF ALGORITHMS</b>	<b>xv</b>
<b>Chapter 1: The Standard Model and perturbative calculations</b>	<b>1</b>
1.1 Overview of the Standard Model . . . . .	1
1.2 Discovery of the Higgs Boson . . . . .	3
1.2.1 Higgs mechanism . . . . .	3
1.2.2 Standard model Higgs . . . . .	10
1.3 Higgs potential and width . . . . .	14
1.3.1 Higgs potential . . . . .	14
1.3.2 Measuring Higgs width . . . . .	16
1.3.3 Experimental constraints . . . . .	17
1.4 Quantum Chromodynamics . . . . .	18
1.4.1 History of QCD . . . . .	18
1.4.2 $SU(N)$ gauge theory . . . . .	21
1.4.3 Perturbative QCD . . . . .	24
1.4.4 Regularisation . . . . .	27
1.4.5 UV renormalisation and IR subtraction . . . . .	30
1.4.6 Asymptotic freedom . . . . .	33
1.4.7 Factorisation . . . . .	36
1.5 Electroweak sector . . . . .	37
1.5.1 Electroweak symmetry breaking . . . . .	37
1.5.2 Custodial symmetry . . . . .	41
1.5.3 Electroweak interactions . . . . .	42
1.6 Theoretical status . . . . .	45
1.6.1 ZZ production at LHC . . . . .	45
1.6.2 Diphoton production at LHC . . . . .	47
<b>Chapter 2: Setup of calculation</b>	<b>49</b>
2.1 Form factor decomposition . . . . .	49
2.2 1-loop amplitude . . . . .	57
2.2.1 Generation of the amplitude . . . . .	57
2.2.2 Reduction to master integrals . . . . .	61
2.2.3 Results . . . . .	67

2.3	2-loop amplitude . . . . .	72
2.3.1	Generation of diagrams . . . . .	72
2.3.2	Class A . . . . .	73
2.3.3	Class B . . . . .	77
<b>Chapter 3: IBP reduction using syzygies</b>		<b>78</b>
3.1	Integration-by-parts reduction . . . . .	78
3.2	Baikov representation . . . . .	84
3.3	Syzygies for IBP reduction . . . . .	88
3.4	Linear algebra based syzygy construction . . . . .	91
3.5	IBP reduction using syzygies . . . . .	95
<b>Chapter 4: Finite basis integrals</b>		<b>98</b>
4.1	Evaluation of Feynman integrals . . . . .	98
4.1.1	Differential equations . . . . .	98
4.1.2	Feynman parametric representation . . . . .	101
4.2	Finite integrals . . . . .	102
4.2.1	Divergences in Feynman parametric representation . . . . .	103
4.2.2	Finite integrals with dimension shifts . . . . .	105
4.2.3	Finite integrals with numerators . . . . .	108
4.2.4	Choice of finite integrals for $gg \rightarrow ZZ$ . . . . .	114
<b>Chapter 5: Compiling the 2-loop amplitude</b>		<b>118</b>
5.1	Inserting reductions into the amplitude . . . . .	118
5.1.1	Multivariate partial fractioning . . . . .	118
5.1.2	Backsubstitution of IBPs . . . . .	120
5.2	Renormalisation, IR subtraction and checks . . . . .	122
5.2.1	UV renormalisation and IR subtraction . . . . .	122
5.2.2	Checks of the calculation . . . . .	125
<b>Chapter 6: Results</b>		<b>128</b>
6.1	Results for the 2-loop $gg \rightarrow ZZ$ amplitude . . . . .	128
<b>Chapter 7: Conclusions</b>		<b>141</b>
<b>APPENDICES</b>		<b>143</b>
	Appendix A QCD Feynman Rules . . . . .	144
	Appendix B Evaluation using Feynman parameters . . . . .	153
	Appendix C Dirac algebra and $\gamma^5$ schemes . . . . .	157
	Appendix D UV renormalisation . . . . .	163
	Appendix E List of denominators . . . . .	170
	Appendix F Numerical checks . . . . .	174
<b>BIBLIOGRAPHY</b>		<b>178</b>

# LIST OF TABLES

Table 1.1.1:	Standard Model Particles . . . . .	2
Table 1.2.1:	Branching ratios for the Standard Model Higgs boson with $m_h = 125.25$ GeV [1]. Numbers reproduced from [2]. . . . .	12
Table 2.3.1:	List of integral families and their propagators for the 2-loop amplitude	74
Table 4.2.1:	Numerical performance of different non-planar integrals for a physical phase-space point. Timings generated with pySecDec [3] using the QMC algorithm [4, 5] on a single Nvidia Tesla V100S GPU, with number of evaluations $neval = 10^7$ . Note that the divergent integrals are only evaluated to $O(\epsilon^0)$ since they start at $\epsilon^{-1}$ . Reproduced from [6]. . . . .	116
Table 6.1.1:	1-loop and 2-loop helicity amplitudes for $gg \rightarrow ZZ$ for the phase-space point $s/m_t^2 = 142/17$ , $t/m_t^2 = -125/22$ , $m_Z^2/m_t^2 = 5/18$ , and $m_t = 1$ , with $\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(1)}$ and $\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(2)}$ defined in Eq. 6.1.2. Only the 8 independent helicity amplitudes (See Eqs. 2.1.20, 2.1.21, and 2.1.22) are shown here. Note that these include only the top-quark contributions from class A diagrams defined in Sec. 2.3.2. The numbers in parentheses denote the uncertainty in the last digit. Reproduced from [6]. . . . .	131
Table F.1:	Numerical poles for the Euclidean phase-space point $s/m_t^2 = -191$ , $t/m_t^2 = -337$ , $m_Z^2/m_t^2 = -853$ , $m_t = 1$ compared against the predicted values. Also shown are the $\epsilon^0$ terms before IR subtraction with the digits in parentheses denoting the uncertainty in the last digit. . . . .	175
Table F.2:	Numerical poles for the physical phase-space point $s/m_t^2 = 142/17$ , $t/m_t^2 = -125/22$ , $m_Z^2/m_t^2 = 5/18$ , $m_t = 1$ compared against the predicted values. Also shown are the $\epsilon^0$ terms before IR subtraction with the digits in parentheses denoting the uncertainty in the last digit. . . . .	176

# LIST OF FIGURES

Figure 1.2.1: $WW$ scattering with the 3 different subprocesses depicted. . . . .	4
Figure 1.2.2: Scalar field $\Phi$ with two different vacuum configurations. For $\mu^2 < 0$ the vacuum state is no longer symmetric and the field acquires a non-zero expectation value. . . . .	6
Figure 1.2.3: $WW$ scattering including the interactions with the Higgs boson. . . . .	9
Figure 1.2.4: Plot of the Standard Model Higgs branching ratios against the Higgs mass. Measured value of Higgs mass is $m_h = 125.25 \pm 0.17$ GeV. Reproduced from [7]. . . . .	11
Figure 1.2.5: Distribution of four lepton invariant mass for the $h \rightarrow ZZ^* \rightarrow l^+l^-l^+l^-$ decay channel for (a) ATLAS (Reproduced from [8]) and (b) CMS (Reproduced from [9]) experiments. . . . .	12
Figure 1.2.6: Distribution of the invariant mass of the photon pair for the $h \rightarrow \gamma\gamma$ decay channel for (a) ATLAS (Reproduced from [8]) and (b) CMS (Reproduced from [9]) experiments. . . . .	13
Figure 1.2.7: Plot of the measured Higgs boson couplings to other Standard Model particles against particle mass. Reproduced from [10]. . . . .	14
Figure 1.3.1: Higgs pair production through gluon fusion at LO. . . . .	16
Figure 1.4.1: Perturbative corrections to $q\bar{q} \rightarrow \gamma$ in QCD represented using Feynman diagrams. . . . .	25
Figure 1.4.2: QCD corrections to $gg \rightarrow h$ . The uncertainty bands represent scale uncertainty for the range $\mu \in \{\frac{m_h}{4}, m_h\}$ with the central value at $\mu = m_h/2$ . Reproduced from [11]. . . . .	26
Figure 1.4.3: 1-loop correction to $q\bar{q} \rightarrow \gamma$ . . . . .	27
Figure 1.4.4: A 3-point massless integral. The incoming momenta $p_1, p_2$ are massless. . . . .	28

Figure 1.4.5: Infrared divergences corresponding to the emission of a real particle. Fig. 1.4.5a shows emission of a gluon with very small momentum i.e. a soft divergence while Fig. 1.4.5b shows the emission of a gluon collinear with the external particle. . . . .	31
Figure 1.4.6: QCD corrections to $q\bar{q} \rightarrow \gamma$ . Fig. 1.4.6b shows the $\alpha_S$ correction through the exchange of a virtual gluon while Fig. 1.4.6c show the $\alpha_S$ correction through the emission of a real gluon. . . . .	32
Figure 1.4.7: Plot showing theoretical prediction for the strong coupling $\alpha_S$ against experimental measurements. Reproduced from [1]. . . . .	35
Figure 2.2.1: Feynman diagrams for the process $gg \rightarrow ZZ$ at 1-loop (LO). 2.2.1a and 2.2.1b are the Higgs exchange diagrams. . . . .	59
Figure 2.2.2: Decomposition of a general 1-loop Feynman integral . . . . .	62
Figure 2.2.3: Comparison of $ \mathcal{M} ^2$ for three different type of contributions. Without including Higgs, the top-quark contribution increases with energy. . . . .	68
Figure 2.2.4: Comparison of $ \mathcal{M} ^2$ for different helicities for the top quark contribution including the Higgs diagrams. . . . .	69
Figure 2.2.5: Ratio of contribution of individual helicities to the total $ \mathcal{M} ^2$ for different helicities for (a) top quark contribution including the Higgs diagrams, and (b) Massless contribution. . . . .	71
Figure 2.3.1: Example Feynman diagrams representing the two classes of diagrams at 2-loops . . . . .	73
Figure 2.3.2: Representative Feynman diagrams in class A with irreducible topologies. The number of master integrals in each topology are 3, 4, 3, 3, 5, 5, and 4 respectively . . . . .	75
Figure 2.3.3: Representative Feynman diagrams in class A with reducible topologies. . . . .	76
Figure 3.1.1: A tadpole graph with the thick loop corresponding to the massive propagator with mass $m$ . . . . .	79
Figure 3.1.2: A two-point function with one massive propagator. The thick line corresponds to the massive propagator. . . . .	81

Figure 3.2.1: A two-point function with massless propagators. The dashed line corresponds to the cut propagators. . . . .	85
Figure 3.5.1: The non-planar topologies for 5-point 2-loop $\gamma\gamma + \text{jet}$ production. . . . .	97
Figure 4.2.1: 3-point integral with 1 massive propagator. . . . .	103
Figure 4.2.2: Examples of divergent and finite integrals in the limit $\epsilon \rightarrow 0$ for a non-planar topology. Thick solid lines represent the top-quark while thick dashed lines represent Z-bosons. Topology (b) contains an irreducible numerator, where $k$ is the difference of the momenta of the edges marked by the thin dash lines. Reproduced from [6]. . . . .	107
Figure 4.2.3: Integrals appearing in the linear combination in Eq. 4.2.20. $I_{1,1}$ is the corner integral of the topology under consideration. $I_{2,1}$ is a second integral in the topology, but with a numerator $(k^2 - m_t^2)$ , where $k$ is equal to the difference of the momenta of the edges marked by the thin dashed lines. Integrals $I_{3,1}, I_{4,1}, I_{5,1}, I_{6,1}, I_{7,1}$ belong to subtopologies. All integrals are defined in $d = 4 - 2\epsilon$ dimensions. Reproduced from [6]. . . . .	113
Figure 4.2.4: Integrals appearing in the linear combination in Eq. 4.2.21. $I_{1,2}$ is the corner integral of the topology under consideration but with a numerator $(k^2 - m_t^2)$ , identical to $I_{2,1}$ from Eq. 4.2.20. $I_{2,2}$ is $I_{1,2}$ but with an extra numerator $(k^2 - m_t^2)$ where $k$ is equal to the difference of the momenta of the edges marked by the thin dashed lines. Integrals $I_{3,2}, I_{4,2}, I_{5,2}, I_{6,2}, I_{7,2}$ are the same as $I_{3,1}, I_{4,1}, I_{5,1}, I_{6,1}, I_{7,1}$ but with an extra numerator $(k^2 - m_t^2)$ . All integrals are defined in $d = 4 - 2\epsilon$ dimensions. Reproduced from [6]. . . . .	114
Figure 5.2.1: Mass counterterm diagrams required at 2-loops. The big dark cross in Fig. 5.2.1b corresponds to the counterterm vertex insertion. . . . .	124
Figure 6.1.1: Comparison of the $\sqrt{s}$ dependence of the unpolarised interference $\mathcal{V}^{(2)}$ with expansion for large and small top-quark mass [12] at fixed $\cos(\theta) = -0.1286$ . The large top-mass expansion is shown in colour red, the small top-mass expansion in blue, and the Padé improved small top-mass expansion in purple. The exact result is shown in black. Note that the error bars have been plotted for the exact result, they are too small to be visible on the plot, however. Reproduced from [6]. . . . .	132

Figure 6.1.2: Comparison of the $\cos(\theta)$ dependence of the unpolarised interference $\mathcal{V}^{(2)}$ with the results expanded in the limit of large top-quark mass for $\sqrt{s} = 247$ GeV (Top Left Panel) and small top-quark mass for $\sqrt{s} = 403$ GeV (Top Right Panel) and $\sqrt{s} = 814$ GeV (Bottom Panel). Reproduced from [6]. . . . .	133
Figure 6.1.3: The $\sqrt{s}$ dependence of 1-loop and 2-loop interferences for polarised $ZZ$ production in gluon fusion at $\cos(\theta) = -0.1286$ . Reproduced from [6].	134
Figure 6.1.4: The $\cos(\theta)$ dependence of 1-loop and 2-loop interferences for polarised $ZZ$ production in gluon fusion at $\sqrt{s}/m_t = 1.426$ . The large top-quark mass expansion [12] (to order $1/m_t^{12}$ ) is shown for comparison. Reproduced from [6]. . . . .	135
Figure 6.1.5: The $\cos(\theta)$ dependence of 1-loop and 2-loop interferences for polarised $ZZ$ production in gluon fusion at $\sqrt{s}/m_t = 2.331$ . The Padé improved small top-quark mass expansion [12] is shown for comparison. Reproduced from [6]. . . . .	136
Figure 6.1.6: The $\cos(\theta)$ dependence of 1-loop and 2-loop interferences for polarised $ZZ$ production in gluon fusion at $\sqrt{s}/m_t = 4.703$ . The small top-quark mass expansion (to order $m_t^{32}$ ) and Padé improved expansion [12] are shown for comparison. Reproduced from [6]. . . . .	137
Figure 6.1.7: The $\sqrt{s}$ dependence of 1-loop and 2-loop interferences for polarised $ZZ$ production in gluon fusion at $\cos(\theta) = -0.1286$ . Here, the top left and bottom right panels of Fig. 6.1.3 are reproduced using Catani's original subtraction scheme [13]. Reproduced from [6]. . . . .	138
Figure 6.1.8: The $\cos(\theta)$ dependence of 1-loop and 2-loop interferences for polarised $ZZ$ production in gluon fusion at $\sqrt{s}/m_t = 1.426$ . The large top-quark mass expansion [12] (to order $1/m_t^{12}$ ) is shown for comparison. Here, the top left and bottom right panels of Fig. 6.1.4 are reproduced using Catani's original subtraction scheme [13]. Reproduced from [6]. . . . .	139
Figure 6.1.9: The $\cos(\theta)$ dependence of 1-loop and 2-loop interferences for polarised $ZZ$ production in gluon fusion at $\sqrt{s}/m_t = 2.331$ . The Padé improved small top-quark mass expansion [12] is shown for comparison. Here, the top left and bottom right panels of Fig. 6.1.5 are reproduced using Catani's original subtraction scheme [13]. Reproduced from [6]. . . . .	139

Figure 6.1.10: The $\cos(\theta)$ dependence of 1-loop and 2-loop interferences for polarised $ZZ$ production in gluon fusion at $\sqrt{s}/m_t = 4.703$ . The small top-quark mass expansion (to order $m_t^{32}$ ) and Padé improved expansion [12] are shown for comparison. Here, the top left and bottom right panels of Fig. 6.1.6 are reproduced using Catani's original subtraction scheme [13]. Reproduced from [6]. . . . .	140
Figure B.1: The integration contour to perform Wick rotation. Note that the poles lie outside the contour. . . . .	154
Figure C.1: The triangle anomaly graph. Here the dark blob vertex represents the axial-vector coupling. . . . .	160
Figure D.1: All the diagrams contributing to 1-loop correction to the gluon propagator, including the counterterm diagram. Requiring that the sum is finite allows the calculation of the counterterm $\delta_G$ . . . . .	167

## LIST OF ALGORITHMS

Algorithm 1:	Syzygies for linear relations without dimension shifts or dots [6]	. . . . . 93
Algorithm 2:	Finite Feynman integrals [6]	. . . . . 111

# Chapter 1

## The Standard Model and perturbative calculations

### 1.1 Overview of the Standard Model

The Standard Model of particle physics provides a fundamental description of the universe at the smallest scales accessible to us. It provides a "unified" prescription of the electromagnetic, weak nuclear, and strong nuclear forces and, along with General Relativity, describes almost all known physical phenomena. Many excellent predictions have been made to date using the Standard Model with one of the most successful being the anomalous magnetic moment  $g - 2$  of the electron which is calculated to 10 digits of relative precision [14, 15]. Incredibly, the experimentally measured value [16], which is also one of the most precise measurements to date and of comparable precision, shows excellent agreement with the Standard Model prediction.

The field of subatomic physics started with the discovery of the electron in 1897 using a simple cathode ray tube. Since then, many more new particles have been discovered, most of them composite. The last puzzle of the Standard Model was solved in July 2012 with the discovery of the Higgs Boson [8, 9]. Predicted in 1964 [17, 18, 19, 20], the Higgs Boson

Fermions			Bosons	
Leptons	$\nu_e$	$\nu_\mu$	$\nu_\tau$	$W^\pm, Z$
	$e^-$	$\mu^-$	$\tau^-$	$\gamma$
Quarks	$u$	$c$	$t$	$g$
	$d$	$s$	$b$	$h$

Table 1.1.1: Standard Model Particles

completed the Standard Model and answered the longstanding question of the origin of gauge boson masses.

The Standard Model roster can be divided into two groups: bosons and fermions. Bosons are integer spin particles and usually appear as force carriers in gauge theories. The photon is the force carrier for the electromagnetic force, gluon is the mediator of strong force, and  $W^\pm, Z$  are the mediators for the weak force; all known force-carriers are spin-1 particles. Fermions are half-integer spin particles; electrons, protons, and neutrons make up almost all of the visible mass in the universe; protons and neutrons are composite though and themselves composed of quarks and gluons bound together through the strong force. All the fermions also have corresponding anti-particles e.g. positron is the anti-particle of the electron. Anti-particles are characterised by one or more of their quantum numbers having the opposite sign compared to the corresponding particles. The positron e.g. is positively charged, opposite to the electron. Neutrons are neutral, but they possess another quantum number called Baryon number which is +1 for the neutron and  $-1$  for the antineutron. Quarks interact through all three forces: strong, weak, and electromagnetic. Electrons, muons, and tau particles do not possess the colour charge and only interact with the weak and electromagnetic forces, while neutrinos are neutral and only interact weakly. The list of Standard Model particles is given in Tab. 1.1.1.

However, despite its many successes, the Standard Model falls short on many accounts. Perhaps the biggest deficiency is the lack of a viable dark matter [21, 22, 23, 24] candidate in the Standard Model. Dark matter accounts for  $\sim 80\%$  of the non-relativistic matter content in the universe which is unexplained by the Standard Model. It also does not

explain Dark energy [25] which makes up  $\sim 70\%$  of the energy density in the universe. The origin of neutrino masses [26] is still a mystery, and while adding a mass term involving right-handed neutrinos is not forbidden in the Standard Model, such a particle would never interact with the rest of the Standard Model particles and is of little consequence. In any case, the extremely small masses of the neutrinos compared to the rest of the Standard Model roster remain unexplained. Other issues include the Hierarchy problem [27, 28], lack of CP violation in the strong sector [29], and matter anti-matter asymmetry [30] which cannot be explained by the existing CP violation in the weak sector. Consequently, it is of extreme importance to have precise theoretical predictions for physical observables in the Standard Model. Comparing these theoretical predictions to experimental measurements, any deviations would signal existence of some new physics; as such, precision calculations are essential for the particle physics program.

## 1.2 Discovery of the Higgs Boson

### 1.2.1 Higgs mechanism

Late 1940s and 1950s saw tremendous progress in the field of particle physics. The Dirac equation predicted the magnetic moment of the electron but the experimental measurements indicated a slight deviation. Gauge theories [31] were eventually developed and were incredibly successful in making precise theoretical predictions, especially with the development of the theory of renormalisation. Quantum Electrodynamics successfully predicted the anomalous magnetic moment of the electron as well as the Lamb Shift paving the way for future work in precision calculations and marking another achievement for Quantum theory. Pions, proposed as the force carriers in the Yukawa potential governing the strong nuclear force, were discovered with the masses predicted by theory. A still unsolved puzzle, however, was the origin of masses for force carriers of the weak nuclear force, which were known from experiments to be massive. No consistent way to generate masses for the gauge bosons while

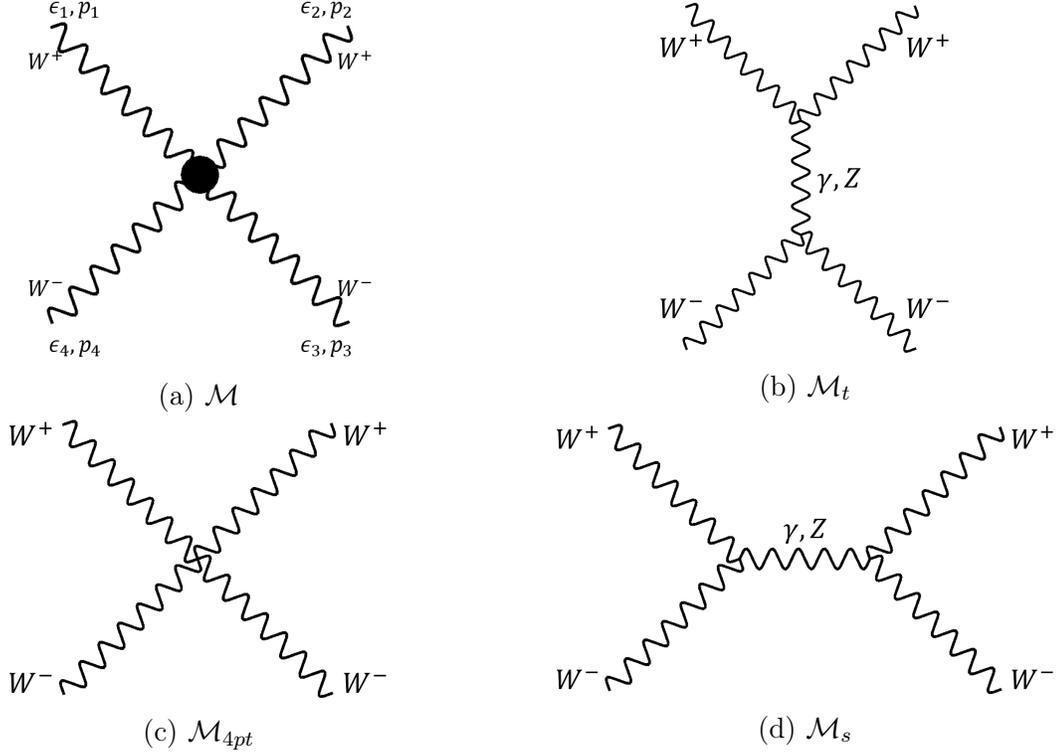


Figure 1.2.1:  $WW$  scattering with the 3 different subprocesses depicted.

preserving the gauge symmetry was known at the time. Another issue was that the scattering of  $W$  and  $Z$  bosons violated unitarity. Scattering amplitude for the longitudinal modes of  $W$  and  $Z$  bosons grows with energy indefinitely. Consider  $W_L^+ W_L^-$  scattering as depicted in Fig. 1.2.1

$$\mathcal{M}(W_L^+ W_L^- \rightarrow W_L^+ W_L^-) = \mathcal{M}_t + \mathcal{M}_{4pt} + \mathcal{M}_s. \quad (1.2.1)$$

Calculating the individual contributions, the 4-point vertex amplitude (Fig. 1.2.1c) is

$$\begin{aligned} \mathcal{M}_{4pt} &= ig_w^2 (2g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho}) \epsilon_L^\mu(p_1) \epsilon_L^\nu(p_2) \epsilon_L^\rho(p_3) \epsilon_L^\sigma(p_4) \\ &= \frac{g^2 (-5 - 12 \cos \theta + \cos 2\theta)}{32} \left( \frac{s}{m_W^2} \right)^2 + \frac{g^2 (1 + 3 \cos \theta)}{2} \left( \frac{s}{m_W^2} \right) \\ &\quad + O \left( \frac{s}{m_W^2} \right)^0, \end{aligned} \quad (1.2.2)$$

amplitude for s-channel  $\gamma, Z$  exchange (Fig. 1.2.1d) is

$$\mathcal{M}_s = -\frac{g^2 \cos \theta}{4} \left(\frac{s}{m_W^2}\right)^2 + \frac{7g^2 \cos \theta}{4} \left(\frac{s}{m_W^2}\right) + O\left(\frac{s}{m_W^2}\right)^0, \quad (1.2.3)$$

and for the t-channel  $\gamma, Z$  exchange (Fig. 1.2.1b) is

$$\mathcal{M}_t = \frac{g^2(-3 + \cos \theta)(1 + \cos \theta)}{16} \left(\frac{s}{m_W^2}\right)^2 + \frac{g^2(3 - \cos \theta)}{8} \left(\frac{s}{m_W^2}\right) + O\left(\frac{s}{m_W^2}\right)^0, \quad (1.2.4)$$

where  $\cos \theta$  is the scattering angle. Adding the three contributions together,

$$\mathcal{M}(W_L^+ W_L^- \rightarrow W_L^+ W_L^-)_{gauge} = \frac{g^2(1 - \cos \theta)}{8} \left(\frac{s}{m_W^2}\right) + O\left(\frac{s}{m_W^2}\right)^0. \quad (1.2.5)$$

The  $E^4/m_w^4$  terms drop out after adding all the different contributions. However, the remainder still grows with energy as  $\sim E^2/m_W^2$ . Non-unitary behaviour of this kind suggested a missing piece. The answer was found through the field of condensed matter physics in spontaneous symmetry breaking [32].

As a consequence of spontaneous symmetry breaking, Nambu-Goldstone bosons [32, 33] are produced as excitations of the quantum field. While these bosons are themselves massless, they are "absorbed" by the gauge fields providing them with an additional longitudinal degree of freedom and a mass term. In certain cases the symmetries are explicitly broken in addition to being spontaneously broken. The resulting Goldstone bosons are no longer massless, but acquire a mass depending on how softly the symmetry is broken. These are referred to as "Pseudo Nambu-Goldstone bosons". An example is pions, which are produced through explicit breaking of the chiral symmetry in QCD due to the quark masses; this results in pions acquiring a small mass. Since the light quark masses are small ( $\sim 5 - 10$  MeV), the pions as a result have a mass that is small compared to the other hadrons.

The Higgs mechanism is best demonstrated through an example. Consider the La-

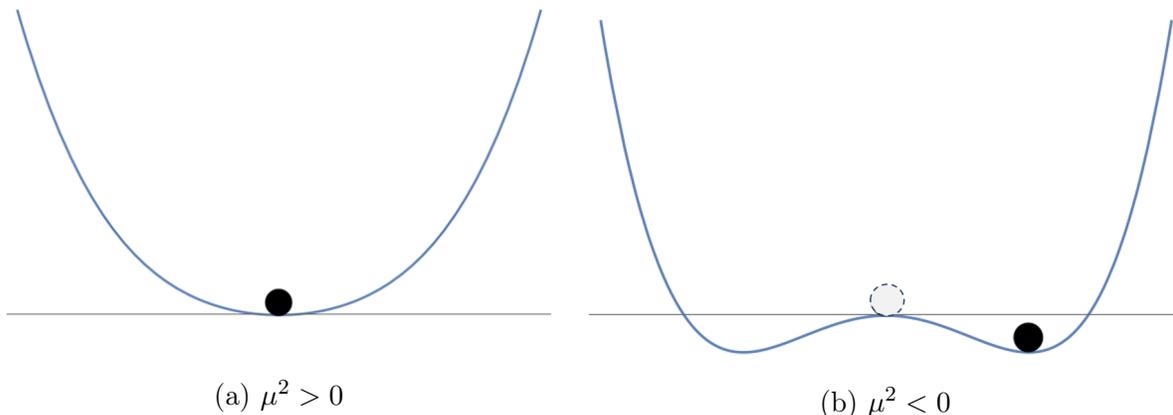


Figure 1.2.2: Scalar field  $\Phi$  with two different vacuum configurations. For  $\mu^2 < 0$  the vacuum state is no longer symmetric and the field acquires a non-zero expectation value.

grangian for a simple theory with a fermion and a spin-1 boson

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \quad (1.2.6)$$

with the covariant derivative  $D_\mu = \partial_\mu + igA_\mu$ . This Lagrangian is invariant under the gauge transformation given by:

$$\begin{aligned} \psi &\rightarrow e^{i\alpha(x)}\psi, \\ A_\mu &\rightarrow A_\mu - \frac{1}{g}\partial_\mu\alpha(x). \end{aligned} \quad (1.2.7)$$

It is easy to see, however, that adding a mass term  $\frac{1}{2}m^2A^2$  will break the gauge symmetry. To ameliorate this, a complex scalar field with the Lagrangian

$$\mathcal{L}_\Phi = (D^\mu\Phi)^*(D_\mu\Phi) - V(\Phi) \quad (1.2.8)$$

can be added with the potential  $V(\Phi)$  given by

$$V(\Phi) = \mu^2\Phi^*\Phi + \lambda(\Phi^*\Phi)^2, \quad \lambda > 0, \quad (1.2.9)$$

with the covariant derivative  $D_\mu$  ensuring that the new scalar terms of the Lagrangian are gauge invariant with the field transformation  $\Phi \rightarrow e^{i\alpha(x)}\Phi$ . This potential presents two interesting possibilities: For the case of  $\mu^2 > 0$ , the minimum of the potential is still at zero. Expanding the field  $\Phi$  around the minimum point  $\Phi = 0$  produces essentially the same Lagrangian in Eq. 1.2.8 and the  $U(1)$  symmetry remains unbroken (Fig. 1.2.2a). For  $\mu^2 < 0$ , however, the minimum of the potential shifts away from zero (Fig. 1.2.2b) and the field acquires a non-zero vacuum expectation value given by

$$\langle \Phi \rangle_0 = \sqrt{-\frac{\mu^2}{2\lambda}} e^{i\alpha} \quad (1.2.10)$$

where  $\alpha$  is some arbitrary phase. We can define the absolute value of this minimum

$$\frac{v}{\sqrt{2}} = \sqrt{-\frac{\mu^2}{2\lambda}} \quad (1.2.11)$$

as the vacuum expectation value. Expanding the field around the new minimum

$$\Phi = \left( \frac{v + \phi}{\sqrt{2}} \right) e^{i\eta(x)}, \quad (1.2.12)$$

where  $\phi$  corresponds to the physical mode and  $\eta$  corresponds to the Goldstone mode, the scalar part of the Lagrangian becomes

$$\begin{aligned} \mathcal{L}_\Phi &= \frac{1}{2} (D^\mu(v + \phi)e^{i\eta(x)})^* (D_\mu(v + \phi)e^{i\eta(x)}) - V\left(\frac{v + \phi}{\sqrt{2}}e^{i\eta(x)}\right) \\ &= \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\phi^2(\partial_\mu\eta)^2 + g\phi^2 A^\mu\partial_\mu\eta + gv^2 A^\mu\partial_\mu\eta + 2gv\phi A^\mu\partial_\mu\eta + g^2v\phi A^2 + \frac{1}{2}g^2\phi^2 A^2 \\ &\quad + \frac{1}{2}g^2v^2 A^2 + \mu^2\phi^2 - \frac{\mu^2}{v}\phi^3 + \frac{\mu^2}{4v^2}\phi^4 + v\phi(\partial_\mu\eta)^2 + \frac{1}{2}v^2(\partial_\mu\eta)^2 + (\text{constant terms}). \end{aligned} \quad (1.2.13)$$

This Lagrangian can be simplified greatly using the transformation

$$A_\mu \rightarrow A_\mu - \frac{1}{g} \partial_\mu \eta(x). \quad (1.2.14)$$

This amounts to fixing the gauge for the vector boson so that  $\alpha(x)$  is fixed to  $\alpha(x) = \eta(x)$ .

With this choice of gauge, the Lagrangian becomes

$$\begin{aligned} \mathcal{L}_\Phi = & \frac{1}{2}(\partial_\mu \phi)^2 - \underbrace{\frac{1}{2}(-2\mu^2)\phi^2}_{\text{Scalar mass term}} + \underbrace{\frac{-\mu^2}{v}\phi^3 + \frac{\mu^2}{4v^2}\phi^4}_{\text{Scalar self-interaction terms}} \\ & + \underbrace{\frac{1}{2}g^2v^2A^2}_{\text{Gauge boson mass term}} + \underbrace{g^2v\phi A^2 + \frac{1}{2}g^2\phi^2 A^2}_{\text{Gauge-Scalar interaction terms}} + (\text{constant terms}). \end{aligned} \quad (1.2.15)$$

Note that the field  $\eta(x)$  completely disappears. The remaining scalar field is actually physical with mass  $m_\phi = \sqrt{-2\mu^2}$ . The gauge boson also acquires a mass  $m_A = gv$ . This is colloquially referred to as the gauge boson "absorbing" the Goldstone mode; the additional degree of freedom required for the longitudinal mode of the now massive gauge boson appears through the Goldstone boson. This is referred to as the "Unitary gauge"; only the physical fields propagate in this gauge.

The Higgs mechanism, described above to generate masses for bosons in a gauge theory, was proposed in three seminal papers in 1964 by Brout and Englert [19], Higgs [17, 18, 34], and Guralnik, Hagen, and Kibble [20]. In the Abelian example above, only one Goldstone boson was generated. For non-Abelian gauge theories the number of Goldstone bosons is equal to the number of broken generators; for a spontaneously broken  $SU(N)$ , for example, there will be  $N$  Goldstone modes. The spontaneously broken  $SU(2)_L \times U(1)_Y$  in the Standard Model generates 3 Goldstone bosons that give mass to the three vector bosons  $W^\pm$  and  $Z$ . The Higgs mechanism can also explain the origin of fermion masses [35]; this can be achieved

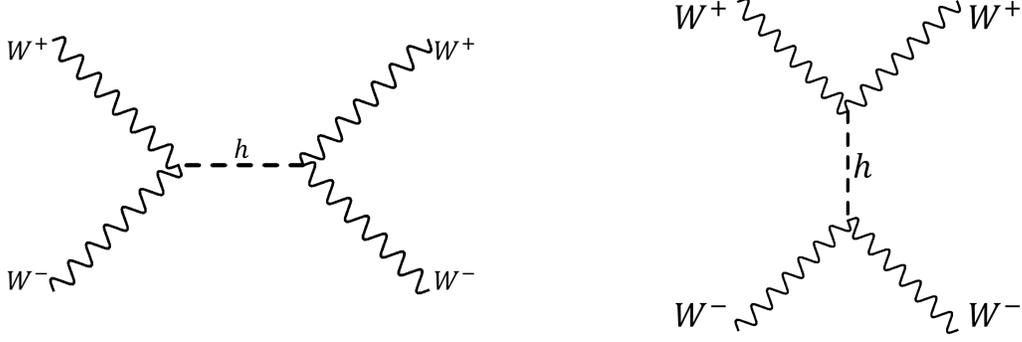


Figure 1.2.3:  $WW$  scattering including the interactions with the Higgs boson.

by adding a Yukawa-type interaction to the Lagrangian in Eq. 1.2.6

$$\mathcal{L} = i\bar{\psi}\not{D}\psi - y_\psi\Phi\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \quad (1.2.16)$$

Expanding  $\Phi$  around the minimum after symmetry breaking, the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= i\bar{\psi}\not{D}\psi - \frac{y_\psi}{\sqrt{2}}(v + \phi)\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \bar{\psi}\left(i\not{D} - \frac{y_\psi v}{\sqrt{2}}\right)\psi - \frac{y_\psi}{\sqrt{2}}\phi\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \end{aligned} \quad (1.2.17)$$

Identifying  $y_\psi v/\sqrt{2} = m_\psi$  as the fermion mass,

$$\mathcal{L} = \bar{\psi}(i\not{D} - m_\psi)\psi - \frac{m_\psi}{v}\phi\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \quad (1.2.18)$$

It's evident from the above equation that the interaction strength between the fermion and the scalar is proportional to the fermion mass.

As was seen in Eq. 1.2.5, the amplitude for  $W_L^+W_L^-$  scattering increases indefinitely with energy. This non-unitary behaviour is fixed after including the Higgs boson interactions

$$\mathcal{M}_h = \frac{-g^2(1 - \cos\theta)}{8} \left(\frac{s}{m_W^2}\right) + O\left(\frac{s}{m_W^2}\right)^0 \quad (1.2.19)$$

which when added to the pure gauge part of the amplitude in Eq. 1.2.5 gives

$$\begin{aligned}\mathcal{M}(W_L^+ W_L^- \rightarrow W_L^+ W_L^-) &= \mathcal{M}_t + \mathcal{M}_{4pt} + \mathcal{M}_s + \mathcal{M}_h \\ &= O\left(\frac{s}{m_W^2}\right)^0\end{aligned}\tag{1.2.20}$$

implying that the Higgs boson unitarises gauge boson scattering.

It was mentioned previously that the longitudinal modes of the gauge bosons are essentially the Goldstone bosons appearing as a consequence of the spontaneously broken symmetry when considering the unitary gauge. This implies that any amplitude involving the longitudinal modes of the gauge bosons can be calculated by replacing the gauge boson with the corresponding Goldstone boson, known as the Goldstone boson equivalence theorem [36, 37].

## 1.2.2 Standard model Higgs

In the Standard Model, the Higgs boson appears as a part of the Higgs doublet which transforms under the  $SU(2)_L \times U(1)_Y$  symmetry of the electroweak sector of the Standard model, details of which are provided in Sec. 1.5. Some of the prominent decay modes for the Standard Model Higgs are shown in Fig. 1.2.4. It is clear that for the measured mass value  $m_h = 125.25$  GeV [1],  $h \rightarrow b\bar{b}$  is the dominant decay mode with a branching ratio of  $\sim 57\%$  (Tab. 1.2.1). However,  $b\bar{b}$  channel has a huge QCD background from production of two jets and identifying those jets as originating from  $b$ -quarks ( $b$ -tagging) is rather challenging; as such this wasn't the ideal discovery channel. Indeed the discovery of Higgs was reported in the subdominant  $h \rightarrow ZZ^* \rightarrow 4l$  and  $h \rightarrow \gamma\gamma$  channels [8, 9] with the resonances appearing in the invariant mass spectra (Fig. 1.2.5 and 1.2.6). The diphoton channel,  $h \rightarrow \gamma\gamma$ , and the  $4l$  channel,  $h \rightarrow ZZ^* \rightarrow 4l$ , are particularly suited for discovery owing to their cleaner signature and more manageable backgrounds. Decay in the  $b\bar{b}$  channel was discovered only a few years back [38, 39]. More recently, the rare decay  $h \rightarrow \mu^+\mu^-$  has been observed [40, 41].

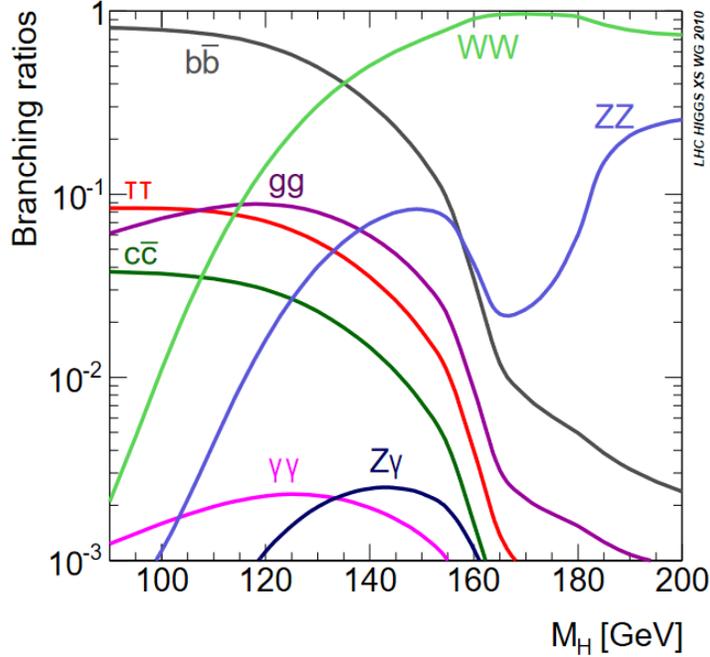


Figure 1.2.4: Plot of the Standard Model Higgs branching ratios against the Higgs mass. Measured value of Higgs mass is  $m_h = 125.25 \pm 0.17$  GeV. Reproduced from [7].

The primary production mode for Higgs boson at the LHC is through gluon fusion via a top quark loop (ggF) which accounts for  $\sim 88\%$  of the total Higgs cross-section at the LHC with  $\sigma_{ggF} = 48.6^{+4.6\%}_{-6.7\%}$  pb [7, 42, 2, 43]. This is largely due to high gluon luminosity at the LHC for low invariant mass (i.e. gluons carry most of the energy of a proton at lower energies probed by the LHC) as well as the large Yukawa coupling of the top quark with the Higgs. Other important production modes are Vector-boson-fusion (VBF) with cross-section  $\sigma_{VBF} = 3.78^{+2.2\%}_{-2.2\%}$ , "Higgstrahlung" where the Higgs is produced in association with a vector boson with  $\sigma_{VH} = 2.25^{+4.8\%}_{-4.4\%}$ , and production in association with a top-quark pair ( $t\bar{t}h$ ) with  $\sigma_{t\bar{t}h} = 0.50^{+6.8\%}_{-9.9\%}$ .

So far the measured properties of the observed particle are consistent with those for the Standard Model Higgs Boson. It is an electrically neutral, spin-0 particle which transforms as a scalar under parity conjugation. Its mass has been measured to be  $m_h = 125.25 \pm 0.17$  [1] GeV. The coupling of the Standard Model Higgs to other particles is proportional to the

Decay Channel	BR	Uncertainty (%)	
$h \rightarrow bb$	0.574	+3.2	-3.3
$h \rightarrow WW$	0.218	+4.2	-4.2
$h \rightarrow gg$	0.0855	+10.2	-10.0
$h \rightarrow \tau\tau$	0.0629	+5.7	-5.6
$h \rightarrow cc$	0.0289	+12.2	-12.2
$h \rightarrow ZZ$	0.0269	+4.2	-4.2
$h \rightarrow \gamma\gamma$	0.00228	+4.9	-4.9
$h \rightarrow Z\gamma$	0.00156	+9.0	-8.8

Table 1.2.1: Branching ratios for the Standard Model Higgs boson with  $m_h = 125.25$  GeV [1]. Numbers reproduced from [2].

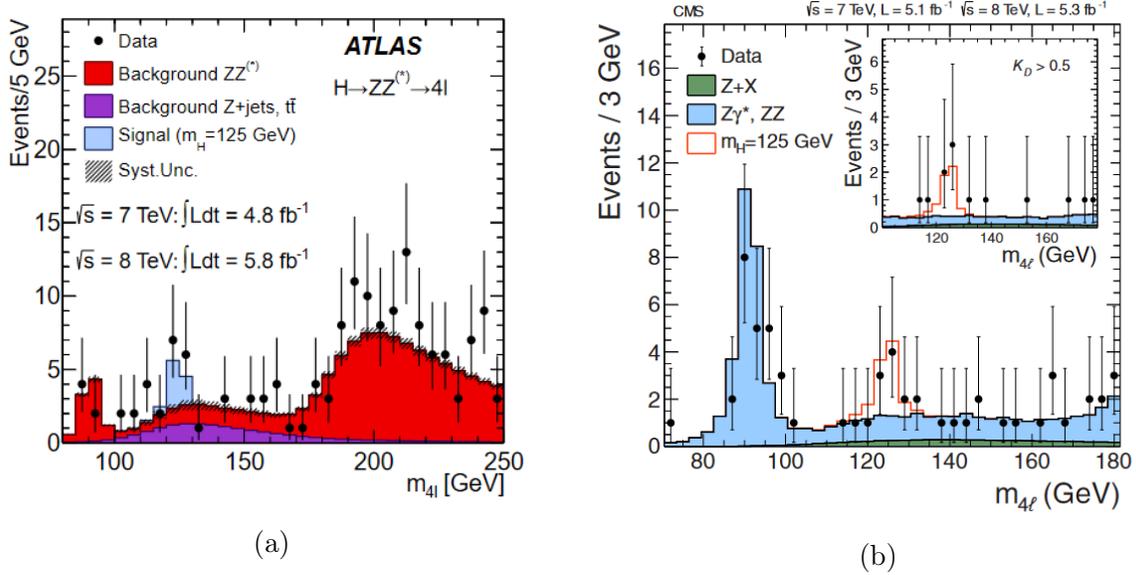


Figure 1.2.5: Distribution of four lepton invariant mass for the  $h \rightarrow ZZ^* \rightarrow l^+l^-l^+l^-$  decay channel for (a) ATLAS (Reproduced from [8]) and (b) CMS (Reproduced from [9]) experiments.

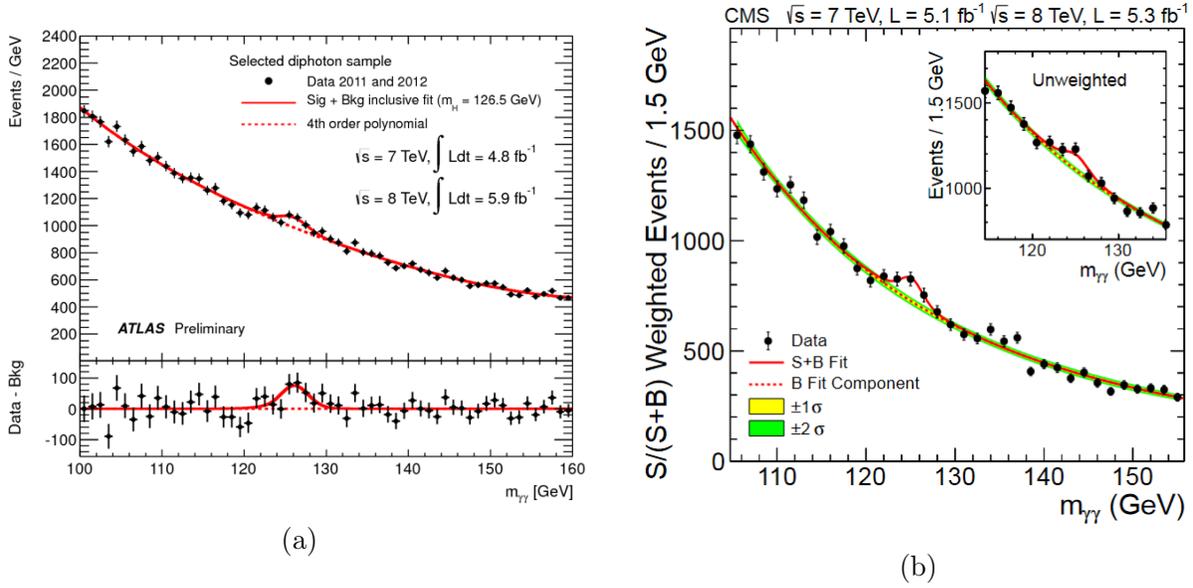


Figure 1.2.6: Distribution of the invariant mass of the photon pair for the  $h \rightarrow \gamma\gamma$  decay channel for (a) ATLAS (Reproduced from [8]) and (b) CMS (Reproduced from [9]) experiments.

particle mass. This is indeed what has been observed at the LHC as seen in Fig. 1.2.7; the straight line through the experimentally measured values indicates that the couplings are consistent with the SM prediction. However, to ascertain the true nature of the Higgs boson, two key properties need to be determined precisely: Higgs potential and decay width.

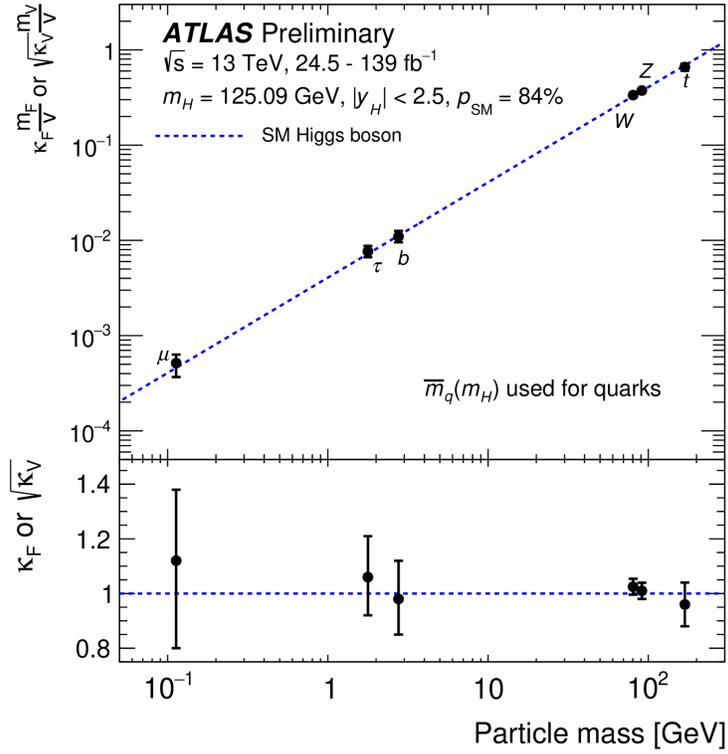


Figure 1.2.7: Plot of the measured Higgs boson couplings to other Standard Model particles against particle mass. Reproduced from [10].

## 1.3 Higgs potential and width

### 1.3.1 Higgs potential

Higgs potential in the gauge-invariant form can be written as

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2, \quad \lambda > 0, \quad (1.3.1)$$

where  $\Phi$  is a complex  $SU(2)$  doublet with the general form

$$\Phi = \begin{pmatrix} \phi_2 + i\phi_3 \\ \phi_0 + i\phi_1 \end{pmatrix}. \quad (1.3.2)$$

Expanding the Higgs field around the vacuum (and ignoring the phase for simplicity)

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix}. \quad (1.3.3)$$

the potential becomes

$$V(h) = v(\lambda v^2 + \mu^2)h + \frac{1}{2}(3v^2\lambda + \mu^2)h^2 + v\lambda h^3 + \frac{\lambda}{4}h^4. \quad (1.3.4)$$

Using Eq. 1.2.11 to substitute  $\mu$  for  $v$ , the potential becomes

$$V(h) = \frac{1}{2}(2\lambda v^2)h^2 + v\lambda h^3 + \frac{\lambda}{4}h^4. \quad (1.3.5)$$

where  $v$  is the vacuum expectation value (vev). From the above equation, the mass of the Higgs is  $m_h^2 = 2\lambda v^2$  which is measured at the LHC. Finally, the potential can be written in terms of Higgs mass and the vacuum expectation value as

$$V(h) = \frac{1}{2}m_h^2 h^2 + \frac{m_h^2}{2v}h^3 + \frac{m_h^2}{8v^2}h^4. \quad (1.3.6)$$

which means that the Higgs trilinear and quartic couplings are

$$\begin{aligned} \lambda_{hhh} &= \frac{3m_h^2}{v} \\ \lambda_{hhhh} &= \frac{3m_h^2}{v^2}. \end{aligned} \quad (1.3.7)$$

The vacuum expectation value appears also in the electroweak sector and can be measured

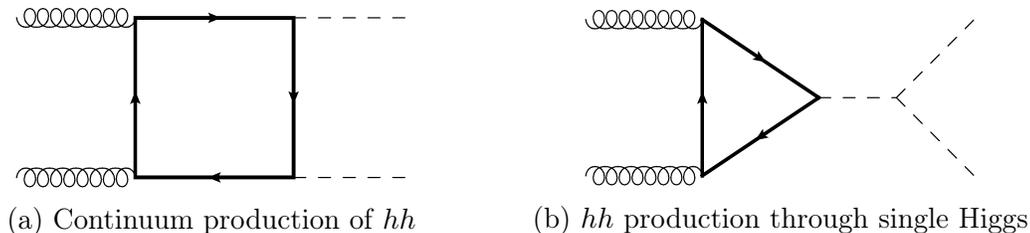


Figure 1.3.1: Higgs pair production through gluon fusion at LO.

precisely through e.g. muon lifetime measurements. With both the mass and the vacuum expectation value precisely measured,  $\mu$  and  $\lambda$  and consequently the Higgs potential are completely fixed. This is not the end of the story however since new physics can potentially alter each of these values as well as introduce higher dimensions terms in the Lagrangian; as such it is of extreme importance to measure the Higgs trilinear and quartic couplings and compare them against their theoretically predicted values.

Measuring the Higgs trilinear and quartic couplings is challenging, though, at the LHC. The trilinear coupling can be measured by studying double Higgs production. There are two modes of production for a pair of Higgs bosons through gluon fusion shown in Fig. 1.3.1; these two modes interfere destructively, with the magnitude of cancellation depending on  $\lambda_{hhh}$ . However, the cross-section at the LHC for  $gg \rightarrow hh$  in SM is only 31.18 fb [44] (NNLL + NNLO in large top mass approximation). The trilinear coupling can be constrained to  $0.5 < \lambda_{hhh} < 4.5$  [45] at the High Luminosity LHC while the quartic coupling  $\lambda_{hhhh}$  is essentially out of reach at the LHC. The prospects look a lot better at the proposed 100 TeV Future Circular Collider and the lepton colliders: ILC, CLIC,  $\mu^+\mu^-$  collider. Best constraint for the trilinear coupling comes from FCC [46] with an estimated precision of 3.4 – 7.8% at 68% CL. For the quartic coupling, achievable precision is 50% at a 14 TeV muon collider [47].

### 1.3.2 Measuring Higgs width

Another crucial property to be measured to ascertain the true nature of the observed Higgs boson is its decay width. This is predicted in SM to be  $\Gamma_h = 4.07$  MeV [1] and any deviation

from this value would imply presence of new physics. However, being such a small value, it cannot be measured directly at the LHC since the detector resolution is too low to measure it precisely. The best constraint obtained from a direct measurement ( $\Gamma_h < 1$  GeV [48, 49, 50]) is orders of magnitude larger than the value predicted in Standard Model of  $\Gamma_h = 4.07$  MeV. Instead, an alternate method was proposed in [51, 52, 53] to use the offshell Higgs signal to constrain the width. The basic idea is as follows: the production cross-section for  $gg \rightarrow h \rightarrow ZZ^* \rightarrow 4l$  can be written as [52]

$$\frac{d\sigma}{dM_{4l}^2} \sim \frac{g_{hgg}^2 g_{hZZ}^2}{(M_{4l}^2 - m_h^2)^2 + m_h^2 \Gamma_h^2}. \quad (1.3.8)$$

where  $g_{hgg}$  and  $g_{hZZ}$  are effective couplings of the Higgs to gluons and Z-bosons. In the above equation, all the quantities independent of  $M_{4l}$  and  $\Gamma_h$  have been suppressed for simplicity. Integrating the above near the resonance peak gives

$$\sigma_{\text{on-shell}} \sim \frac{g_{hgg}^2 g_{hZZ}^2}{\Gamma_h}. \quad (1.3.9)$$

Assuming now that the couplings  $g_{hgg}$  and  $g_{hZZ}$  are scaled with a factor  $\xi^{\frac{1}{4}}$  and the width is scaled with  $\xi$ , the cross-section in the above equation remains unchanged. Away from the resonance peak, however, the denominator in Eq. 1.3.8 is dominated by  $M_{4l}$  so the cross-section scales linearly with  $\xi$  under the above scaling

$$\sigma_{\text{off-shell}} \sim \xi g_{hgg}^2 g_{hZZ}^2. \quad (1.3.10)$$

Comparing the number of events, then, in the on-shell and off-shell production regions can be used to calculate an upper bound on  $\Gamma_h$ .

### 1.3.3 Experimental constraints

As mentioned previously,  $\Gamma_h$  cannot be measured directly at the resonance peak at the LHC

since it is limited by detector resolution. The off-shell method, however, can provide stringent constraint on the width. An upper bound of  $\gamma_h < 88$  MeV was calculated in [52] with the possibility of the constraint being improved to  $\Gamma_h < 10 \Gamma_h^{SM}$  with more data. In fact, recent analysis [54] with Run II data has constrained the width much further to  $3.2_{-2.2}^{+2.8}$  MeV with the 95% CL constraint  $0.08 < \Gamma_h < 9.16$  MeV both of which are consistent with the SM value.

## 1.4 Quantum Chromodynamics

### 1.4.1 History of QCD

The discovery of the atomic nucleus, and consequently the proton, in the early 20th century led to an obvious problem: the nucleus was too heavy to be accounted for just by the protons. Among the prevalent suggestions was the electrically neutral bound states of electrons and protons, referred to as neutrons, residing inside the nucleus. It was shown, however, that an electron cannot be confined inside the nucleus. This issue was resolved with the discovery of the neutron in 1932. The neutron was found to have certain remarkable properties. Other than the electric charge, it was very similar to the proton; its mass was extremely close to that of the proton and it had the same spin of  $1/2$ . Furthermore, it was found that the strong nuclear force had the same interaction strength for protons and neutrons, i.e. the interaction strength between a pair of protons was the same as between a pair of neutrons or a proton-neutron pair. This led to the development of isospin symmetry, as an analogue of spin for electrons. Protons and neutrons are treated as two "isospin" states of the same particle, the "nucleon", with pions, carriers of the strong nuclear force, acting much like the

classic spin operators in a harmonic oscillator.

$$\begin{aligned}
 \pi^+|n\rangle &=|p\rangle \\
 \pi^-|p\rangle &=|n\rangle \\
 \pi^0|p, n\rangle &=|p, n\rangle.
 \end{aligned}
 \tag{1.4.1}$$

Here  $\pi^\pm$  act like the well-known raising and lowering operators  $J^\pm$  while  $\pi^0$  acts like the total spin operator  $J$ .

The "isospin" model was very successful in explaining the strong nuclear force in early days especially after the discovery of pions. As more new particles were discovered, they were formed into isospin multiplets e.g. nucleon doublet  $(p, n)$  with  $I = 1/2$ , Sigma triplet  $(\Sigma^+, \Sigma^0, \Sigma^-)$  with  $I = 1$ , etc. A major success of the isospin model was the prediction of rho mesons as the vector bosons which were eventually discovered experimentally. With the observation of the "strange" decays of Kaons, however, a new quantum number "Strangeness" was introduced for the particles [55, 56] and the "Eightfold way" was proposed by Gell-Mann [57] and Ne'eman [58] to classify the known particles. The isospin symmetry was extended to an  $SU(3)$  flavour symmetry with particles being grouped into octets and decuplets.

Eventually, the quark model was proposed by Gell-Mann [59] and Zweig [60] to explain the observed spectrum of particles with hadrons being composed of smaller particles called quarks. Three quarks were postulated to transform under the  $SU(3)_F$  flavour symmetry. However, while the quark model was successful in explaining the spectrum of hadrons, it gave rise to a new question. Specifically, some of the hadrons seemed to be violating Fermi-Dirac statistics e.g. the  $\Delta^{++}$  baryon was proposed to be made up of three up quarks,

$$|\Delta^{++}\rangle = |u^\uparrow u^\uparrow u^\uparrow\rangle,
 \tag{1.4.2}$$

with all three up quarks having  $J_z = +1/2$ . Since the three up quarks are identical fermions,

they cannot occupy the same quantum state [61, 62]. The solution to this problem was introduced as the "colour charge" [63] being an additional degree of freedom possessed by quarks. The  $\Delta^{++}$  baryon can then be expressed as an anti-symmetric combination of three up quarks with different "colours",

$$|\Delta^{++}\rangle = \frac{1}{\sqrt{6}} \left( |u_r^\uparrow u_g^\uparrow u_b^\uparrow\rangle - |u_r^\uparrow u_b^\uparrow u_g^\uparrow\rangle + |u_b^\uparrow u_r^\uparrow u_g^\uparrow\rangle - |u_b^\uparrow u_g^\uparrow u_r^\uparrow\rangle + |u_g^\uparrow u_b^\uparrow u_r^\uparrow\rangle - |u_g^\uparrow u_r^\uparrow u_b^\uparrow\rangle \right), \quad (1.4.3)$$

making the resulting combination "colourless". Analogous to quarks, anti-quarks possess the anti-colour charges  $\bar{r}, \bar{g}, \bar{b}$ . This results in mesons, which are combinations of quarks and anti-quarks, being colourless, e.g.

$$|\pi^+\rangle = \frac{1}{\sqrt{3}} (|u_r d_{\bar{r}}\rangle + |u_g d_{\bar{g}}\rangle + |u_b d_{\bar{b}}\rangle). \quad (1.4.4)$$

The quarks now form a colour triplet,

$$|u\rangle = \begin{pmatrix} |u_r\rangle \\ |u_g\rangle \\ |u_b\rangle \end{pmatrix}, \quad (1.4.5)$$

that transforms under the new  $SU(3)_C$  group. Unlike the  $SU(3)_F$  flavour symmetry, however,  $SU(3)_C$  colour is an exact symmetry.

In [63], the  $SU(3)_C$  colour symmetry of the quarks was first considered as a Yang-Mills gauge theory [31]. Gauge bosons for this theory were called "Gluons", acting as glue holding the quarks together in hadrons. An interesting feature of this theory was that gluons carried the colour charge as well, unlike photons in Quantum Electrodynamics. This leads to the phenomena of asymptotic freedom and confinement, discussed later in this section.

### 1.4.2 $SU(N)$ gauge theory

A general  $SU(N)$  group has  $N^2 - 1$  generators; for  $SU(3)$  the number of generators is 8 corresponding to the 8 different gluons. The generators of a general  $SU(N)$  group can be represented using  $T^a$  where the index  $a \in \{1, \dots, N^2 - 1\}$  refers to the  $SU(N)$  charge in the adjoint representation. These  $T^a$  are traceless and hermitian.

$$\text{Tr}(T^a) = 0, \quad (1.4.6)$$

$$(T^a)^\dagger = T^a, \quad (1.4.7)$$

and satisfy the following commutation relation:

$$[T^a, T^b] = i f^{abc} T^c. \quad (1.4.8)$$

$f^{abc}$  are the totally anti-symmetric structure constants

$$f^{abc} = -2i \text{Tr}([T^a, T^b] T^c). \quad (1.4.9)$$

Further details of the  $SU(N)$  algebra are provided in Appendix A.3.

The Lagrangian for the gluon field can be written as

$$\mathcal{L}_g = -\frac{1}{4} G^{a,\mu\nu} G_{\mu\nu}^a, \quad (1.4.10)$$

where the field strength tensor is

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c. \quad (1.4.11)$$

It is straightforward to see that the Lagrangian in Eq. 1.4.10 is invariant under the gauge

transformation

$$G_\mu^a \rightarrow G_\mu^a - \partial_\mu \theta^a(x) - g_s f^{abc} \theta^b(x) G_\mu^c. \quad (1.4.12)$$

Expanding the Lagrangian in Eq. 1.4.10, we get the kinetic term

$$-\frac{1}{4}(\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu})(\partial_\mu G_\nu^a - \partial_\nu G_\mu^a), \quad (1.4.13)$$

the 3-point self-interaction term

$$-\frac{g_s}{2} f^{abc}(\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) G_\mu^b G_\nu^c, \quad (1.4.14)$$

and the 4-point self-interaction term

$$+\frac{g_s^2}{4} f^{abe} f^{cde} G^{a,\mu} G^{b,\nu} G_\mu^c G_\nu^d. \quad (1.4.15)$$

Unlike with photons in Quantum Electrodynamics, gluons interact with themselves which leads to the phenomena of asymptotic freedom and confinement, discussed later in Sec. 1.4.6.

Details on the QCD Lagrangian and the Feynman rules are given in Appendix A.

The above Lagrangian for the gluon field poses a problem though. Specifically, the propagator of the gluon cannot be derived from this Lagrangian using a naive approach.

This is evident from considering the kinetic term in the Lagrangian

$$\begin{aligned} \mathcal{L}_{kin} &= -\frac{1}{4}(\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu})(\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) \\ &= \frac{1}{2} G^{a,\nu} (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) \delta^{ab} G^{b,\mu} + \dots \end{aligned} \quad (1.4.16)$$

The propagator  $D^{ab,\mu\nu}(x-y)$  is defined using

$$\delta^{ac}(g_{\mu\rho}\partial^2 - \partial_\mu\partial_\rho)D^{bc,\nu\rho}(x-y) = i\delta^{ab}\delta_\mu^\nu\delta^{(4)}(x-y), \quad (1.4.17)$$

which in momentum space becomes

$$\delta^{ac}(-g_{\mu\rho}p^2 + p_\mu p_\rho)\tilde{D}^{bc,\nu\rho}(p) = i\delta^{ab}\delta_\mu^\nu. \quad (1.4.18)$$

Here,  $(-g_{\mu\rho}p^2 + p_\mu p_\rho)$  is singular due to gauge invariance and hence cannot be inverted to obtain the propagator. This can be resolved by adding a gauge fixing term to the Lagrangian

$$\mathcal{L}_{gauge-fixing} = -\frac{1}{2\xi}(\partial_\mu G^{a,\mu})^2. \quad (1.4.19)$$

which, of course, renders the Lagrangian in Eq. 1.4.10 no longer gauge invariant. Eq. 1.4.18 now becomes

$$\delta^{ac}\left(-g_{\mu\rho}p^2 + \left(1 - \frac{1}{\xi}\right)p_\mu p_\rho\right)\tilde{D}^{bc,\nu\rho}(p) = i\delta^{ab}\delta_\mu^\nu \quad (1.4.20)$$

which, when inverted, gives

$$\tilde{D}^{ab,\mu\nu}(p) = \frac{i\delta^{ab}}{p^2}\left(-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right). \quad (1.4.21)$$

In the above equation  $\xi$  is a free parameter used to fix the gauge. This family of gauges is known as the  $R_\xi$  gauges. Some common choices are "Feynman-'t Hooft gauge"  $\xi = 1$  which is the most popular choice in particular for higher-order corrections owing to the simplicity of the resulting expressions, and "Landau gauge" ( $\xi = 0$ ). The physical observables, however, are gauge-invariant and as such  $\xi$  dependence must drop out at the end.

This procedure of gauge-fixing is made more transparent using the Path-integral formalism. Note that gauge-fixing is a requirement for quantisation in this case. A consequence

of this specific gauge-fixing procedure is the appearance of Faddeev-Popov ghost fields [64] which are required for preserving manifest Lorentz invariance as well as unitarity. The Faddeev-Popov Lagrangian is given by

$$\mathcal{L}_{ghost} = (\partial^\mu c^{*a})(\partial_\mu c^a) - g_s f^{abc} (\partial^\mu c^{*a}) c^b G_\mu^c. \quad (1.4.22)$$

The ghosts live in the adjoint representation, like the gluons, and interact only with gluons through a 3-point vertex, appearing as internal lines in Feynman diagrams. Such diagrams are essential to obtain physical results e.g. the gluon self-energy correction at 1-loop (see Appendix D). Feynman rules are given in Appendix A.2.

The name "ghost" arises from the fact that these fields are not physical and are simply a tool to make sense of the path integral and preserve unitarity for the specific gauge choice; they act to cancel the unphysical degrees of freedom appearing from ambiguity in gauge choice. This is also evident from the fact that the ghosts violate the Spin-Statistics theorem; they appear as spin-0 particles (scalar), however they are anti-commuting (fermions). Indeed other choices such as the Axial gauge [65] do not require the presence of such fields. It must be pointed out that the gauge-fixing procedure above, and consequently the appearance of Faddeev-Popov ghosts, is not simply an artefact of a *non-abelian* gauge theory; while the ghosts do appear for Quantum Electrodynamics as well, they do not couple to the photon due to the abelian nature of the theory and hence are irrelevant for any meaningful calculations.

### 1.4.3 Perturbative QCD

Calculating scattering amplitudes and cross-sections in quantum field theory can be very challenging. Very few theories are exactly solvable; for almost all phenomenological purposes, perturbation theory is used. The observable or quantity of interest is expanded in a small parameter, usually the coupling strength of the interaction. This is justified in most cases, e.g. the coupling constant of QED, the fine structure constant,  $\alpha \simeq 1/137$  and the coupling

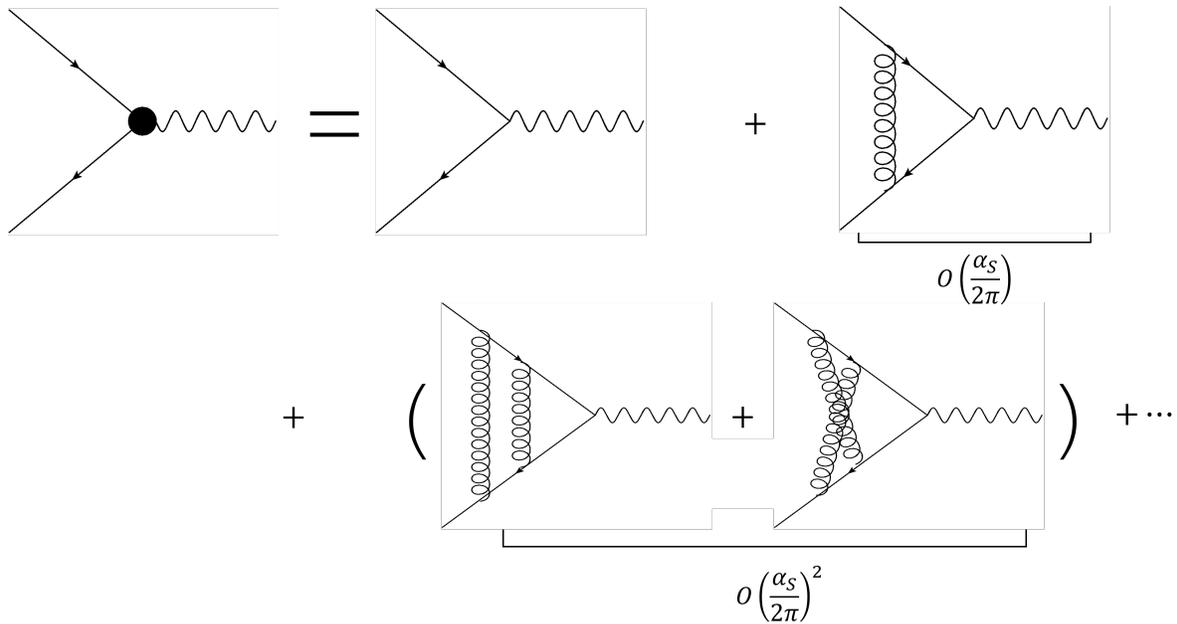


Figure 1.4.1: Perturbative corrections to  $q\bar{q} \rightarrow \gamma$  in QCD represented using Feynman diagrams.

constant of QCD  $\alpha_S \simeq 0.118$ . For QCD, the cross-section can be expanded in the strong coupling constant  $\alpha_S$  as:

$$\sigma = \sigma_0 + \left(\frac{\alpha_S}{2\pi}\right) \sigma_1 + \left(\frac{\alpha_S}{2\pi}\right)^2 \sigma_2 + \left(\frac{\alpha_S}{2\pi}\right)^3 \sigma_3 + \dots \quad (1.4.23)$$

where  $\sigma_0, \sigma_1, \sigma_2, \dots$  are the contributions from each order in perturbative expansion (e.g. Fig. 1.4.1). Standard technique to calculate these is by the use of Feynman diagrams which provide a highly intuitive pictorial representation of the scattering process in consideration in addition to providing a convenient tool to calculate scattering amplitudes.

While at first glance it might seem that the higher-order corrections are increasingly less important given each successive term is multiplied by higher powers of the coupling constant, they can be large in many cases, perhaps best demonstrated via Higgs boson production through gluon fusion. Fig. 1.4.2 shows the QCD corrections to this process. It is clear that the perturbative corrections here are extremely important; the Next-to-Leading Order (NLO) correction is about as large as the Leading Order (LO) term and even the Next-to-Next-to-

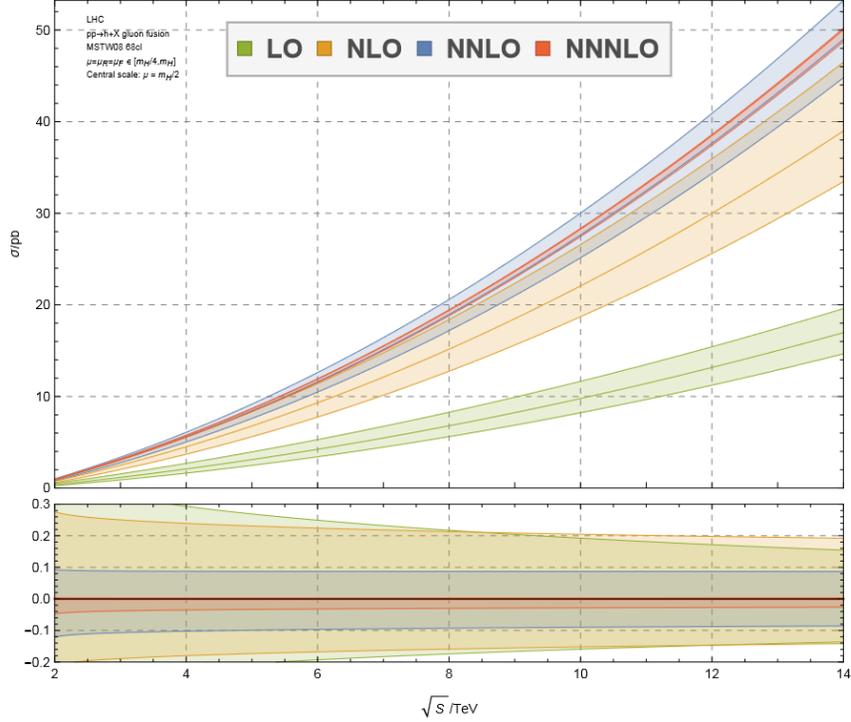


Figure 1.4.2: QCD corrections to  $gg \rightarrow h$ . The uncertainty bands represent scale uncertainty for the range  $\mu \in \{\frac{m_h}{4}, m_h\}$  with the central value at  $\mu = m_h/2$ . Reproduced from [11].

Leading Order term is significant, only at N3LO does the calculation become stable. The uncertainty bands in the figure are estimated by varying the scale  $\mu$ ; this dependence on scale enters through renormalisation 1.4.5 and factorisation 1.4.7, and physical observables should not depend on it. In a sense, then, the dependence on the scale  $\mu$  encodes some information about the missing higher order terms and can be used to estimate the uncertainty. However, the uncertainty bands at LO utterly fail to capture the NLO correction; only at N3LO the uncertainty bands become small enough and it lies entirely within the NNLO correction which is suggestive of the series converging (note that the perturbative series is expected to be asymptotic and not convergent).

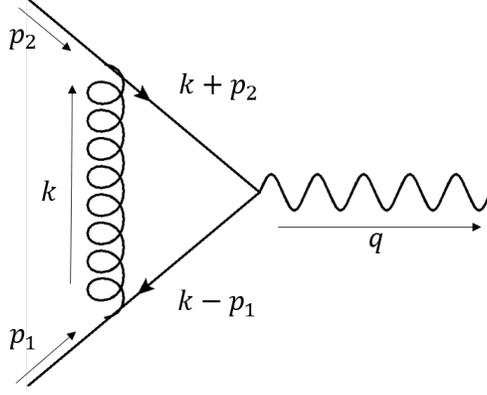


Figure 1.4.3: 1-loop correction to  $q\bar{q} \rightarrow \gamma$

### 1.4.4 Regularisation

The type of corrections shown in Fig. 1.4.1 are commonly referred to as the "virtual corrections" owing to the presence of additional virtual particles. Considering the diagram corresponding to the second term in the perturbative expansion in Fig. 1.4.1 and inserting Feynman rules, it is found that the external momenta are insufficient to constrain the momenta of the internal edges or the "loop". This loop momentum  $k$  in Fig. 1.4.3 is a free quantity that parameterises intermediate states. Quantum Mechanics dictates that all the intermediate states should be summed over; in this case the loop momentum  $k$  needs to be integrated over all possible values leading to the expression

$$\mathcal{M} \sim \int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} - \not{p}_1)\gamma^\mu(\not{k} + \not{p}_2)}{(k^2)(k - p_1)^2(k + p_2)^2}. \quad (1.4.24)$$

A closer look at the above integral shows that it diverges for high values of loop momenta:

$$\int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} - \not{p}_1)\gamma^\mu(\not{k} + \not{p}_2)}{(k^2)(k - p_1)^2(k + p_2)^2} \xrightarrow[k \rightarrow \infty]{\text{Wick rot.}} \int d^4k_E \frac{k_E^2}{k_E^6} \sim \int^\Lambda \frac{dk_E}{k_E}, \quad (1.4.25)$$

where Wick rotation (see Appendix B) has been performed to transform to Euclidean space and the integration domain has been cutoff with a parameter  $\Lambda$  to regulate the divergence. This kind of divergence that manifests for high values of loop momenta is commonly referred

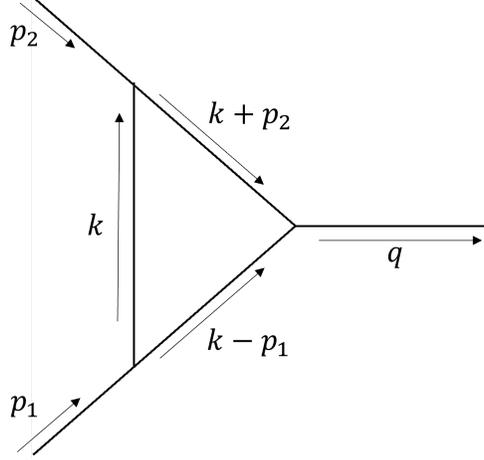


Figure 1.4.4: A 3-point massless integral. The incoming momenta  $p_1, p_2$  are massless.

to as Ultraviolet (UV) divergence. In this example a cutoff regulator was used; other choices exist e.g. Pauli-Villars regularisation.

In fact this diagram has two different kinds of divergences originating from different regions of loop momentum. The second kind of divergence becomes apparent in the region of small loop-momentum. Considering only the scalar integral from Fig. 1.4.4 with  $p_1^2 = p_2^2 = 0$ ,

$$I = \int \frac{d^4k}{i\pi^2} \frac{1}{(k^2)(k-p_1)^2(k+p_2)^2}. \quad (1.4.26)$$

For loop momentum in the soft limit  $k \rightarrow 0$ ,

$$I \sim \int \frac{d^4k}{i\pi^2} \frac{1}{(k^2)(2k \cdot p_1)(2k \cdot p_2)} \sim \int_{k \rightarrow 0} \frac{dk}{k}. \quad (1.4.27)$$

This integral also has a logarithmic divergence similar to Eq. 1.4.25, albeit of a very different nature. This kind of divergence for small loop momentum is called a "Soft divergence". There is another possibility for this integral when the loop momentum goes collinear to either  $p_1$  or  $p_2$ .

$$I \sim \int_{k \rightarrow p_1} d^4k \frac{1}{(k^2)(k-p_1)^2}. \quad (1.4.28)$$

These are called "Collinear divergences". The soft and collinear divergences are collectively referred to as Infrared divergences.

As shown in Eq. 1.4.25, one way to regulate the UV divergence is by cutting the integral off at some large scale  $\Lambda$ ; this is "Cutoff regularisation" scheme. The divergences then appear in the scale  $\Lambda$ . This scheme can also be used to regulate soft divergences by cutting off the loop momentum at some low energy scale. The most commonly used scheme, however, is dimensional regularisation [66, 67, 68, 69, 70]. In this scheme, ordinarily 4-dimensional objects are assumed to be  $d$ -dimensional where  $d = 4 - 2\epsilon$  is a complex valued quantity in general. At the end, the limit  $\epsilon \rightarrow 0$  is taken with poles appearing as  $1/\epsilon^n$  ( $n > 0$ ). Consider the UV divergent integral

$$I = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 - m^2)(k + p)^2} \sim \int_{k \rightarrow \infty} \frac{d^d k}{k^4}. \quad (1.4.29)$$

This integral is finite for  $d < 4$ . We can then analytically continue the integral to  $d = 4 - 2\epsilon$  with the divergences now appearing as poles in  $1/\epsilon$ . IR divergences can be similarly regulated:

$$I \sim \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)(2k \cdot p_1)(2k \cdot p_2)} \sim \int_{k \rightarrow 0} \frac{d^d k}{k^4}. \quad (1.4.30)$$

This integral converges for  $d > 4$ . Again, we can analytically continue the integral to  $d = 4 - 2\epsilon$ .

This provides a unified prescription to deal with both UV and IR divergences and has allowed for much of the progress in perturbative calculations. A huge advantage of using dimensional regularisation is that it preserves gauge symmetries unlike cutoff regularisation.

In Conventional Dimensional Regularisation, all the momenta are treated in  $d$ -dimensions as well as all the Dirac matrices etc. This has certain disadvantages e.g. it leads to spurious structures that vanish in  $d = 4$ ; this is discussed in more detail in sec. 2. An alternative approach is the 't Hooft-Veltmann scheme [68] where the loop momenta are treated in  $d$ -dimensions while the external momenta are treated in  $d = 4$  dimensions. This leads

to significant simplifications in many situations and prevents appearance of such spurious structures [71, 72].

Another consequence of dimensional regularisation is that the integral measure in a loop integral gains an additional mass dimension of  $(d - 4)L$  where  $L$  is the number of loops, along with the couplings gaining a mass dimension as well. It is conventional to redefine the coupling to render it dimensionless by introducing a dimensionful parameter  $\mu$  such that, for example,  $g'_s = g_s \mu^{\frac{4-d}{2}}$  with  $\mu$  known as the "t Hooft scale". This is done to ensure that all the terms in the Lagrangian have proper dimensions  $d$ .

### 1.4.5 UV renormalisation and IR subtraction

Since the physical observables are finite, these UV and IR divergences need to cancel in some way. The process of removing the UV singularities is called "Renormalisation". The Lagrangian is modified such that the "bare" parameters are replaced by "renormalised" parameters. E.g.

$$m_0 \bar{\psi}_0 \psi_0 \rightarrow (Z_m Z_\psi) m_R \bar{\psi}_R \psi_R \quad (1.4.31)$$

with the renormalisation constants  $Z_m$  and  $Z_\psi$  defined according to

$$\begin{aligned} m_R &= \frac{1}{Z_m} m_0 \\ \psi_R &= \frac{1}{\sqrt{Z_\psi}} \psi_0. \end{aligned} \quad (1.4.32)$$

The UV divergences are absorbed into the renormalisation constants during this redefinition. The renormalised parameters are finite implying that the bare parameters are also divergent. The renormalisation constants are written such that they have a trivial part and a counterterm e.g.  $Z_m = 1 + \delta_m$  where 1 refers to the tree level term and  $\delta_m$  is the counterterm and cancels against the poles order by order in perturbation theory. In principle there are an in-

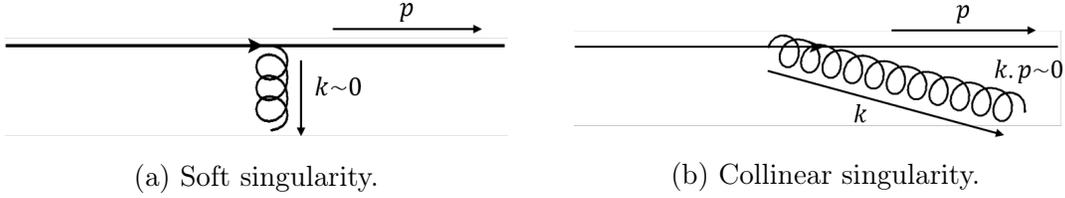


Figure 1.4.5: Infrared divergences corresponding to the emission of a real particle. Fig. 1.4.5a shows emission of a gluon with very small momentum i.e. a soft divergence while Fig. 1.4.5b shows the emission of a gluon collinear with the external particle.

finite number of renormalisation schemes; the divergences are fixed, however any finite term can be added to the renormalisation constant. Some of the commonly used schemes are the "On-shell" scheme where the renormalised parameters are chosen using on-shell properties of the particles and the modified "Minimal Subtraction" ( $\overline{MS}$ ) scheme [73, 74]; these are the two schemes used in this work. Details of the renormalisation procedure for this calculation are discussed in Sec. 5.2 and Appendix D.

In general, new divergences appear at every order in perturbation theory. These divergences must all be absorbed during the renormalisation procedure to render the result finite. For many theories new counterterms are required at every order implying that an infinite number is required to renormalise the theory at all orders, which results in a loss of predictive power. Such theories are called non-renormalisable. To show that a theory is renormalisable and requires only a finite number of counterterms is highly non-trivial; requirements for a renormalisable theory were given in [75, 76, 77] with the renormalisability of non-abelian gauge theories with spontaneous symmetry breaking (e.g. Standard Model) proven in [78].

The IR poles cancel against poles appearing in real emission diagrams, which have soft and collinear poles similar to the loop amplitudes through the emission of a soft and collinear particle respectively (see Fig. 1.4.5). For the case in Fig. 1.4.5, the propagator for the initial particle is

$$\frac{1}{(p+k)^2} = \frac{1}{2p \cdot k} \tag{1.4.33}$$

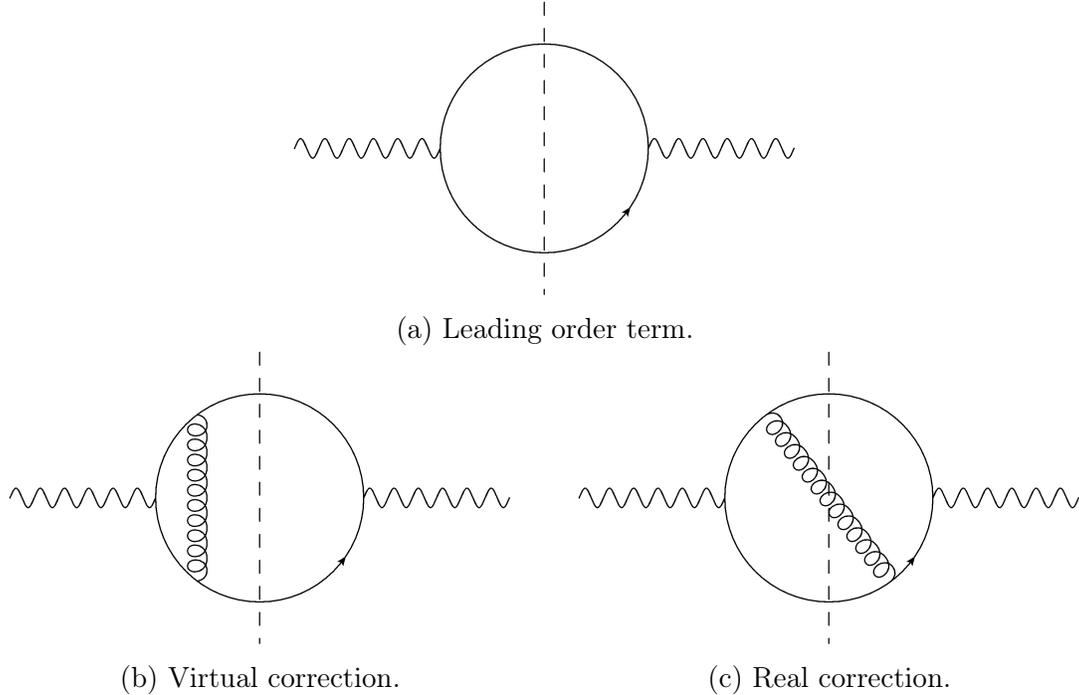


Figure 1.4.6: QCD corrections to  $q\bar{q} \rightarrow \gamma$ . Fig. 1.4.6b shows the  $\alpha_S$  correction through the exchange of a virtual gluon while Fig. 1.4.6c show the  $\alpha_S$  correction through the emission of a real gluon.

for a massless particle. For both  $k \rightarrow 0$  (soft) and  $k \rightarrow xp$  (collinear) the propagator has a singularity. This is a real physical particle, hence the name real corrections. These divergences appear during phase-space integration over the external momenta and can be regulated using dimensional regularisation just like UV poles. It must be noted that these divergences don't appear for massive particles. Considering the cut diagram in Fig. 1.4.6c, a gluon is emitted from a quark line. Since both virtual and real corrections occur at the order  $\alpha_S$ , it is necessary to add both to get the correct result. This is most readily seen through Fig. 1.4.6; the amplitude-squared can be thought of as  $\gamma \rightarrow \gamma$  with the real and virtual corrections simply being different cut configurations. Bloch-Nordsieck theorem [79, 80] showed that the IR divergences for QED must cancel once all configurations are included i.e. real correction diagrams in addition to the virtual corrections. This was extended to non-abelian gauge theories in [81, 82, 83]. Observables for which this is possible are called IR safe

observables. However, computing amplitude-squared as in Fig. 1.4.6 is highly impractical for all but the simplest processes. The standard technique is to subtract the poles from the virtual corrections and add them to the real corrections. Sum of both is finite and produces the full cross-section:

$$\begin{aligned}\sigma_{NLO} &= \sigma_{LO} + \int_N d\sigma_V + \int_{N+1} d\sigma_R \\ &= \sigma_{LO} + \int_N \left( d\sigma_V - \int_1 d\sigma_S \right) + \int_{N+1} (d\sigma_R + d\sigma_S) .\end{aligned}\tag{1.4.34}$$

Here  $d\sigma_S$  is the subtraction term; it must fulfill two conditions: it must have the same singular behaviour as  $d\sigma_R$  such that the sum  $d\sigma_R + d\sigma_S$  is finite for  $d \rightarrow 4$ , and it must be integrable in  $d$ -dimensions over the 1-particle phase-space and produce the infrared divergences to cancel against the virtual corrections. The pole structure for real and virtual corrections is fixed; there is freedom to choose the finite part of  $d\sigma_S$ , however. Many subtraction schemes exist with the most widely used ones at NLO being the Catani-Seymour dipole subtraction [13, 84], the Frixione-Kunzst-Signer (FKS) subtraction [85, 86], and the Nagy-Soper subtraction [87], and have been automated successfully [88, 89]. However, the situation at NNLO is far from settled; the reader is referred to [90] for a detailed review of the schemes utilised for NNLO calculations.

## 1.4.6 Asymptotic freedom

Unlike Quantum Electrodynamics, Quantum Chromodynamics is a non-abelian gauge theory i.e. the force carriers (gluons) also carry the charge and, as a consequence, can interact with each other. This leads to the phenomenon of "asymptotic freedom" [91, 92] at high energies i.e. the coupling strength of strong interaction becomes smaller with increasing energy. On the flip-side, the coupling increases with decreasing energy leading to the phenomenon of "confinement". This energy dependence of the renormalised coupling strength arises as a consequence of the regularisation and renormalisation procedure. However, the bare coupling

should remain constant irrespective of any renormalisation procedure which leads to the equation

$$\frac{d\alpha_{S,0}}{d\mu} = 0. \quad (1.4.35)$$

Writing the bare coupling  $\alpha_{S,0}$  in terms of the renormalised coupling  $\alpha_S$  results in

$$\frac{d\alpha_S(\mu)}{d\log(\mu^2)} = \beta(\alpha_S). \quad (1.4.36)$$

The Beta-function can be expanded order-by-order in  $\alpha_S$  as

$$\beta(\alpha_S) = -\alpha_S \left( \left( \frac{\alpha_S}{2\pi} \right) \beta_0 + \left( \frac{\alpha_S}{2\pi} \right)^2 \beta_1 + O(\alpha_S^3) \right). \quad (1.4.37)$$

The first few coefficients are given by

$$\begin{aligned} \beta_0 &= \frac{11C_A - 4T_F n_f}{6} \\ \beta_1 &= \frac{17C_A^2 - 10C_A T_F n_f - 6C_F T_F n_f}{6} \end{aligned} \quad (1.4.38)$$

where  $n_f$  is the number of active flavours. For QCD,  $C_A = 3$  and  $T_F = 1/2$  which implies  $\beta_0 > 0$  for  $n_f < 17$  i.e. the coupling decreases with energy for low number of active flavours; this is the origin of asymptotic freedom. This behaviour is experimentally confirmed as shown in Fig. 1.4.7.

For QED, on the other hand,  $\beta_0 = -2/3$  which means that the coupling grows with energy. Solving Eq. 1.4.36 to analytically solve for the running coupling,

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \frac{\alpha(\mu_0)}{2\pi} \beta_0 \log \frac{\mu^2}{\mu_0^2}} \quad (1.4.39)$$

It is clear that the above equation has a pole at very high energy, referred to as Landau

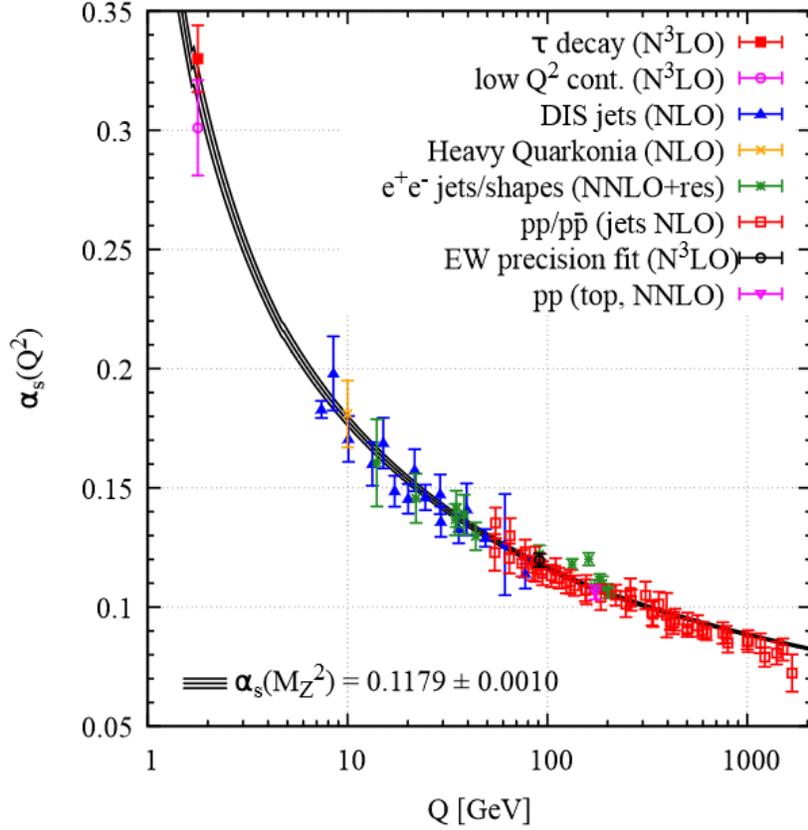


Figure 1.4.7: Plot showing theoretical prediction for the strong coupling  $\alpha_S$  against experimental measurements. Reproduced from [1].

pole [93], where the running coupling diverges. For QED, the coupling increases with energy and the divergence appears at a very high energy ( $\sim 10^{286}$  eV); for QCD this happens as the energy decreases. It makes sense, however, to interpret this as the transition from perturbative to non-perturbative regime when the coupling  $\alpha > 4\pi$ . Past this point, perturbation theory and, as a consequence, Eq.1.4.39 are no longer valid. The scale at which perturbation theory breaks down for QCD is  $\lambda_{QCD} \simeq 200$  MeV; for energies lower than this, non-perturbative techniques such as Chiral Perturbation Theory and Lattice Gauge Theory need to be used for calculations.

### 1.4.7 Factorisation

In collider experiments, often composite particles like protons are collided at high energies and the outgoing particles are studied to measure the observables e.g. proton-proton collisions at the LHC. At such high energies, however, the constituent partons [94] are the ones actually undergoing scattering governed by the short-distance physics which is modelled using perturbation theory. But the behaviour of these partons inside the proton, specifically their momentum distribution (Parton Distribution Functions), cannot be described using perturbative QCD. To perform a complete calculation of the cross-section, both short and long distance physics is required. That the full calculation can be factorised into the perturbative and non-perturbative parts which can then be evaluated independently is crucial to this. The scale separating the two is called the factorisation scale ( $\mu_F$ ). A scattering cross-section for the collision of two hadrons  $h_1 h_2 \rightarrow cd$  (e.g.  $pp$  at the LHC) can be written in a factorised form as

$$d\sigma^{h_1 h_2 \rightarrow cd} = \int_0^1 dx_1 dx_2 \sum_{a,b} f_{a/h_1}(x_1, \mu_F^2) f_{b/h_2}(x_2, \mu_F^2) d\hat{\sigma}^{ab \rightarrow cd}(Q^2, \mu_F^2) \quad (1.4.40)$$

where  $a, b$  are the colliding partons and  $Q$  is the scattering energy. The  $f_{a/h_1}, f_{b/h_2}$  in the above equations are the Parton Distribution Functions (PDFs) and specify the probability of finding the parton  $a, b$  in the hadron  $h_1, h_2$  with momentum fraction  $x_1, x_2$  respectively. All the unresolved particles that lead to IR poles below the factorisation scale  $\mu_F$  are absorbed in the PDF definitions. PDFs are process-independent but cannot be calculated using the regular perturbation theory methods. Traditionally they are determined using data [95, 96, 97] based on the DGLAP evolution equations [98, 99, 100]. There has been significant progress, however, in calculating PDFs through first principles using Lattice methods.

Factorisation has been proven for many processes in gauge theory [101, 102, 103, 104]. Like the renormalisation scale, the actual calculation must be independent of the factorisation scale. However, a scale dependence still remains owing to truncation of the series, and should

disappear if all the orders are calculated. This scale dependence manifests itself in the Parton Distribution Functions as well as the hard-scattering cross-section (Eq. 1.4.40).

The detectors have finite resolution which means that they cannot measure particles with arbitrarily low energies, neither can they distinguish between two particles extremely close together. Physical observables then need to be devised such that they respect such physical restrictions and are infrared safe i.e. additional soft or collinear radiations do not affect them. One such observable is a jet which is just a collection of final-state partons combined together according to a specific algorithm. They are designed to be simple to use in both experiments and theory as well as insensitive to non-perturbative effects. A way to imagine a jet is to consider a final state parton which radiates additional partons with radiation within a certain cone around the radiating particle being part of the jet. Two major kinds of algorithms are cone algorithms and sequential recombination algorithms. An in-depth discussion of jets is out of scope for this work and the reader is instead referred to [105, 106, 107] for details.

## 1.5 Electroweak sector

### 1.5.1 Electroweak symmetry breaking

The electroweak sector of the Standard Model describes the electromagnetic and weak interactions through the  $SU(2)_L \times U(1)_Y$  gauge symmetry [108, 109, 110, 111]. The L in  $SU(2)_L$  refers to "left", i.e. the gauge bosons associated with this symmetry interact only with left-handed particles and Y refers to "hypercharge", the additional quantum number possessed by SM fermions. Using the general  $SU(N)$  formula for the number of generators, there are 3  $SU(2)_L$  gauge bosons  $W_1, W_2, W_3$  along with  $B$  for  $U(1)_Y$ .

The kinetic term for the gauge bosons can be written as

$$\mathcal{L}_{kin} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{i,\mu\nu}W_{\mu\nu}^i \quad (1.5.1)$$

with the field strength tensors given by

$$\begin{aligned}
B^{\mu\nu} &= \partial^\mu B^\nu - \partial^\nu B^\mu \\
W_i^{\mu\nu} &= \partial^\mu W_i^\nu - \partial^\nu W_i^\mu + g_2 \epsilon_{ijk} W_j^\mu W_k^\nu
\end{aligned} \tag{1.5.2}$$

where  $g_2$  is the coupling strength of  $SU(2)_L$ . To generate masses for these bosons, the  $SU(2)_L \times U(1)_Y$  symmetry must be spontaneously broken through Higgs mechanism. The Higgs part of the Lagrangian can be written as

$$\mathcal{L}_{higgs} = (D^\mu \Phi)^\dagger (D_\mu \Phi) - \mu^2 (\Phi^\dagger \Phi) - \lambda (\Phi^\dagger \Phi)^2, \tag{1.5.3}$$

where the covariant derivative  $D_\mu$  is such that the above Lagrangian is invariant under the  $SU(2)_L \times U(1)_Y$  gauge symmetry

$$D_\mu = \partial_\mu - ig_1 Y B_\mu - i \frac{g_2}{2} \sigma_i W_{i,\mu}. \tag{1.5.4}$$

which can be written as

$$D_\mu = \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} - \frac{i}{2} \begin{pmatrix} g_1 B_\mu + g_2 W_{3,\mu} & g_2 (W_{1,\mu} - i W_{2,\mu}) \\ g_2 (W_{1,\mu} + i W_{2,\mu}) & g_1 B_\mu - g_2 W_{3,\mu} \end{pmatrix} \tag{1.5.5}$$

with the Hypercharge  $Y$  set to  $1/2$ . The gauge fields themselves transform under infinitesimal transformations as

$$\begin{aligned}
B_\mu &\rightarrow B_\mu - \partial_\mu \theta(x) \\
W_{i,\mu} &\rightarrow W_{i,\mu} - \partial_\mu \theta_i(x) - g_2 \epsilon^{ijk} \theta_j(x) W_{k,\mu},
\end{aligned} \tag{1.5.6}$$

similar to Eq. 1.4.12. Inserting Eq. 1.5.5 into Eq. 1.5.3 and expanding the Higgs field around

the vacuum (keeping only the physical Higgs field)  $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix}$ , the gauge bosons gain masses:

$$\mathcal{L}_{gauge-mass} = \frac{v^2}{8} (g_2^2(W_1^2 + W_2^2) + (g_1 B_\mu - g_2 W_{3,\mu})^2) \quad (1.5.7)$$

It is clear that two of the  $SU(2)_L$  bosons  $W_1, W_2$  acquire a mass. The third boson  $W_3$  does not acquire a mass itself, but rather the combination  $g_1 B_\mu - g_2 W_{3,\mu}$  does. At this point it makes sense to define certain quantities. The electric charge of a particle can be defined as

$$Q = Y + I_3, \quad (1.5.8)$$

and the electromagnetic coupling  $e$  can be defined as

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}. \quad (1.5.9)$$

Parameterising the couplings with an angle  $\theta_W$ ,

$$g_1 = g_2 \tan \theta_W, \quad (1.5.10)$$

the fields  $B_\mu$  and  $W_{3,\mu}$  can be rotated as

$$\begin{aligned} A_\mu &= \frac{g_2 B_\mu + g_1 W_{3,\mu}}{\sqrt{g_1^2 + g_2^2}} = \cos \theta_W B_\mu + \sin \theta_W W_{3,\mu} \\ Z_\mu &= \frac{-g_1 B_\mu + g_2 W_{3,\mu}}{\sqrt{g_1^2 + g_2^2}} = -\sin \theta_W B_\mu + \cos \theta_W W_{3,\mu}. \end{aligned} \quad (1.5.11)$$

It is also useful to define the linear combinations  $W_\pm = W_1 \pm iW_2$ . The mass terms then

become

$$\mathcal{L}_{gauge-mass} = \frac{g_2^2 v^2}{8} \left( W_+ W_- + \frac{1}{\cos^2 \theta_W} Z^2 \right) \quad (1.5.12)$$

A couple things are clear from the above equation.  $Z$  boson is predicted to be more massive than the  $W^\pm$  bosons since  $\cos^2 \theta_W < 1$ . Also, only 3 bosons gain mass with  $A_\mu$  remaining massless. This can be seen in another way; the  $Q$  operator acting on the vacuum state gives

$$\begin{aligned} Q|0\rangle &= (Y + I_3)|0\rangle \\ &= \begin{pmatrix} Y + \frac{1}{2} & 0 \\ 0 & Y - \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \quad (1.5.13)$$

since  $Y = 1/2$  for the Higgs field, meaning that  $Q$  is conserved compared to, for instance,  $Y$ :

$$Y|0\rangle = \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}. \quad (1.5.14)$$

This implies that the  $SU(2)_L \times U(1)_Y$  gauge symmetry spontaneously breaks down to  $U(1)_{EM}$  with the gauge boson of the  $U(1)_{EM}$  symmetry, photon, remaining massless. Using Eqs. 1.5.9 and 1.5.10, the masses of the gauge bosons can be written in terms of electric charge  $e$ :

$$\mathcal{L}_{gauge-mass} = \frac{e^2 v^2}{8 \sin^2 \theta_W} \left( W_+ W_- + \frac{1}{\cos^2 \theta_W} Z^2 \right). \quad (1.5.15)$$

The gauge boson masses are then

$$\begin{aligned} m_W^2 &= \frac{e^2 v^2}{4 \sin^2 \theta_W} \\ m_Z^2 &= \frac{e^2 v^2}{4 \sin^2 \theta_W \cos^2 \theta_W}. \end{aligned} \quad (1.5.16)$$

Since  $e$  and  $v$  are known, measuring one of the gauge boson masses fixes the other.

## 1.5.2 Custodial symmetry

It is useful to define the  $\rho$ -parameter

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_W}. \quad (1.5.17)$$

At tree level, in SM,  $\rho = 1$ ; this is guaranteed by the so-called "Custodial symmetry" [112, 113]. Before spontaneous symmetry breaking, the Higgs field has a global  $SO(4)$  symmetry

$$\Phi^\dagger \Phi = \begin{pmatrix} \phi_2 - i\phi_3 & \phi_0 - i\phi_1 \end{pmatrix} \begin{pmatrix} \phi_2 + i\phi_3 \\ \phi_0 + i\phi_1 \end{pmatrix} = \sum_{i=1}^4 \phi_i^2. \quad (1.5.18)$$

Alternatively, it can be referred to as a global  $SU(2)_L \times SU(2)_R$  symmetry since the two are isomorphic. After spontaneous symmetry breaking,  $\phi_0 = h + v$  and the  $SO(4)$  is broken to  $SO(3)$

$$\Phi^\dagger \Phi \rightarrow (h + v)^2 + \sum_{i=1}^3 \phi_i^2 \quad (1.5.19)$$

with the Goldstone bosons corresponding to  $W^\pm, Z$  transforming as a triplet under the residual  $SO(3)$ . This  $SO(3)$  isn't exact however and is broken by the mixing between the  $W_3$  and  $B$  fields implying that the corrections to the  $\rho$  parameter must be proportional to the mixing angle  $\sin^2 \theta_W$ , which is indeed found through explicit calculations. On the other hand, non-degenerate fermion doublets coupling to the Higgs also break the custodial symmetry where the symmetry breaking is proportional to the difference in the Yukawa couplings e.g. for the  $t - b$  doublet, the correction to the  $\rho$ -parameter vanishes in the limit  $y_t = y_b$ . Thus the custodial symmetry prevents the gauge bosons from acquiring arbitrarily large radiative corrections for their masses.

### 1.5.3 Electroweak interactions

Consider again the gauge-boson kinetic terms in Eq. 1.5.1. Expanding the field strength tensors gives the gauge boson interaction terms e.g. the 3-point vertices

$$\begin{aligned}
& -ig_2 \cos \theta_W (\partial^\mu Z^\nu (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) + Z^\nu (W^{+,\mu} \partial_\mu W_\nu^- + \dots)) \\
& -ie (\partial^\mu A^\nu (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) + A^\nu (W^{+,\mu} \partial_\mu W_\nu^- + \dots)) ,
\end{aligned} \tag{1.5.20}$$

and the 4-point vertices

$$\begin{aligned}
& + \frac{g_2^2}{2} W^{+,\mu} W^{-,\nu} W_\mu^+ W_\nu^- + g_2^2 \cos^2 \theta_W Z^\mu Z_\mu W^{+,\nu} W_\nu^- \\
& - e^2 A^\mu A_\mu W^{+,\nu} W_\nu^- - eg_2 \cos \theta_W Z^\mu A_\mu W^{+,\nu} W_\nu^- .
\end{aligned} \tag{1.5.21}$$

It is interesting to note that the all- $W$  interactions involve purely the  $SU(2)_L$  coupling  $g_2$  implying that this is a pure "weak" interaction while those involving the photon field  $A_\mu$  have the coupling  $e$  implying the electromagnetic nature of the interaction, the fact that the photon is coupling to the electrically charged  $W^\pm$ -bosons. The  $Z$ -boson interactions also depend purely on the weak coupling  $g_2$  but only the purely weak part of  $Z_\mu = -\sin \theta_W B_\mu + \cos \theta_W W_{3,\mu}$  contributes. Also,  $Z$  being electrically neutral, does not couple to the photon.

Considering now the fermions, the Lagrangian can be written as

$$\mathcal{L}_{fermion} = \bar{\Psi} (i\gamma^\mu D_\mu^f) \Psi \tag{1.5.22}$$

where  $\Psi$  is the doublet  $\Psi = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$  and the covariant derivative for fermions  $D_\mu^f$  is

$$D_\mu^f = \partial_\mu - ig_1 Y B_\mu - i\frac{g_2}{2} P_L \sigma_i W_{i,\mu} . \tag{1.5.23}$$

Taking the example of the (left-handed)  $e - \nu_e$  doublet  $L_e = \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}$ , the interaction term can be written as

$$\mathcal{L}_e = i\bar{L}_e(\not{\partial} - ig_1 Y_e \not{B} - i\frac{g_2}{2} \sigma_i \not{W}_i)L_e + i\bar{e}_R(\not{\partial} - ig_1 Y_{e_R} \not{B})e_R + i\bar{\nu}_R(\not{\partial} - ig_1 Y_{\nu_R} \not{B})\nu_R. \quad (1.5.24)$$

Since the right-handed fields  $e_R, \nu_R$  do not form part of an  $SU(2)_L$  doublet,  $I_3 = 0$  for both. For the neutrino this implies that  $Y_{\nu_R} = 0$  from the relation  $Q = Y + I_3$ , meaning that the right-handed neutrino does not interact with through weak interactions at all. And since it is not charged under  $SU(3)_C$  either, it is completely invisible in the Standard Model. In principle nothing forbids its presence, however it is impossible to detect in SM and does not interact with any other SM particles making it irrelevant to any meaningful physical observable. This does not mean, though, that the right-handed neutrinos are pointless; in fact many new physics models predict their interactions. Since neutrinos have been shown to have masses through oscillation experiments [114, 115, 1], assuming that they gain their masses through the Higgs field necessitates the presence of right-handed fields. The other hypercharges can also be inferred similarly; since  $Q_e = -1$ ,  $Y_e = -1/2$  which can also be inferred using  $Q_\nu = 0$  and  $I_3(\nu_{e,L}) = 1/2$ . For the right-handed electron,  $Y_{e_R} = Q_e = -1$ .

Expanding the above Lagrangian in terms of the rotated  $A-Z$  basis and using  $Q = Y + I_3$

gives

$$\begin{aligned}
\mathcal{L}_e &= i\bar{L}_e \not{\partial} L_e + i\bar{e}_R \not{\partial} e_R + \frac{e}{2\sin\theta_W} \bar{L}_e \begin{pmatrix} 0 & W^- \\ W^+ & 0 \end{pmatrix} L_e \\
&+ e\bar{L}_e \begin{pmatrix} Q_e A + \frac{1}{2\sin\theta_W \cos\theta_W} (1 - 2Q_e \sin^2\theta_W) \not{Z} & 0 \\ 0 & \frac{-1}{2\sin\theta_W \cos\theta_W} \not{Z} \end{pmatrix} L_e \\
&+ eQ_e \bar{e}_R A e_R - eQ_e \tan\theta_W \bar{e}_R \not{Z} e_R \\
&= i\bar{L}_e \not{\partial} L_e + i\bar{e}_R \not{\partial} e_R + \frac{e}{2\sin\theta_W} \underbrace{(\bar{\nu}_{e,L} W^+ e_L + \bar{e}_L W^- \nu_{e,L})}_{\text{Charged current weak interaction}} - e \underbrace{(\bar{e}_R A e_R + \bar{e}_L A e_L)}_{\text{Electromagnetic interaction}} \\
&+ \frac{e}{\sin 2\theta_W} \underbrace{((1 - 2Q_e \sin^2\theta_W) \bar{e}_L \not{Z} e_L - 2Q_e \sin^2\theta_W \bar{e}_R \not{Z} e_R + \bar{\nu}_{e,L} \not{Z} \nu_{e,L})}_{\text{Neutral current weak interaction}}. \quad (1.5.25)
\end{aligned}$$

where, with abuse of notation,  $e$  is the fundamental electric charge. A few things to note from the above equations: The  $W^\pm$ -bosons act as raising and lowering operators on an  $SU(2)$  doublet;  $Z$ -boson acts like the  $I_3$  operator, however due to mixing between the  $B - W_3$  fields there is also a photon like interaction that, unsurprisingly, depends on the strength of this mixing;  $\nu_{e,L}$  has no electromagnetic interactions, as expected, since it is electrically neutral.

The interactions above can be written in a more condensed form (omitting the pure EM term) for a general fermion doublet  $\Psi = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$  as

$$\mathcal{L}_{weak} = \frac{e}{2\sin\theta_W} (\bar{\psi}_2 W^+ \psi_1 + \bar{\psi}_1 W^- \psi_2) + \frac{e}{2\sin\theta_W \cos\theta_W} \bar{\psi}_f \not{Z} (v_f + a_f \gamma^5) \psi_f. \quad (1.5.26)$$

with the purely EM interaction term being ignored for brevity. Here  $v_f, a_f$  refer to the vector and axial coupling of the  $Z$ -boson to the fermion  $f$  and are given by

$$\begin{aligned}
v_f &= I_3 - 2Q_f \sin^2\theta_W \\
a_f &= -I_3. \quad (1.5.27)
\end{aligned}$$

## 1.6 Theoretical status

### 1.6.1 ZZ production at LHC

Higher-order perturbative calculations are an essential part of the modern particle physics program. One of the most important processes in that regard is the production of a pair of vector bosons where the vector boson pair could correspond to any combination  $VV = ZZ, WW, \gamma\gamma, WZ, Z\gamma, W\gamma$ . The specifics below refer only to  $gg \rightarrow ZZ$  production since that is the primary focus of this work.

$ZZ$  production is key to many new physics searches as well as precision physics [116, 117, 118, 119, 120]. In addition, it is a significant background to the crucial Higgs signal process  $gg \rightarrow h \rightarrow VV \rightarrow leptons$  [121, 122, 54, 123]. It was also observed that it interfere quite significantly ( $\sim 10\%$ ) with offshell Higgs production [51, 124] which is important for indirect width measurement as discussed in Sec. 1.3.2. As such, precise theoretical predictions for this process are indispensable. Since gluons don't couple directly to vector bosons, gluon initiated channels always appear at a higher order in the perturbation series. E.g.  $q\bar{q} \rightarrow ZZ$  already starts at tree level while  $gg \rightarrow ZZ$  starts at 1-loop order which is formally NNLO for the hadronic process  $pp \rightarrow ZZ$ . Nevertheless, due to high gluon luminosity at LHC it accounts for most of the NNLO correction ( $\sim 60\%$ ) [125] to  $pp \rightarrow ZZ$ . The NLO corrections to  $gg \rightarrow ZZ$  were also found to be significant [126] increasing the total  $pp \rightarrow ZZ$  cross-section by ( $\sim 5\%$ ) [127]. Consequently, pair production of vector bosons has been one of the focal points of perturbative QCD efforts.

The  $qq' \rightarrow VV$  processes are all known at NLO in QCD [128, 129, 130, 131, 132, 133] as well as NLO EW for most processes [134, 135, 136, 137, 138, 139, 140, 141]. This was possible in large part to the so-called "NLO revolution" based on unitarity methods that automated 1-loop calculations [142, 143, 144, 145, 146, 147, 148]. Gluon initiated processes

are all also known at 1-loop [149, 150, 151, 152, 153, 154, 155, 156, 157, 158]. The 2-loop amplitudes are not universally known, however. Complexity at 2-loops increases enormously compared to 1-loop and as such 2-loop processes are calculated on a case-by-case basis. So far no convenient and computationally feasible way to automate the 2-loop processes has been developed. Nevertheless, there has been significant progress in the past decade. In particular for  $gg \rightarrow ZZ$ , the 2-loop contributions including only massless quarks in the loops are known [159, 160]. This does not provide the complete picture, however, and corrections due to massive quark loops are deemed important as a consequence of the Goldstone boson equivalence theorem [36, 37]. This is especially true for loops involving the top quark whose large coupling to Higgs and the longitudinal modes of the gauge bosons (as a consequence of the equivalence theorem) could result in significant corrections in high invariant mass region. The longitudinal modes for the  $Z$ -bosons are, moreover, particularly interesting since they provide a window into potential new physics manifesting through loop effects as well as they provide a way to measure the anomalous  $t\bar{t}Z$  coupling [161, 162].

Calculating the 2-loop contributions to  $gg \rightarrow ZZ$  with massive internal quarks provide significant challenges. Naively, due to addition of one more variable to the problem compared to the massless case, an increase in complexity is expected. However, it is much worse and the traditional methods to calculate multiloop amplitudes are insufficient. The 2-loop contributions from top quark loops were calculated in an approximation where the top quark was considered much heavier than every other scale, colloquially referred to as the heavy top-mass Limit, in [163, 164]. Since the expansion has a finite radius of convergence, it cannot be used in all the regions of phase space, particularly for higher invariant mass. This can be improved with the use of Padé approximants as in [165] to extend the expansion over a larger phase space region. Another approximation using an expansion around the top quark pair production threshold together with the heavy top limit, improved with Padé approximants, was used in [166] for contributions relevant to Higgs production. In [12], the heavy top limit was used together with small top-mass limit to access the regions of higher

invariant mass and improved using Padé approximants to access the intermediate region. The amplitudes involving Higgs as a mediator have been computed with exact top mass dependence [167, 168, 169, 170]. More recently, contributions to  $WW$  production from third generation quark loops ( $b, t$ ) were computed with exact mass dependence in [171]. It must be pointed out that the 2-loop amplitudes for  $gg \rightarrow ZZ$  with exact top quark dependence were also calculated independently in [172] which appeared a couple months after this work was completed.

### 1.6.2 Diphoton production at LHC

Another important class of processes at LHC is the production of a pair of photons. It provides a crucial channel for new physics searches [173, 174] as well as Higgs production. Similar to  $ZZ$ , continuum production of  $\gamma\gamma$  is also an important background to Higgs signal process. In addition, transverse momentum of the diphoton pair is sensitive to potential new physics whether through Higgs production or through new heavy resonances. It is, thus, a crucial part of the particle physics program [175] and better theoretical predictions are always desirable.

The NNLO QCD corrections to  $pp \rightarrow \gamma\gamma$  are known already [176, 177, 178, 179]. The ingredients for 3-loop amplitudes for four particle scattering became accessible only in the recent years [180, 181] with the full amplitudes for  $qq \rightarrow \gamma\gamma$  appearing very recently [182]. Including 1-jet in addition,  $pp \rightarrow \gamma\gamma + jet$  is known only at NLO [183, 184]. For both  $pp \rightarrow \gamma\gamma$  at N3LO and  $pp \rightarrow \gamma\gamma + jet$  at NNLO, the missing component was the 2-loop amplitudes for diphoton plus jet production where they are required to subtract the IR poles for the former. However,  $2 \rightarrow 3$  particle scattering amplitudes at 2-loop provided a great challenge and required the development of powerful new techniques. These advances, discussed in detail in this work, led finally to the calculation of first full  $2 \rightarrow 3$  amplitudes, first in leading colour approximations [185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195], for simpler helicity configuration [196], and then very recently in full colour for all helicities [197, 198].

Calculation of both  $gg \rightarrow ZZ$  and  $qq(g) \rightarrow \gamma\gamma + jet$  2-loop amplitudes required incredible advances in the technology and are the focus of this work along with the results.

# Chapter 2

## Setup of calculation

### 2.1 Form factor decomposition

Let us consider pair production of  $Z$ -bosons through gluon fusion,

$$g(p_1) + g(p_2) \rightarrow Z(p_3) + Z(p_4). \quad (2.1.1)$$

The gluons  $(p_1, p_2)$  are considered to be incoming and the  $Z$ -bosons  $(p_3, p_4)$  considered to be outgoing so that momentum conservation implies

$$p_1 + p_2 = p_3 + p_4. \quad (2.1.2)$$

The external particles are also considered to be on-shell with the momenta then satisfying the on-shell conditions

$$p_1^2 = p_2^2 = 0, \quad p_3^2 = p_4^2 = m_Z^2. \quad (2.1.3)$$

Formally, the scattering amplitude for this process can be written as

$$\mathcal{M} = \mathcal{M}_{\mu\nu\rho\sigma}^{ab}(p_1, p_2, p_3, p_4) \epsilon_{\lambda_1}^{a,\mu}(p_1) \epsilon_{\lambda_2}^{b,\nu}(p_2) \epsilon_{\lambda_3}^{*\rho}(p_3) \epsilon_{\lambda_4}^{*\sigma}(p_4) \quad (2.1.4)$$

stripping away the polarisation vectors  $\epsilon_{\lambda_i}(p_i)$  for the gluons and the  $Z$ -bosons with  $\lambda_i$  denoting the particle helicity. For brevity, the colour and Lorentz indices on the polarisation vectors are henceforth suppressed and the abbreviation  $\epsilon_i \equiv \epsilon_{\lambda_i}(p_i)$  is instead used. Note that the amplitude  $\mathcal{M}$ , or equivalently  $\mathcal{M}_{\mu\nu\rho\sigma}^{ab}(p_1, p_2, p_3, p_4)$ , is the full scattering amplitude; no perturbative expansion has been performed yet. In general, it is desirable to maintain the full Lorentz structure of the amplitude since complete polarisation information can be useful to include e.g. decays of the outgoing particles. In special cases, however, the calculation can be immensely simplified by the use of explicit representations for the polarisation vectors  $\epsilon_{\lambda_i}(p_i)$ , for example using the spinor-helicity formalism [199, 200, 201, 202, 203, 147]; the most well-known example is perhaps the Parke-Taylor formula [204, 205] where the amplitude collapses to a remarkably simple 1-line expression.

In general, evaluating  $\mathcal{M}_{\mu\nu\rho\sigma}^{ab}(p_1, p_2, p_3, p_4)$  directly can be an extremely difficult task, especially for multi-loop amplitudes. The expressions involve a large number of tensor loop integrals, and while techniques exist to handle them in a systematic way, most of the powerful modern methods involve scalar loop integrals where the open Lorentz indices are contracted with loop or external momenta. As such, it is highly beneficial to separate the Lorentz structure from the amplitude. Lorentz invariance dictates that the amplitude  $\mathcal{M}_{\mu\nu\rho\sigma}^{ab}(p_1, p_2, p_3, p_4)$  can be decomposed in terms of all possible, in this case rank-4, Lorentz tensors. These Lorentz tensors can be composed of the independent external momenta as well as general process-independent tensors such as the metric tensor  $g_{\mu\nu}$  and the totally anti-symmetric Levi-Civita tensor  $\epsilon_{\mu\nu\rho\delta}$ . For the process at hand, a total of 138 tensor structures appear in

the decomposition [159]:

$$\begin{aligned}
\mathcal{M}^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) &= a_1 g^{\mu\nu} g^{\rho\sigma} + a_2 g^{\mu\rho} g^{\nu\sigma} + a_3 g^{\mu\sigma} g^{\nu\rho} \\
&+ \sum_{i,j=1}^3 ( a_{1,ij} g^{\mu\nu} p_i^\rho p_j^\sigma + a_{2,ij} g^{\mu\rho} p_i^\nu p_j^\sigma + a_{3,ij} g^{\mu\sigma} p_i^\nu p_j^\rho \\
&\quad + a_{4,ij} g^{\nu\rho} p_i^\mu p_j^\sigma + a_{5,ij} g^{\nu\sigma} p_i^\mu p_j^\rho + a_{6,ij} g^{\rho\sigma} p_i^\mu p_j^\nu ) \\
&+ \sum_{i,j,k,l=1}^3 a_{ijkl} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma .
\end{aligned} \tag{2.1.5}$$

In the above, parity-odd tensors involving the Levi-Civita tensor have been dropped. This is because of Bose symmetry and charge-parity conservation for this process [153]. Note that the colour indices have been dropped in the above equation for brevity.

Fundamentally, the decomposition in Eq. 2.1.5 is completely independent of the underlying process and should hold for any  $2 \rightarrow 2$  scattering process involving four vector bosons as well as satisfying Bose symmetry and conserving charge-parity. All the physics is now contained in the coefficients  $a_{ij..}$  which are referred to as "form factors". It would be quite difficult to calculate all 138 form factors; thankfully, that is not the case and certain identities can be exploited to reduce this number. Transversality of the gluon polarisation vectors (from Ward identities) implies

$$\epsilon_1 \cdot p_1 = 0, \quad \epsilon_2 \cdot p_2 = 0. \tag{2.1.6}$$

Further reduction can be achieved by explicitly choosing a gauge for the external particles such as

$$\epsilon_1 \cdot p_2 = 0, \quad \epsilon_2 \cdot p_1 = 0, \quad \epsilon_3 \cdot p_3 = 0, \quad \epsilon_4 \cdot p_4 = 0, \tag{2.1.7}$$

which reduces the total number of independent form factors to 20. The above gauge choice

for the polarisation vectors corresponds to the following polarisation sums:

$$\begin{aligned}
\sum_{\text{pol}} \epsilon_1^{a,\mu} \epsilon_1^{*b,\nu} &= \sum_{\text{pol}} \epsilon_2^{a,\mu} \epsilon_2^{*b,\nu} = \left( -g^{\mu\nu} + \frac{p_1^\mu p_2^\nu + p_2^\mu p_1^\nu}{p_1 \cdot p_2} \right) \delta^{ab}, \\
\sum_{\text{pol}} \epsilon_3^\mu \epsilon_3^{*\nu} &= -g^{\mu\nu} + \frac{p_3^\mu p_3^\nu}{p_3 \cdot p_3}, \\
\sum_{\text{pol}} \epsilon_4^\mu \epsilon_4^{*\nu} &= -g^{\mu\nu} + \frac{p_4^\mu p_4^\nu}{p_4 \cdot p_4}.
\end{aligned} \tag{2.1.8}$$

It is useful to define the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_2 - p_3)^2. \tag{2.1.9}$$

Since they are Lorentz-invariant, it is convenient to express the amplitude in terms of these quantities. Momentum conservation (Eq. 2.1.2) allows us to write down a relation between the Mandelstam variables:

$$s + t + u = 2m_Z^2 \tag{2.1.10}$$

The amplitude can then be written in terms of the remaining 20 form factors as

$$\mathcal{M}^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \sum_{i=1}^{20} A_i(s, t, m_Z^2, m_1^2, m_2^2, \dots) T_i^{\mu\nu\rho\sigma}, \tag{2.1.11}$$

where the  $A_i$  are the form factors depending only on the Mandelstam variables  $(s, t)$ , mass of the  $Z$ -boson ( $m_Z$ ), and masses of the internal particles denoted by  $m_1, m_2, \dots$ . The remaining

20 tensors  $T_i$  are as follows:

$$\begin{aligned}
T_1^{\mu\nu\rho\sigma} &= g^{\mu\nu} g^{\rho\sigma}, & T_2^{\mu\nu\rho\sigma} &= g^{\mu\rho} g^{\nu\sigma}, & T_3^{\mu\nu\rho\sigma} &= g^{\mu\sigma} g^{\nu\rho}, & T_4^{\mu\nu\rho\sigma} &= p_1^\rho p_1^\sigma g^{\mu\nu}, \\
T_5^{\mu\nu\rho\sigma} &= p_1^\rho p_2^\sigma g^{\mu\nu}, & T_6^{\mu\nu\rho\sigma} &= p_1^\sigma p_2^\rho g^{\mu\nu}, & T_7^{\mu\nu\rho\sigma} &= p_2^\rho p_2^\sigma g^{\mu\nu}, & T_8^{\mu\nu\rho\sigma} &= p_1^\sigma p_3^\nu g^{\mu\rho}, \\
T_9^{\mu\nu\rho\sigma} &= p_2^\sigma p_3^\nu g^{\mu\rho}, & T_{10}^{\mu\nu\rho\sigma} &= p_1^\rho p_3^\nu g^{\mu\sigma}, & T_{11}^{\mu\nu\rho\sigma} &= p_2^\rho p_3^\nu g^{\mu\sigma}, & T_{12}^{\mu\nu\rho\sigma} &= p_1^\sigma p_3^\mu g^{\nu\rho}, \\
T_{13}^{\mu\nu\rho\sigma} &= p_2^\sigma p_3^\mu g^{\nu\rho}, & T_{14}^{\mu\nu\rho\sigma} &= p_1^\rho p_3^\mu g^{\nu\sigma}, & T_{15}^{\mu\nu\rho\sigma} &= p_2^\rho p_3^\mu g^{\nu\sigma}, & T_{16}^{\mu\nu\rho\sigma} &= p_3^\mu p_3^\nu g^{\rho\sigma}, \\
T_{17}^{\mu\nu\rho\sigma} &= p_1^\rho p_1^\sigma p_3^\mu p_3^\nu, & T_{18}^{\mu\nu\rho\sigma} &= p_1^\rho p_2^\sigma p_3^\mu p_3^\nu, & T_{19}^{\mu\nu\rho\sigma} &= p_2^\rho p_1^\sigma p_3^\mu p_3^\nu, & T_{20}^{\mu\nu\rho\sigma} &= p_2^\rho p_2^\sigma p_3^\mu p_3^\nu.
\end{aligned} \tag{2.1.12}$$

To extract the form factors from the amplitude, projection operators  $P_i^{\mu\nu\rho\sigma}$  can be constructed such that they fulfill

$$\sum_{\text{pol}} P_i^{\mu\nu\rho\sigma} \epsilon_{1\mu}^* \epsilon_{2\nu}^* \epsilon_{3\rho} \epsilon_{4\sigma} \epsilon_{1\mu'} \epsilon_{2\nu'} \epsilon_{3\rho'}^* \epsilon_{4\sigma'}^* \mathcal{M}^{\mu'\nu'\rho'\sigma'} = A_i. \tag{2.1.13}$$

To find these projection operators, they can be decomposed in terms of the  $T_i^{\mu\nu\rho\sigma}$ :

$$P_i^{\mu\nu\rho\sigma} = \sum_{j=1}^{20} B_{ij}(s, t, m_Z^2) (T_j^{\mu\nu\rho\sigma})^\dagger, \quad i = 1, \dots, 20. \tag{2.1.14}$$

Note that the coefficients  $B_{ij}$  are functions of only the external kinematic configuration, independent of the internal particles. The actual expressions for  $B_{ij}$  are too complicated to be provided here and the reader is instead referred to the VVamp project website. To calculate them is relatively straightforward, however. Inserting Eqs. 2.1.11 and 2.1.14 into Eq. 2.1.13,

$$\sum_{j=1}^{20} B_{ij}(s, t, m_Z^2) (T_j^{\mu\nu\rho\sigma})^\dagger T_k^{\mu'\nu'\rho'\sigma'} \sum_{\text{pol}} (\epsilon_{1\mu}^* \epsilon_{2\nu}^* \epsilon_{3\rho} \epsilon_{4\sigma} \epsilon_{1\mu'} \epsilon_{2\nu'} \epsilon_{3\rho'}^* \epsilon_{4\sigma'}^*) = \delta_{ik}. \tag{2.1.15}$$

Polarisation sums from Eq. 2.1.8 can then be inserted into the above equation and the resulting linear system solved to obtain the coefficients  $B_{ij}$ . Not all 20 form factors are independent

though. With identical bosons in both the initial and final states, the process exhibits Bose symmetry which implies that the amplitude must be invariant under the exchange of the incoming gluons or the outgoing  $Z$ -bosons [159]:

$$\begin{aligned} 1 \leftrightarrow 2 & : p_1 \leftrightarrow p_2, \quad \epsilon_{\lambda_1}(p_1) \leftrightarrow \epsilon_{\lambda_2}(p_2), \\ 3 \leftrightarrow 4 & : p_3 \leftrightarrow p_4, \quad \epsilon_{\lambda_3}(p_3) \leftrightarrow \epsilon_{\lambda_4}(p_4). \end{aligned}$$

This symmetry results in certain relations between the form factors. First, there are the identities:

$$A_7 = A_4, \quad A_{12} = -A_{11}, \quad A_{13} = -A_{10}, \quad A_{14} = -A_9, \quad A_{15} = -A_8, \quad A_{20} = A_{17}. \quad (2.1.16)$$

Second, the following relations under the swapping of the Mandelstam invariants  $t \leftrightarrow u$ :

$$\begin{aligned} A_1(s, t) &= A_1(s, u), & A_4(s, t) &= A_4(s, u), & A_7(s, t) &= A_7(s, u), \\ A_{16}(s, t) &= A_{16}(s, u), & A_{17}(s, t) &= A_{17}(s, u), & A_{20}(s, t) &= A_{20}(s, u), \\ A_2(s, t) &= A_3(s, u), & A_5(s, t) &= A_6(s, u), & A_8(s, t) &= A_{13}(s, u), \\ A_9(s, t) &= A_{12}(s, u), & A_{10}(s, t) &= A_{15}(s, u), & A_{11}(s, t) &= A_{14}(s, u), \\ A_{18}(s, t) &= A_{19}(s, u). \end{aligned} \quad (2.1.17)$$

It must be noted that everything to this point has been treated in general  $d$ -dimensions i.e. the external momenta and consequently the form factors and the projection operators are all in  $d$ -dimensions. This has the advantage of being compatible with dimensional regularisation and all the techniques used for solving loop integrals.

At this point, it makes sense to switch to physically relevant 4-dimensional observables. A convenient choice is the so-called "helicity amplitudes", derived by explicitly fixing the helicities for the external particles. Note that such helicity amplitudes are inherently 4-dimensional quantities. It is rather straightforward to derive the expressions for the helicity

amplitudes in terms of the form factors  $A_i$ . It must be pointed out that since the  $Z$ -bosons are massive, their helicities are frame dependent; in this case, the momenta are considered to be in the center-of-momentum frame for both the incoming and outgoing states. This allows the momenta to be parameterised according to

$$\begin{aligned} p_1^\mu &= \frac{\sqrt{s}}{2} (1, 0, 0, 1), & p_3^\mu &= \frac{\sqrt{s}}{2} (1, \beta \sin \theta, 0, \beta \cos \theta), \\ p_2^\mu &= \frac{\sqrt{s}}{2} (1, 0, 0, -1), & p_4^\mu &= \frac{\sqrt{s}}{2} (1, -\beta \sin \theta, 0, -\beta \cos \theta), \end{aligned} \quad (2.1.18)$$

where  $\beta = \sqrt{1 - 4m_Z^2/s}$  and  $\theta$  is the angle between the direction of the incoming gluon ( $p_1$ ) and the outgoing  $Z$ -boson with momentum  $p_3$ . The polarisation vectors can then be written as following [6]:

$$\begin{aligned} \epsilon_\pm^\mu(p_1) &= \frac{1}{\sqrt{2}} (0, \mp 1, -i, 0), \\ \epsilon_\pm^\mu(p_2) &= \frac{1}{\sqrt{2}} (0, \pm 1, -i, 0), \\ \epsilon_\pm^\mu(p_3) &= \frac{1}{\sqrt{2}} (0, \mp \cos \theta, -i, \pm \sin \theta), & \epsilon_0^\mu(p_3) &= \frac{\sqrt{s}}{2m_Z} (\beta, \sin \theta, 0, \cos \theta), \\ \epsilon_\pm^\mu(p_4) &= \frac{1}{\sqrt{2}} (0, \pm \cos \theta, -i, \mp \sin \theta), & \epsilon_0^\mu(p_4) &= \frac{\sqrt{s}}{2m_Z} (\beta, -\sin \theta, 0, -\cos \theta). \end{aligned} \quad (2.1.19)$$

It can be easily shown that the above choice of polarisation vectors satisfies the polarisation sums in Eqs. 2.1.6, 2.1.7, and 2.1.8.

A straightforward calculation would show that there are a total of  $2^2 \times 3^2 = 36$  helicity amplitudes for this process. However, parity transformation reduces the number of independent helicity amplitudes by a factor of two [72]:

$$\mathcal{M}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} = (-1)^{\delta_{\lambda_3 0} + \delta_{\lambda_4 0}} \mathcal{M}_{-\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4}. \quad (2.1.20)$$

Above identity under parity transformation holds for a general external kinematics. For the special case at hand where the gluons and  $Z$ -bosons are on-shell, following symmetry

relations also hold [153, 12]:

$$\begin{aligned}
\mathcal{M}_{++++} &= \mathcal{M}_{+--+} \\
\mathcal{M}_{+---} &= \mathcal{M}_{-++-} \\
\mathcal{M}_{++\pm 0} &= \mathcal{M}_{++0\pm} \\
\mathcal{M}_{+-\pm 0} &= -\mathcal{M}_{+-0\mp},
\end{aligned} \tag{2.1.21}$$

where the identities in Eq. 2.1.16 have been used to derive the above relations. Another set of relations can be derived if the variables  $s, \theta$  are eliminated in favour of  $\beta, t$  using  $\theta = \arccos((t - u)/(\beta s)) \in [0, \pi]$ ,  $u = 2m_Z^2 - s - t$ , and  $s = 4m_Z^2/(1 - \beta^2)$ :

$$\begin{aligned}
\mathcal{M}_{++++}(\beta, t) &= \mathcal{M}_{+--+}(-\beta, t), \\
\mathcal{M}_{+--+}(\beta, t) &= \mathcal{M}_{+---}(-\beta, t), \\
\mathcal{M}_{+\pm+0}(\beta, t) &= \mathcal{M}_{+\pm-0}(-\beta, t).
\end{aligned} \tag{2.1.22}$$

Eqs. 2.1.20, 2.1.21, and 2.1.22 together reduce the total number of independent helicity amplitudes to 8. The explicit expressions for the helicity amplitudes in terms of the form factors  $A_i$ , as defined through Eq. 2.1.19, are provided in an ancillary file with [6].

Before taking all the symmetries into account, there are 20 form factors  $A_i$ , while there are only 18 helicity amplitudes. In fact, looking at the analytic forms of the projection operators, they have spurious poles in the space-time dimension  $d$  of the form  $\frac{1}{d-4}$ . This is clearly not physical and suggests some redundancy between the form factors. It is not unimaginable that for  $d = 4$  certain linear relations appear between the form factors. This phenomenon was already observed in [159] where allowing the  $Z$ -bosons to decay to fermions and considering specific fermion helicities, only 18 independent "helicity amplitudes" were found. These helicity amplitudes, corresponding to the helicities of the final-state fermions, were found to be gauge-invariant physically relevant quantities, unlike the form factors  $A_i$ ;

explicit calculation showed that while the form factors  $A_i$  are dependent on the specific regularisation scheme for  $\gamma^5$ , the helicity amplitudes written in [159] are not. Choosing specific helicities for the external particles also forces them to be four-dimensional. This prescription where external particles are treated in  $d = 4$  while all the internal quantities are treated in general  $d$  is colloquially referred to as the 't Hooft-Veltmann scheme.

Another disadvantage of using the form factors defined as in Eq. 2.1.13 is that they are not orthogonal. An alternate approach was used in [12] to remedy this using the Gram-Schmidt orthogonalisation procedure. Starting from the 20 tensors as in Eq. 2.1.12, the Gram-Schmidt procedure results in 20 new tensors that are orthogonal linear combinations of the original tensors. Crucially, 2 of the linear combinations vanish in  $d \rightarrow 4$  limit with only 18 remaining, further indicating the redundancy in the original decomposition. A general prescription to construct physically relevant form factors in  $d$ -dimensions was provided in [71, 72]. The authors used the 't Hooft-Veltmann prescription, considering all objects in  $d$ -dimensions except the external momenta which are treated in 4-dimensions. Lastly, it must be pointed out that in this work only the contribution from top-quarks in the loop are considered.

## 2.2 1-loop amplitude

### 2.2.1 Generation of the amplitude

The form factor decomposition in the previous subsection was general and valid to all orders in perturbation theory. Here, the amplitude in Eq. 2.1.4 is perturbatively expanded in the strong coupling constant  $\alpha_S$  as

$$\mathcal{M} = \mathcal{M}_0 + \left(\frac{\alpha_S}{2\pi}\right) \mathcal{M}_1 + O(\alpha_S^2). \quad (2.2.1)$$

Here  $\mathcal{M}_0$  corresponds to the Leading-Order amplitude.  $Z$ -bosons don't have "colour" charge and hence don't directly couple to gluons. This process, thus, starts at 1-loop through closed

quark loops. `Qgraf` [206] is used to generate the Feynman diagrams. All the quarks other than the top quark are treated as massless and only the Higgs is allowed to couple to the top quark. A total of 10 diagrams are found out of which 2 are zero due to colour where a single gluon couples to a closed quark loop. Of the remaining 8, 2 are mediated via offshell production of the Higgs boson (Figs. 2.2.1a and 2.2.1b); remaining 6 are the "box" type contributions. Note that only 2 of these "box" diagrams are truly independent, the rest can be obtained through simple crossings of the external legs. In particular, Fig. 2.2.1d can be obtained by crossing the incoming gluons and Fig. 2.2.1e can be obtained by crossing the outgoing Z-bosons. This corresponds to making the change  $t \leftrightarrow u$  in the analytical expressions. We can further exchange the Z-bosons in Fig. 2.2.1d to obtain Fig. 2.2.1f. However, since the Z-bosons are onshell (Eq. 2.1.3), and have the same invariant mass, crossing both the gluons and the Z-bosons leaves the diagram unchanged which implies the diagrams in Figs. 2.2.1c and 2.2.1d are equal to Figs. 2.2.1f and 2.2.1e respectively.

The next step is to insert Feynman rules in the diagrams to obtain the amplitudes. The Feynman rules employed here are given in the appendix A. Custom `FORM` [207, 208, 209] scripts are used to insert the Feynman rules into the diagrams to generate algebraic expressions. Colour structure of the remaining non-zero diagrams is quite simple; it is essentially a gluon propagator like structure, so the colour factor is simply given by

$$\mathcal{C} = \mathcal{C}_0 \delta_{ab}. \quad (2.2.2)$$

For the tree-level propagator,  $\mathcal{C}_0$  is just 1. At 1-loop,  $\mathcal{C}_0 = T_F = \frac{1}{2}$  for the diagram involving only a closed quark loop. At higher loops, more complicated structures appear in  $\mathcal{C}_0$  i.e. the quadratic Casimirs  $C_F, C_A$ .

The Z-boson couples to the fermion through both a vector and an axial-vector coupling

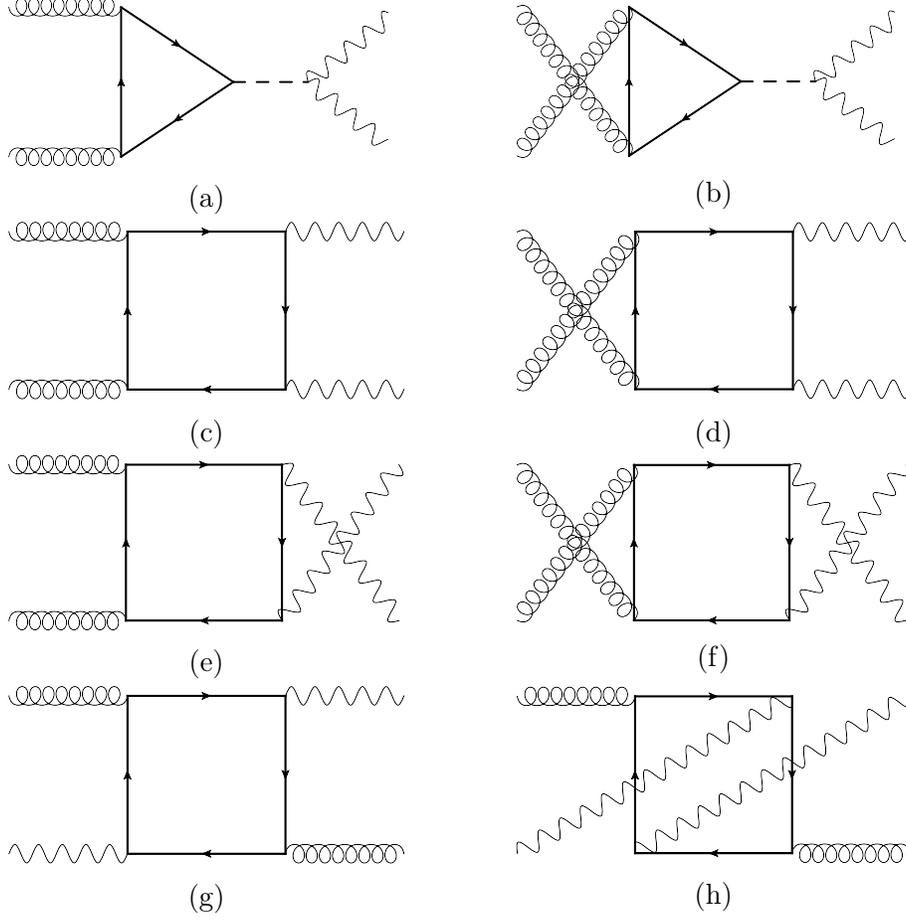


Figure 2.2.1: Feynman diagrams for the process  $gg \rightarrow ZZ$  at 1-loop (LO). 2.2.1a and 2.2.1b are the Higgs exchange diagrams.

with the vertex written as

$$\begin{aligned}
 \mathcal{V}_\mu^{Vff\bar{f}} &= ie \left[ L_{f\bar{f}}^Z \gamma_\mu \left( \frac{1 - \gamma_5}{2} \right) + R_{f\bar{f}}^Z \gamma_\mu \left( \frac{1 + \gamma_5}{2} \right) \right] \\
 &= i \frac{e}{2 \sin \theta_W \cos \theta_W} \gamma_\mu (v_t + a_t \gamma_5)
 \end{aligned} \tag{2.2.3}$$

from the EW couplings in Eq. 1.5.26, where the coupling to the left-handed fermion is  $L_{f\bar{f}}^Z = (I_3^f - q_f \sin^2 \theta_W) / (\sin \theta_W \cos \theta_W)$  and to the right-handed fermion is  $R_{f\bar{f}}^Z = -q_f \sin \theta_W / \cos \theta_W$ . Here  $e$  is the positron charge and  $q_f$  is the electric charge of the fermion in terms of  $e$ . Without repeating the general forms of the vector and axial-vector couplings  $v_t, a_t$ , for top quark they are given by  $\theta_W$  by  $v_t = \frac{1}{2} - \frac{4}{3} \sin^2 \theta_W$  and  $a_t = -\frac{1}{2}$ , respectively, in terms of the weak mixing

angle  $\theta_W$ , as defined in Eq. 1.5.10.

At this point it is necessary to discuss the  $\gamma$ -matrices, in particular the  $\gamma^5$  matrix.  $\gamma$ -matrices are an essential part of most scattering amplitude calculations in Standard Model. A brief description of their properties is provided in Appendix C. While the  $\gamma$ -matrices that satisfy the anti-commutation relation  $\{\gamma^\mu, \gamma^\nu\} = 2g_{\mu\nu}$  can be consistently treated in  $d$ -dimensions, the  $\gamma^5$  matrix, as defined, is inherently a 4-dimensional object. As such, it is not clear how to incorporate it in Conventional Dimensional Regularisation (CDR) [66, 67, 68, 69] where objects are treated in  $d$ -dimensions, along with the usual  $\gamma$ -matrices. There are several schemes used to work around this issue; see Appendix C for details. This calculation employs Kreimer's [210, 211, 212] anti-commuting  $\gamma^5$  scheme to properly treat  $\gamma^5$  in  $d$ -dimensions. In this scheme, the anti-commuting property of  $\gamma^5$  is kept while traces are no longer assumed to be cyclic. In other schemes such as the 't Hooft, Veltman, Breiten-Lohner, and Maison scheme [213, 214, 215, 68], the anti-commutation relation is given up and cyclicity of trace is kept. This scheme necessarily treats 4 and  $d - 4$  dimensional objects separately. A fundamental disadvantage of this is that the Ward identities are not preserved requiring addition of extra counterterms to the Lagrangian. The reader is referred to Appendix C for a more detailed discussion of  $\gamma$ -matrices and different  $\gamma^5$  schemes.

The two Z-bosons coupled to a fermion line can in general generate 3 different contributions: vector-vector ( $v_t^2$ ), axial-vector ( $a_t v_t$ ), and axial-axial ( $a_t^2$ ), and it is useful to split the amplitude into these 3 parts. The vector-vector and axial-axial contributions are identical for massless quarks [153]. This can be understood by looking at the traces involved; for traces involving an even number of  $\gamma^5$ 's, the  $\gamma^5$ 's disappear as a result of the anti-commutation relations and the fact that  $(\gamma^5)^2 = 1$ . The resulting trace is identical to that of the vector-vector case (with a possible overall sign difference). This then means that the difference must be proportional to the quark mass. E.g.

$$Tr \left[ (\not{p}_1 + m) \gamma^\mu (\not{p}_2 + m) \gamma^\nu \right] - Tr \left[ (\not{p}_1 + m) \gamma^\mu \gamma^5 (\not{p}_2 + m) \gamma^\nu \gamma^5 \right] = 8m^2 g^{\mu\nu} \quad (2.2.4)$$

The axial-vector ( $a_t v_t$ ) contribution to this amplitude vanishes identically as a consequence of Bose symmetry and charge-parity conservation for this process [153]. The absence of any terms containing the  $\epsilon$ -tensor in Eq. 2.1.5 can also be explained through this since the Levi-Civita tensor violates parity and hence any such terms are forbidden through charge-parity conservation.

The projection operators from Eq. 2.1.13 are then applied to obtain analytical expressions for individual form factors.

## 2.2.2 Reduction to master integrals

The form factors contain only scalar loop integrals without any open Lorentz indices. The list of scalar loop integrals isn't the minimal list of integrals that needs to be evaluated, however. Indeed it can be shown that at 1-loop, all the integrals can be related to a minimal set of integrals often referred to as the basis integrals or "master integrals". In fact, any 1-loop integral can be written as a linear combination of basis integrals [216, 217, 218]

$$I = c_{1,i} A_i + c_{2,i} B_i + c_{3,i} C_i + c_{4,i} D_i + \mathcal{R} + O(d-4) \quad (2.2.5)$$

as depicted in Fig. 2.2.2, where  $A_i, B_i, C_i, D_i$  are all possible 1-, 2-, 3-, and 4-propagator scalar Feynman integrals

$$A_0(m_1^2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1}, \quad (2.2.6)$$

$$B_0(p^2, m_1^2, m_2^2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1 D_2}, \quad (2.2.7)$$

$$C_0(p_1^2, p_2^2, s_{12}, m_1^2, m_2^2, m_3^2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1 D_2 D_3}, \quad (2.2.8)$$

$$D_0(p_1^2, p_2^2, p_3^2, p_4^2, s_{12}, s_{23}, m_1^2, m_2^2, m_3^2, m_4^2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1 D_2 D_3 D_4}, \quad (2.2.9)$$

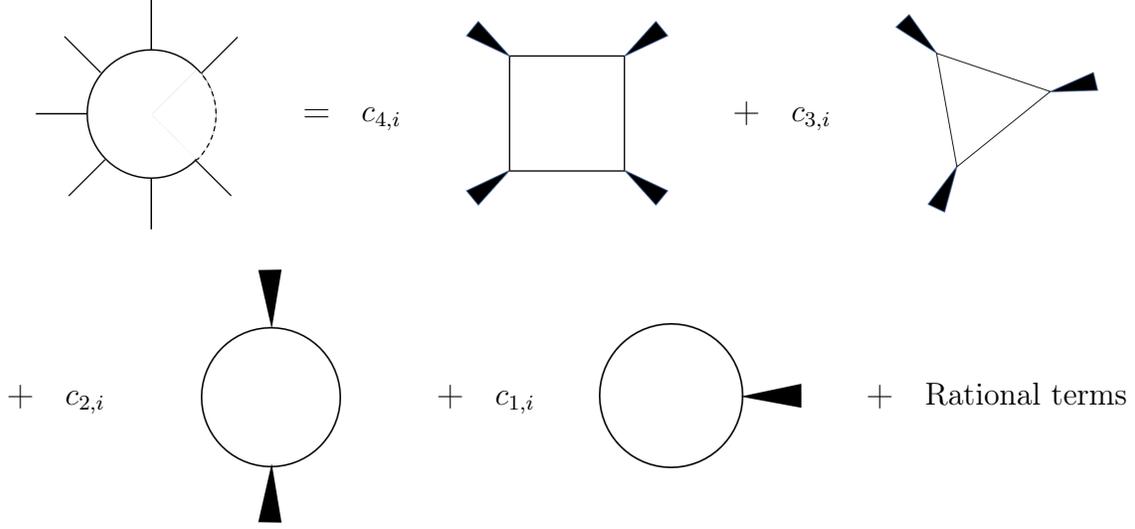


Figure 2.2.2: Decomposition of a general 1-loop Feynman integral

with

$$\begin{aligned}
D_1 &= k^2 - m_1^2, \\
D_2 &= (k + p_1)^2 - m_2^2, \\
D_3 &= (k + p_1 + p_2)^2 - m_3^2, \\
D_4 &= (k + p_1 + p_2 + p_3)^2 - m_4^2.
\end{aligned} \tag{2.2.10}$$

$\mathcal{R}$  is the so-called "Rational term" generated from the  $O(d-4)$  terms appearing in the expansions for the scalar integrals  $A_i, B_i, C_i, D_i$  multiplied by  $\frac{1}{d-4}$ . For example, assuming  $B_0(p^2, m) = \frac{1}{\epsilon} + b_0(p^2, m) + \epsilon b_1(p^2, m)$  where  $\epsilon = (4-d)/2$ ,

$$\begin{aligned}
\frac{1}{d-4}(B_0(p_1^2, m) - B_0(0, m)) &= \frac{1}{d-4}(b_0(p^2, m) - b_0(0, m)) + \frac{1}{d-4}\epsilon(b_1(p^2, m) - b_1(0, m)) \\
&= \frac{1}{d-4}(b_0(p^2, m) - b_0(0, m)) - \frac{1}{2}(b_1(p^2, m) - b_1(0, m)) \\
&= \frac{1}{d-4}(b_0(p^2, m) - b_0(0, m)) + \mathcal{R}.
\end{aligned} \tag{2.2.11}$$

The assertion in Eq. 2.2.5 is in fact more general. Any  $N$ -point function can be written as a linear combination of  $d$  and lower point functions.

The first approach to do this was proposed by Passarino-Veltmann in [219]. Passarino-Veltmann reduction exploits Lorentz invariance of tensor integrals to find relations between them. Consider the following 1-loop 2-propagator tensor integral as an example:

$$I^\mu(p, m) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2 - m^2)(k + p)^2}. \quad (2.2.12)$$

Here Conventional Dimensional Regularisation (CDR) is being employed to regulate the divergences. From Lorentz invariance, the only rank-1 Lorentz tensor possible for this integral is  $p^\mu$ . Using this, the integral can be written as

$$I^\mu(p, m) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2 - m^2)(k + p)^2} = p^\mu B_1(p^2, m), \quad (2.2.13)$$

where the scalar part has been defined as  $B_1(p^2, m)$ , which can be related to the original integral by contracting with  $p_\mu$  as

$$\begin{aligned} p^2 B_1(p^2, m) &= p_\mu I^\mu(p, m) = \int \frac{d^d k}{(2\pi)^d} \frac{k \cdot p}{(k^2 - m^2)(k + p)^2} \\ &= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{(k + p)^2 - (k^2 - m^2) - (m^2 + p^2)}{(k^2 - m^2)(k + p)^2} \\ &= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)} - \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k + p)^2} \\ &\quad - \frac{1}{2}(p^2 + m^2) \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k + p)^2} \\ &= \frac{1}{2} (A_0(m) - A_0(0) - (p^2 + m^2)B_0(p^2, m, m)). \end{aligned} \quad (2.2.14)$$

$A_0(0)$  can be shown to vanish identically in dimensional regularisation (see Sec. 3.1);  $A_0(m)$  and  $B_0(p^2, m)$  are the standard 1-loop scalar integrals from Eqs. 2.2.6 and 2.2.7, also known

as Passarino-Veltmann functions. Higher rank tensor integrals can be similarly decomposed:

$$I^{\mu\nu}(p, m) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - m^2)(k + p)^2} = g^{\mu\nu} B_{00}(p^2, m) + p^\mu p^\nu B_{11}(p^2, m). \quad (2.2.15)$$

Note the appearance of  $g^{\mu\nu}$  as a rank-2 tensor. The  $B_{00}, B_{11}$  functions in the above equations can be similarly written in terms of  $A_0(m)$  and  $B_0(p^2, m)$ . These, along with the 3 and 4 propagator Passarino-Veltmann functions  $C_0$  and  $D_0$ , form a basis of integrals at 1-loop. The basis integrals can then be evaluated either analytically using a multitude of methods (see Sec. 4.1), or numerically using several libraries e.g. `LoopTools` [220], `QCDLoop` [221], `OneLOop` [222], `Golem` [223], `Collier` [224], `Package-X` [225]. Passarino-Veltmann reduction has been used to calculate a number of 1-loop processes including the 1-loop amplitude for  $gg \rightarrow ZZ$  in [150, 153, 154].

The conventional Passarino-Veltmann reduction, however, has certain limitations:

1. In Eq. 2.2.14, for instance,  $B_1$  cannot be solved for the case where  $p^2 = 0$ . In particular, the reduction procedure for integrals with  $\geq 2$  propagators involves inverting the so-called "Gram matrices" and these Gram matrices become singular in certain regions of phase-space. This can lead to severe numerical instabilities and makes it challenging to write efficient numerical algorithms.
2. Applying the reduction procedure to higher tensor rank integrals is a rather cumbersome procedure and leads to an explosion in the number of terms in intermediate stages. This results in severe algebraic complexity in the case of analytic calculation and loss of precision in the case of numerical calculation.
3. For processes with large number of diagrams and/or external legs the integral-by-integral reduction procedure is extremely tedious.

Several new approaches have been developed and refined in the past few decades to resolve the above issues; the reader is referred to [148, 226] for an overview of the modern

1-loop amplitude methods. By far the most popular method for 1-loop computations is the OPP method proposed by Ossola-Papadopoulos-Pittau in [143], closely related to the ideas of generalised unitarity cuts [227, 142]. Further progress regarding efficient numerical implementations and automation [228, 229, 144, 230, 231] lead to the so-called "NLO revolution" as a result of complete automation of 1-loop amplitudes. There are a number of public libraries for integral reduction based on OPP method: `CutTools` [232], `Samurai` [233], `Ninja` [234]. The computation of the amplitudes and NLO corrections has been automated in many publicly available codes such as `BlackHat` [235], `HELAC-NLO` [236], `MadGraph` [237], `GoSam2.0` [238], and `OpenLoops2` [239].

Traditional Passarino-Veltmann reduction relies on step-by-step procedure to reduce high rank tensor integrals to find the coefficients of the master integrals. Using the OPP method, the coefficients can be calculated directly as follows. Consider the integrand of a general 4-point Feynman diagram

$$\mathcal{I}(\bar{k}) = \frac{N(k)}{\bar{D}_1 \bar{D}_2 \bar{D}_3 \bar{D}_4} \quad (2.2.16)$$

where  $\bar{D}_i$  are the propagators in  $d$ -dimensions  $\bar{D}_i = (\bar{k} + q_i)^2 - m_i^2$  with  $\bar{k}$  being the  $d$ -dimensional loop momentum,  $k$  the 4-dimensional loop momentum, and  $q_i, m_i$  being the momentum flowing through and mass of the edge respectively. Note that the numerator  $N(k)$  is strictly 4-dimensional; this creates a mismatch between the numerator and the denominator that leads to the rational terms. For now, it suffices to consider everything in  $d = 4$ ; the rational terms are computed later. The result of Eq. 2.2.5 can be extended to

integrands as shown in [240] with the numerator decomposed as

$$\begin{aligned}
N(k) &= (d(1234) + \tilde{d}(1234; k)) \\
&+ (c(123) + \tilde{c}(123; k)) D_4 + (c(124) + \tilde{c}(124; k)) D_3 + \dots \\
&+ (b(12) + \tilde{b}(12; k)) D_3 D_4 + (b(13) + \tilde{b}(13; k)) D_2 D_4 + \dots \\
&+ (a(1) + a(1; k)) D_2 D_3 D_4 + (a(2) + a(2; k)) D_1 D_3 D_4 + \dots \\
&+ \tilde{P}(k) D_1 D_2 D_3 D_4.
\end{aligned} \tag{2.2.17}$$

The coefficients  $d(1234), c(123)$ , etc. are independent of the loop-momentum and drop out of the integration. These are the true coefficients same as  $c_{i,j}$  in Eq. 2.2.5. The other loop-momentum dependent coefficients  $\tilde{d}(1234; k)$ , etc. are "spurious" and vanish after integration. In principle the problem now reduces to calculating the above equation for sufficient values of  $k$  and inverting the system to find the coefficients. This approach is, however, very inefficient since it requires inverting large systems and may lead to numerical instabilities. Instead, the values of loop-momentum can be chosen such that some of the propagators go on-shell e.g.

$$D_1 = D_2 = D_3 = D_4 = 0. \tag{2.2.18}$$

Denoting the solution to the above constraints by  $k = k_0^\pm$  and substituting into Eq. 2.2.17 results in

$$d(1234) + \tilde{d}(1234; k_0^\pm) = N(k_0^\pm). \tag{2.2.19}$$

It turns out that  $\tilde{d}(1234; k_0^+) = -\tilde{d}(1234; k_0^-) = \tilde{d}(1234) T(k_0^+)$ , where the general tensor

$T(k)$  is known. This allows the computation of the coefficients:

$$\begin{aligned} d(1234) &= \frac{1}{2} (N(k_0^+) + N(k_0^-)) \\ \tilde{d}(1234) &= \frac{1}{2T(k_0^+)} (N(k_0^+) - N(k_0^-)) \end{aligned} \quad (2.2.20)$$

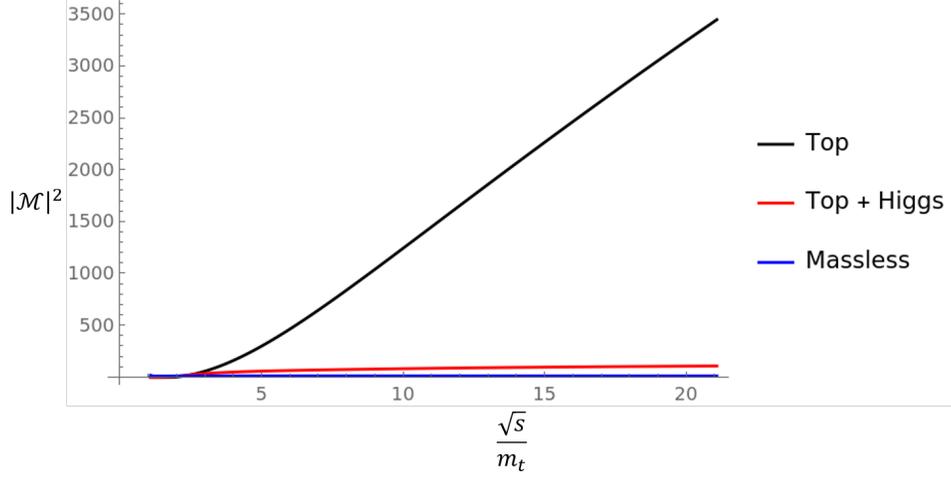
Once the 4-point coefficients are known, the lower point coefficients can be calculated by setting fewer propagators on shell. Effectively, the on-shell conditions convert the linear system into a block-triangular form that can be solved much more easily. Details of the above procedure as well as derivation can be found in [143].

The OPP and generalised unitarity methods resolve the issues with singular Gram determinants as well as explosion of terms. A huge advantage is that the procedure can be applied to sum of Feynman diagrams instead of each individual Feynman integral. The method is not generally applicable beyond 1-loop, however. The decomposition in Eqs. 2.2.14 or 2.2.17 works because the resultant numerator can be written as a linear combination of the propagator factors in the denominator [240]. This is not possible in general beyond 1-loop; such numerators are called "Irreducible Scalar Products". A much more general approach called "Integration-By-Parts" (IBP) reduction [241] is most commonly used for processes beyond 1-loop. Details of IBP reduction and its application to this work are provided in Sec. 3.

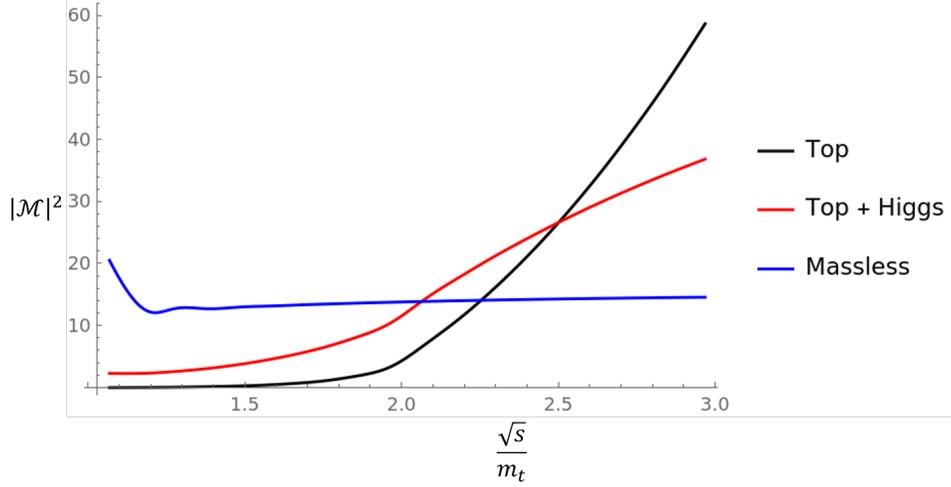
### 2.2.3 Results

The 1-loop amplitudes after calculation are compared to published results in [12]; agreement is found for each form factor and helicity amplitude. Below are some results for the 1-loop amplitude.

In Fig. 2.2.3a, the amplitude-squared  $|\mathcal{M}|^2$  is compared for 3-different contributions: Only the top-quark contribution, top-quark contribution with Higgs exchange diagrams included, and the contribution of a massless up-type quark. It is clear that the top-quark contribution (in black) grows with energy. After including the Higgs mediated diagrams, however, the



(a)



(b)

Figure 2.2.3: Comparison of  $|\mathcal{M}|^2$  for three different type of contributions. Without including Higgs, the top-quark contribution increases with energy.

growth is arrested and the amplitude-squared plateaus; this is the unitarising effect of Higgs. Another important observation is that even after including the Higgs mediated diagrams, the contribution from top-quark dominates over the massless contribution. This further underlines the importance of a full 2-loop computation for this process. Fig. 2.2.3b shows the same data but zoomed in to the region of smaller invariant mass. While the massless contribution is fairly uniform and dominates in the low-energy region, massive (top-quark) contribution starts to increase past  $\sqrt{s} = 2m_t$  when the top quarks in the loop can be

produced on-shell.

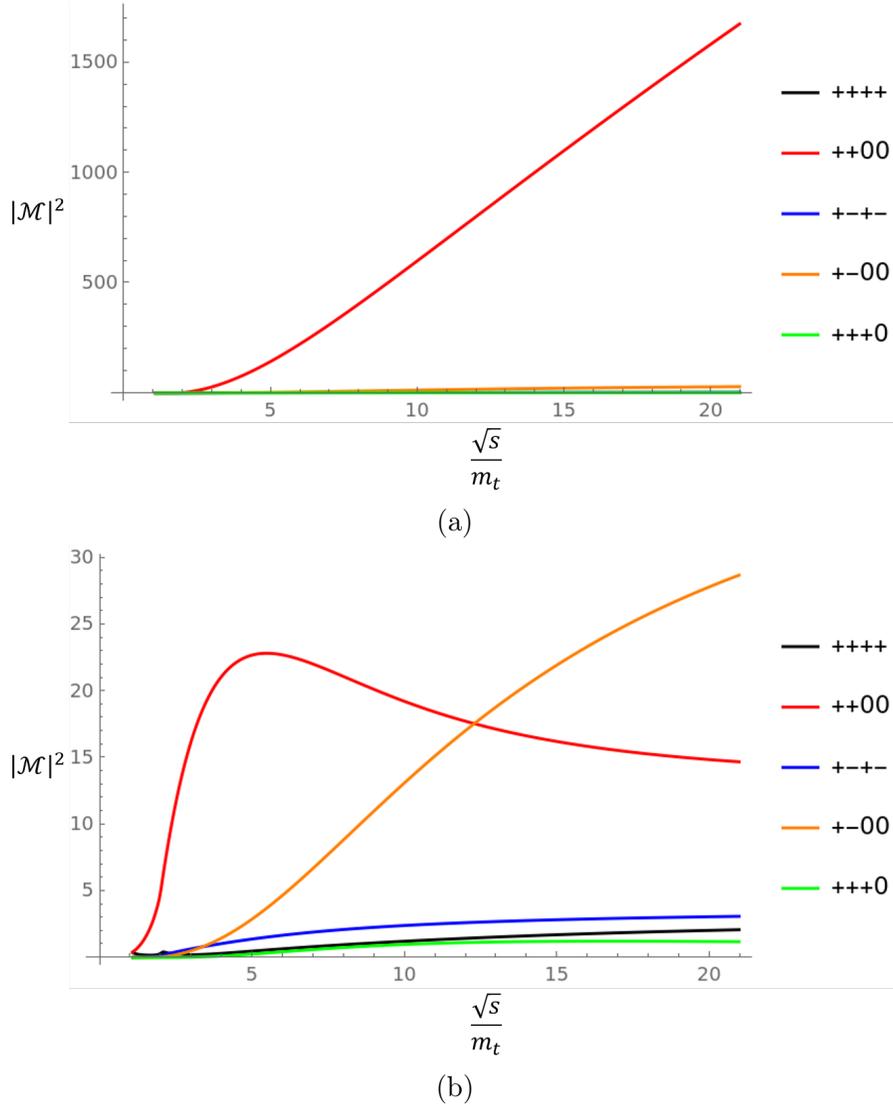
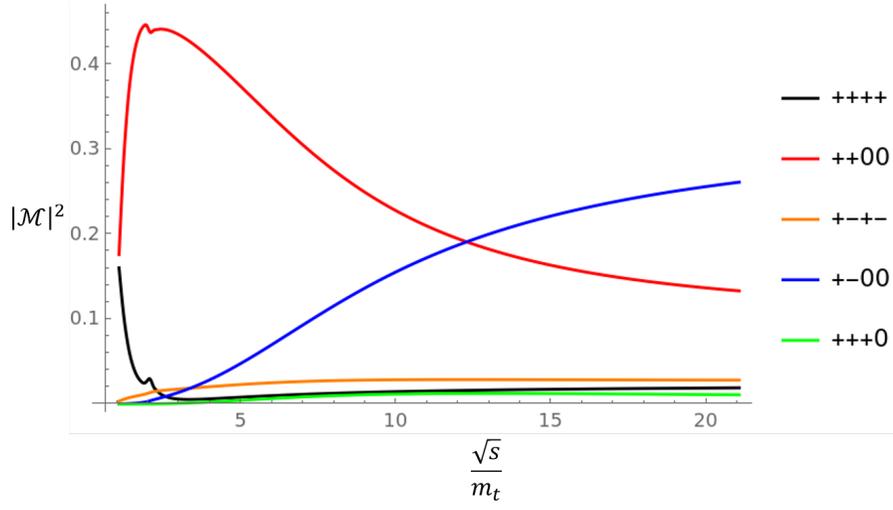


Figure 2.2.4: Comparison of  $|\mathcal{M}|^2$  for different helicities for the top quark contribution including the Higgs diagrams.

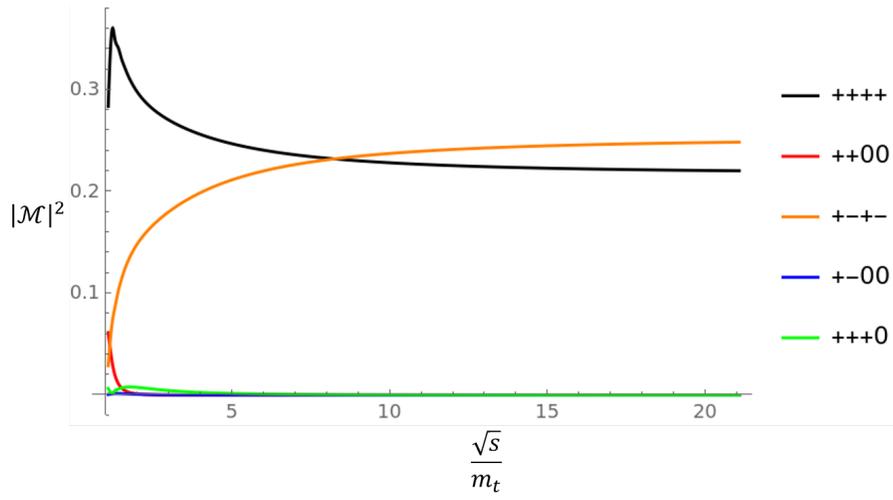
This unitarising behaviour and the threshold effect for  $\sqrt{s} = 2m_t$  is even more apparent in the individual helicity amplitudes as shown in Fig. 2.2.4. Fig 2.2.4a shows the individual helicity amplitudes without the Higgs mediated diagrams included. In this case the "++00" mode is dominant and increases with energy, much like Fig. 2.2.3a. This is easily understood through the Equivalence theorem where the "00" mode corresponds to scattering of two scalars with the coupling proportional to top mass. Including the Higgs diagrams

(Fig. 2.2.4b) it is clear that the  $++00$  mode does not increase indefinitely with energy. In fact, at very high energies, the  $+ - 00$  mode takes over and becomes the dominant mode. Another interesting feature of the above plot is the stark threshold effect observed in the  $++00$  mode; this is absent in the  $+ - 00$  mode as a consequence of a mismatch in the helicities, and consequently the spin-states of the top quarks being produced on-shell.

Fig. 2.2.5 shows the ratio of individual helicities to the total  $|\mathcal{M}|^2$  for both the top (Fig. 2.2.5a) and massless (Fig. 2.2.5b) contributions. For the massive case, the longitudinal mode  $00$  has the largest contribution, unsurprisingly with the  $+ - 00$  helicity being the dominant mode for high energies. For the massless case, however, the longitudinal modes are almost negligible and the  $++++$  and  $+ - + -$  helicities contribute the most.



(a)



(b)

Figure 2.2.5: Ratio of contribution of individual helicities to the total  $|\mathcal{M}|^2$  for different helicities for (a) top quark contribution including the Higgs diagrams, and (b) Massless contribution.

## 2.3 2-loop amplitude

### 2.3.1 Generation of diagrams

At 2-loops, contributions to this process involving  $s$ -channel Higgs production are already known [167, 168, 169, 170] and those for the massless quarks were computed in [159, 160]. As such, only the top-quark contribution is considered here. The diagrams for the 2-loop amplitude are generated using `Qgraf` [206]. A total of 166 diagrams are generated that contain at least 1 top-quark propagator. A total of 49 diagrams vanish from colour since they have a single gluon coupled to a closed quark loop. The remaining diagrams can broadly be split into two different "classes":

**Class A:** In diagrams of class A, both  $Z$ -bosons are attached to the same closed fermion loop. As described previously, the anti-commuting, Kreimer's  $\gamma^5$  scheme is used to handle  $\gamma^5$  in  $d$ -dimensions. In general, a reading point prescription is used to ensure that all traces are read from the same point since cyclicity is not preserved. For a closed fermion loop, however, in case of an even number of  $\gamma^5$ 's, it is rather straightforward to eliminate the  $\gamma^5$  using the anti-commutation identities. This allows for a trivial implementation of the  $\gamma^5$  scheme for this class of diagrams.

**Class B:** In this class of diagrams, the  $Z$ -bosons couple to different closed fermion loops. The vector-vector ( $v_t^2$ ) contribution for these diagrams can be shown to vanish due to Furry's theorem. The vector-axial ( $a_t v_t$ ) piece vanishes identically, as discussed previously, as a result of charge-parity conservation for this process. However, the axial-axial ( $a_t^2$ ) piece vanishes only for a sum over a degenerate  $SU(2)_L$  doublet. Since the third generation quarks ( $t, b$ ) are not degenerate, a finite term remains from the top-bottom mass splitting. For these diagrams, there is a single  $\gamma^5$  in each fermion loop which is non-trivial to handle and requires a careful application of the anti-commuting  $\gamma^5$  scheme. It must be pointed out that these diagrams are effectively 1-loop\*1-loop type diagrams and as such are a lot easier to reduce

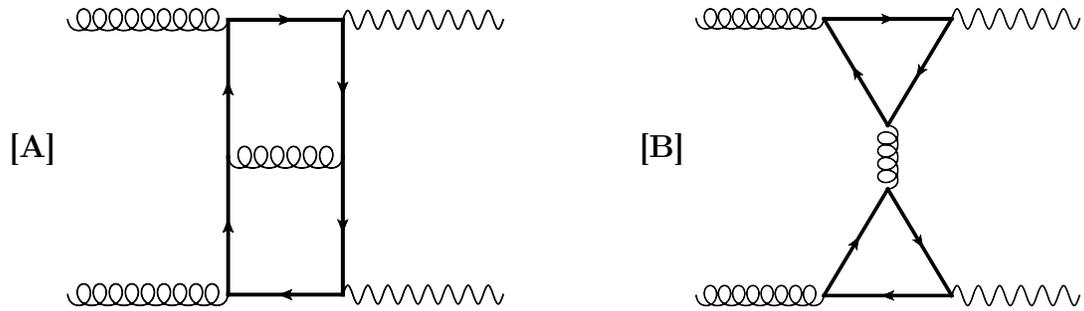


Figure 2.3.1: Example Feynman diagrams representing the two classes of diagrams at 2-loops

to master integrals than **Class A**.

### 2.3.2 Class A

After generating the diagrams in class A, the Feynman rules are inserted for which custom FORM scripts are employed. The resulting algebraic expressions are required to be interpreted as linear combinations of Feynman integrals. For this, **Reduze 2** [242, 243, 244, 245] is employed to map them to the 4 different integral families shown in Tab. 2.3.1.

An "integral family" is essentially a set of Feynman propagators that can be used to classify the integrals appearing in one or more topologies. The concept of an "integral family" is discussed in a bit more detail in Sec. 3. For the purpose of this section it suffices to say that an integral family is a set of Feynman propagators grouped together such that all possible scalar products between the momenta in the integral family (both loop momenta and external momenta) are uniquely determined in terms of the inverse propagators. E.g. considering an integral family with  $L$  loop momenta and  $E$  independent external momenta, the number of distinct scalar products (dependent on loop momentum) is  $N = L(L + 1)/2 + EL$ . This is the number of propagators that an integral family must contain. For a 4-point topology at 1-loop, this number is 4; at 2-loops, it is 9. Note that for a 4-point topology at 2-loops, only 7 propagators can be present assuming all the vertices connect 3 edges. The 2 "extra" propagators, even though they do not form a part of the topology are required to specify all the scalar products.

A	B	C	D
$k_1^2 - m_t^2$	$k_1^2$	$k_1^2 - m_t^2$	$k_1^2$
$(k_1 + p_1)^2 - m_t^2$	$(k_1 + p_1)^2$	$(k_1 + p_1)^2 - m_t^2$	$(k_1 + p_1)^2$
$(k_1 + p_1 + p_2)^2 - m_t^2$	$(k_1 + p_1 + p_2)^2$	$(k_1 + p_1 - p_3)^2 - m_t^2$	$(k_1 + p_1 + p_2)^2$
$(k_1 + p_4)^2 - m_t^2$	$(k_1 + p_4)^2$	$(k_1 + p_4)^2 - m_t^2$	$k_2^2 - m_t^2$
$k_2^2 - m_t^2$	$k_2^2 - m_t^2$	$k_2^2 - m_t^2$	$(k_2 + p_1 + p_2)^2 - m_t^2$
$(k_2 + p_1)^2 - m_t^2$	$(k_2 + p_1 - p_3)^2 - m_t^2$	$(k_2 + p_1 - p_3)^2 - m_t^2$	$(k_2 + p_4)^2 - m_t^2$
$(k_2 + p_1 + p_2)^2 - m_t^2$	$(k_2 - p_3)^2 - m_t^2$	$(k_2 + p_4)^2 - m_t^2$	$(k_2 - k_1)^2 - m_t^2$
$(k_2 + p_4)^2 - m_t^2$	$(k_2 + p_4)^2 - m_t^2$	$(k_2 - k_1)^2$	$(k_2 - k_1 + p_2)^2 - m_t^2$
$(k_1 - k_2)^2$	$(k_1 + k_2 + p_4)^2 - m_t^2$	$(k_1 - k_2 + p_1)^2$	$(k_2 - k_1 + p_4)^2 - m_t^2$

Table 2.3.1: List of integral families and their propagators for the 2-loop amplitude

For simplification of the expressions, Feynman gauge is used ( $\xi = 1$ ) for internal gluons. After mapping the diagrams to the above integral families, a total of 29247 integrals are found in the amplitude. In the worst case, the integrals have up to 4 inverse propagators in the numerator. The algebraic expressions at this stage of the calculation are quite large, occupying  $\sim 50$  GB of disk space.

Usually, topologies that have many common denominators are combined together in one integral family. Even so, there can still be common propagators between topologies from different topologies. This, however, leads to redundancy in the definition of integrals and certain symmetry relations can be written down that relate integrals from different integral families. Such relations aren't limited to different families, though, and similar relations between different "sectors" of the same integral family exist as well as within a sector. Using such relations generated using `Reduze 2`, the total number of integrals in the diagrams can be reduced to 4504. Adding diagrams together and considering only form factors, this number drops to 1584 due to further cancellations and the total size of the expressions drastically reduces to  $< 1$  GB. This really underlines the importance of working with objects like form factors instead of individual diagrams as well as the use of symmetry relations; the resulting algebraic expressions are a lot more manageable and the number of Feynman integrals appearing in the expressions reduces drastically.

Unlike the case of 1-loop, at 2-loops the master integrals are not known a priori. Addi-

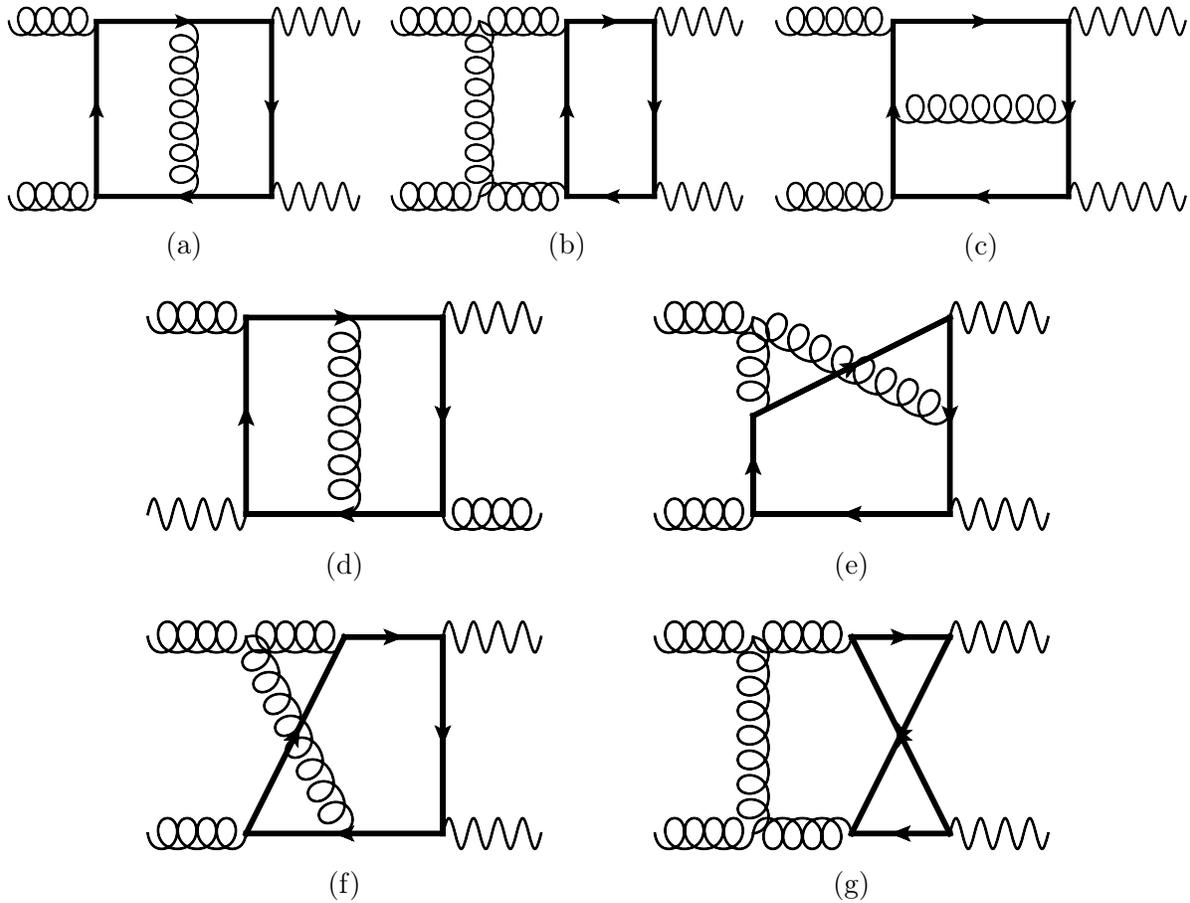


Figure 2.3.2: Representative Feynman diagrams in class A with irreducible topologies. The number of master integrals in each topology are 3, 4, 3, 3, 5, 5, and 4 respectively

tionally, the methods traditionally applied at 1-loop e.g. Passarino-Veltmann reduction or OPP method are not adequate beyond 1-loop. Conventional approach to reduce integrals beyond 1-loop is Integration-By-Parts reduction [241]. `Reduze 2` is used to perform a numerical reduction (i.e. with numbers substituted for kinematic variables) to find the list of masters. A total of 264 master integrals are found out of which 172 are not related via any crossing or exchange of external legs. In terms of topologies, there are 85 irreducible ones i.e. the topologies containing master integrals and which cannot be completely expressed in terms of integrals from subtopologies, with the worst topology having 6 propagators and 6 master integrals. Of the 13 top-level topologies i.e. the topologies with 7 propagators, 7 are

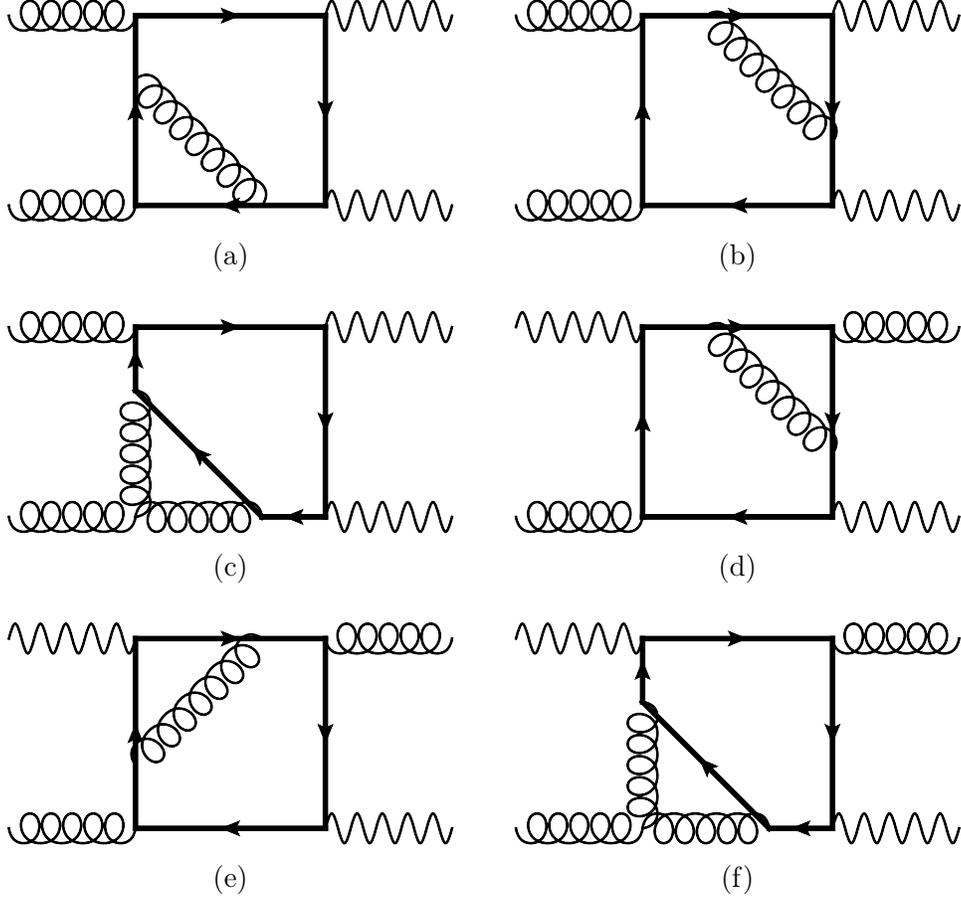


Figure 2.3.3: Representative Feynman diagrams in class A with reducible topologies.

irreducible (Fig. 2.3.2) and the other 6 are reducible (Fig. 2.3.3).

For diagrams of **Class A** two different colour structures appear given by the two quadratic Casimir invariants  $C_F$  and  $C_A$  i.e. the colour structure of a diagram can be written as

$$\mathcal{C} = (a_1 C_F + a_2 C_A) \delta_{ab}. \quad (2.3.1)$$

Interestingly, it is observed that the diagrams contributing to the  $C_F$  piece are purely planar while a mix of planar and non-planar diagrams contributes to the  $C_A$  piece. Since the planar diagrams are expected to be simpler to reduce than the non-planar diagrams, the  $C_F$  piece is expected to be simpler.

Full IBP reduction for this class of diagrams is extremely challenging and the conventional

approach proved inadequate. For instance, `Reduze 2` was used to perform the reduction for this process for the case of massless internal quarks in [159]. However, it was only able to solve reductions for the massive case only for integrals with 1 inverse propagator while the full problem requires the solution for integrals with up to 4 inverse propagators. The new developments required to solve this problem are discussed in Chapter 3.

### 2.3.3 Class B

Diagrams in **Class B** have a single  $\gamma^5$  in each loop which leads to a non-trivial structure and requires careful application of the reading point prescription. Details of the  $\gamma^5$  scheme and the reading point prescription are given in Appendix C.

Since these diagrams are effectively 1-loop\*1-loop, they are significantly simpler and hence treated separately from **Class A**. As described previously, the only surviving terms are of the axial-axial type ( $a_t^2$ ). These should also be proportional to  $m_t^2$  assuming all the other quark flavours are massless. The colour factor for this class of diagrams is particularly simple since this is just a product of two 1-loop diagrams.

$$\mathcal{C} = T_F^2 \delta_{ab}. \quad (2.3.2)$$

In the context of this process, the application of the anti-commuting scheme is relatively simple since every trace must be read from the same point due to the presence of only 1  $\gamma^5$  in each loop. Passarino-Veltmann reduction is used along with the 1-loop tool `Package-X` [225] for manipulation of tensor objects to reduce the amplitudes to master integrals. Exact results for these diagrams were previously presented in [165] with which full agreement is found.

# Chapter 3

## IBP reduction using syzygies

### 3.1 Integration-by-parts reduction

A general  $L$ -loop scalar Feynman integral with  $N$  propagators can be represented by

$$I(\nu_1, \dots, \nu_N) = \int \left( \prod_{l=1}^L \frac{d^d k_l}{i\pi^{\frac{d}{2}}} \right) \prod_{i=1}^N \frac{1}{(q_i^2 - m_i^2)^{\nu_i}} \quad (3.1.1)$$

where  $k_1, \dots, k_L$  are the loop momenta,  $p_1, \dots, p_E$  are the independent external momenta,  $q_i$  are the momenta of the propagators (linear combinations of loop and external momenta),  $m_i$  are the masses of the propagators,  $\nu_i \leq 0$  are (integer) exponents of the propagators, and  $d = 4 - 2\epsilon$  is the number of space-time dimensions. Here, non-positive powers  $\nu_i$  of the propagators are also allowed, that is, an *integral family* is considered allowing for general propagators. Note that in an integral family, all possible scalar products between the momenta (both loop and external) are fixed in terms of the kinematic invariants and inverse propagators. This requires the total number of propagators in an integral family to be equal to  $L(L+1)/2 + LE$ . Feynman integrals often contain fewer propagators than this; the remaining propagators can appear in the numerator as irreducible scalar products. E.g. the 4-point topologies at 2-loops considered in this work (Figs. 2.3.2 and 2.3.3) have a maximum of 7 distinct propagators with the remaining 2 being used to express the irreducible scalar

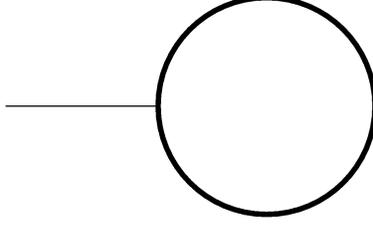


Figure 3.1.1: A tadpole graph with the thick loop corresponding to the massive propagator with mass  $m$ .

products.

The total derivative of an integral in dimensional regularisation vanishes. This allows the following linear relations between different integrals [241]

$$0 = \int \left( \prod_{l=1}^L \frac{d^d k_l}{i\pi^{d/2}} \right) \frac{\partial}{\partial k_j^\mu} \left( v^\mu \prod_{i=1}^N \frac{1}{(q_i^2 - m_i^2)^{\nu_i}} \right), \quad (3.1.2)$$

where  $v_\mu$  could be any linear combination of loop and external momenta. The surface term from the integration of the above integrand can be shown to vanish for small values of  $d$ . Extending the formula to general  $d$  using analytic continuation leads to the relations known as "Integration-By-Parts" identities.

It is instructive to look at an example. Consider a Feynman integral with just one propagator in  $d$  dimensions e.g. in Fig. 3.1.1

$$I = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 - m^2)} \quad (3.1.3)$$

often referred to as a "tadpole" integral. Taking the derivative of the integrand with respect to the loop momentum (and dropping the  $i\pi^{d/2}$  factor for brevity), as in Eq. 3.1.2, yields

$$\int d^d k \frac{\partial}{\partial k_\mu} \frac{k_\mu}{(k^2 - m^2)} = \int d^d k \left[ \left( \frac{\partial k_\mu}{\partial k_\mu} \right) \frac{1}{(k^2 - m^2)} + k_\mu \frac{\partial}{\partial k_\mu} \frac{1}{(k^2 - m^2)} \right] \quad (3.1.4)$$

$$= (d - 2) \int d^d k \frac{1}{(k^2 - m^2)} - 2m^2 \int d^d k \frac{1}{(k^2 - m^2)^2}. \quad (3.1.5)$$

The left hand side of the above equation, which is a total derivative, can be shown to be vanish for small  $d$ . This leads to the following linear relation between integrals

$$(d - 2) \int d^d k \frac{1}{(k^2 - m^2)} = 2m^2 \int d^d k \frac{1}{(k^2 - m^2)^2}. \quad (3.1.6)$$

This relation can be generalised for an arbitrary exponent of the propagator. Denoting with  $I(\nu)$  the integral with a general exponent  $\nu$  for the propagator,

$$I(\nu) = \int d^d k \frac{1}{(k^2 - m^2)^\nu}, \quad (3.1.7)$$

the general relation is

$$(d - 2\nu) I(\nu) = 2m^2 \nu I(\nu + 1). \quad (3.1.8)$$

The above relation implies that in this tower of integrals  $I(\nu)$ , only one is independent and the rest are simply related to it. It must be pointed out that any of these integrals can be chosen to be the independent integral or, as is referred to more commonly, the "master" integral. Some choices might offer certain advantages over others e.g. finiteness in  $d = 4$  dimensions. We will explore this in more detail in chapter (ref chapter finite integral). It is interesting to note that setting  $m = 0$  in Eq. 3.1.8 gives zero i.e.  $\forall \nu \neq d/2, I(\nu) = 0$ . In other words, massless tadpole integrals vanish in dimensional regularisation. This is a part of the more general statement: scaleless integrals i.e. integrals with no dependence on external kinematic variables vanish in dimensional regularisation.

In the above example, it was possible to derive an analytical solution for the IBP relation; this is in general not possible in practice. Consider a slightly more complicated example, the 2-point integral with one massive and one massless propagator (Fig. 3.1.2). An integral

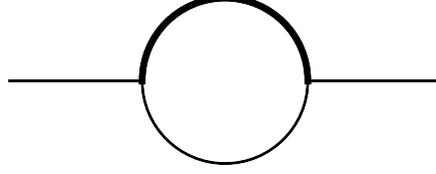


Figure 3.1.2: A two-point function with one massive propagator. The thick line corresponds to the massive propagator.

with general exponents for this topology can be written as

$$I(\nu_1, \nu_2) = \int d^d k \frac{1}{(k^2 - m^2)^{\nu_1} ((k+p)^2)^{\nu_2}} \quad (3.1.9)$$

where  $p$  is the external momentum entering the loop. Taking derivative with respect to the loop-momentum vector  $k^\mu$  and choosing  $v^\mu = k^\mu, p^\mu$ ,

$$\begin{aligned} 0 = \int d^d k \frac{\partial}{\partial k_\mu} \frac{k_\mu}{(k^2 - m^2)^{\nu_1} ((k+p)^2)^{\nu_2}} &= (d - 2\nu_1 - \nu_2) I(\nu_1, \nu_2) - 2m^2 \nu_1 I(\nu_1 + 1, \nu_2) \\ &\quad - \nu_2 I(\nu_1 - 1, \nu_2 + 1) + \nu_2 (p^2 - m^2) I(\nu_1, \nu_2 + 1), \end{aligned} \quad (3.1.10)$$

$$\begin{aligned} 0 = \int d^d k \frac{\partial}{\partial k_\mu} \frac{p_\mu}{(k^2 - m^2)^{\nu_1} ((k+p)^2)^{\nu_2}} &= (\nu_1 - \nu_2) I(\nu_1, \nu_2) - \nu_1 I(\nu_1 + 1, \nu_2 - 1) \\ &\quad + \nu_1 (p^2 + m^2) I(\nu_1 + 1, \nu_2) + \nu_2 I(\nu_1 - 1, \nu_2 + 1) \\ &\quad - \nu_2 (p^2 - m^2) I(\nu_1, \nu_2 + 1). \end{aligned} \quad (3.1.11)$$

It is clear that a trivial analytical solution akin to Eq. 3.1.8 is not possible in this situation.

To make the equations a bit simpler, it helps to start with  $\nu_2 = 0$  which gives

$$0 = (d - 2\nu_1) I(\nu_1, 0) - 2m^2 \nu_1 I(\nu_1 + 1, 0), \quad (3.1.12)$$

$$0 = \nu_1 I(\nu_1, 0) - \nu_1 I(\nu_1 + 1, -1) + \nu_1 (p^2 + m^2) I(\nu_1 + 1, 0). \quad (3.1.13)$$

In the above equations, Eq. 3.1.12 is just the solution for the tadpole integrals (Eq. 3.1.8),

while Eq. 3.1.13 provides solutions for integrals with negative exponents for the second propagator i.e. for the propagator appearing in the numerator. E.g. choosing  $I(1, 0)$  as the master integral and setting  $\nu_1 = 1$  results in

$$I(2, 0) = \frac{d-2}{2m^2} I(1, 0), \quad (3.1.14)$$

$$I(2, -1) = \frac{d(p^2 + m^2) - 2p^2}{2m^2} I(1, 0). \quad (3.1.15)$$

The above equations provide solutions for  $I(2, 0)$  and  $I(2, -1)$ . Repeated application of Eqs. 3.1.12 and 3.1.13 will allow the solution of integrals  $I(\nu, -1) \forall \nu \geq 1$ .

Setting, instead,  $\nu_1 = 0$ ,

$$0 = (d - \nu_2) I(0, \nu_2) - \nu_2 I(-1, \nu_2 + 1) + \nu_2(p^2 - m^2) I(0, \nu_2 + 1), \quad (3.1.16)$$

$$0 = -\nu_2 I(0, \nu_2) + \nu_2 I(-1, \nu_2 + 1) - \nu_2(p^2 - m^2) I(0, \nu_2 + 1). \quad (3.1.17)$$

Since scaleless integrals vanish in dimensional regularisation,  $I(0, \nu_2) = 0 \forall \nu_2 \geq 0$ . The above equations then imply

$$I(-1, \nu_2) = 0. \quad (3.1.18)$$

Now the case where  $\nu_1, \nu_2 \neq 0$  can be considered. Setting  $\nu_1 = \nu_2 = 1$  results in the following equations:

$$0 = (d - 3) I(1, 1) - 2m^2 I(2, 1) - I(0, 2) + (p^2 - m^2) I(1, 2), \quad (3.1.19)$$

$$0 = -I(2, 0) + (p^2 + m^2) I(2, 1) + I(0, 2) - (p^2 - m^2) I(1, 2). \quad (3.1.20)$$

Here  $I(0, 2) = 0$  (since it is scaleless) and  $I(2, 0)$  is given in Eq. 3.1.14. This leaves  $I(1, 1)$ ,  $I(2, 1)$ , and  $I(1, 2)$  undetermined. With 3 remaining integrals and only 2 equations, it appears that the above 3 integrals cannot all be expressed in terms of the tadpole integral.

Indeed, another master integral is required with both  $\nu_1, \nu_2 \geq 1$ . Choosing  $I(1, 1)$  as the master integral in addition to  $I(1, 0)$ , the identities can be written as

$$0 = (d - 3) I(1, 1) - 2m^2 I(2, 1) + (p^2 - m^2) I(1, 2), \quad (3.1.21)$$

$$0 = -I(2, 0) + (p^2 + m^2) I(2, 1) - (p^2 - m^2) I(1, 2). \quad (3.1.22)$$

Solving the above equations gives  $I(2, 1)$  and  $I(1, 2)$  in terms of the chosen master integrals  $I(1, 0)$  and  $I(1, 1)$ .

To summarise, the reduction procedure is started by choosing the lowest value of  $\nu = \nu_1 + \nu_2 + \dots = 1$ . For  $\nu_1 = 1, \nu_2 = 0$ , the system gets reduced to that of a massive tadpole while for  $\nu_1 = 0, \nu_2 = 1$  the equations lead to scaleless integrals that vanish. In the next step, the value of  $\nu$  is increased to 2. Starting with  $\nu_1 = \nu_2 = 1$ , the solutions require another master integral. However, it can be shown that for higher values of  $\nu$ , all the other integrals can be expressed in terms of these 2 master integrals. This systematic way of generating and solving linear relations for integer exponents  $\nu_i$  starting from lowest values of  $\nu = \nu_1 + \nu_2 + \dots$  was first presented by Laporta [246]. Many public codes based on this algorithm are available for this purpose [247, 248, 243, 249, 250].

**Reduze 2**, based on Laporta's algorithm was used to perform the reductions for  $gg \rightarrow ZZ$  at 2-loops for the case of massless internal quarks in [159]. It was found that it was insufficient for the process at hand, however. In fact, even for the planar topologies which are expected to be significantly simpler than the non-planar topologies, the reductions for integrals with up to one inverse propagator, denoted by  $s \leq 1$ , were incredibly difficult and required several months of CPU time. The integrals with  $s = 2$  were totally out of reach with conventional methods.

In particular, conventional IBP solvers have the following major issue.

1. Perhaps the most significant drawback is the presence of integrals with doubled propagators, also referred to as dots, in IBP identities. As is evident from Eqs. 3.1.21

and 3.1.22, IBP identities generated this way necessarily contain these doubled propagators for the conventional choice of vectors  $v^\mu = (k_1^\mu, \dots, p_1^\mu, \dots)$ . These integrals don't often appear in the amplitude and as such their reductions aren't required.

2. While it is possible to avoid generating solutions for such integrals with doubled propagators, reduction procedure for just the integrals required for the amplitude still proceeds through row-reduction of linear systems containing these integrals. This means that a much larger linear system than is optimal is being solved, further slowing down the process.

A solution was first provided to the above issues in [251] to avoid generating such integrals with doubled propagators with specific choices of the generating vectors  $v_\mu$  using a Gröbner basis approach. An alternate approach using linear algebra was instead proposed in [252], which forms the basis of the approach used in this work. Before discussing these solutions in detail, however, it is useful to introduce an alternate representation for the Feynman integrals.

## 3.2 Baikov representation

Consider the general  $L$ -loop integral family with  $E$  independent external momenta and  $N$  edges (propagators) from Eq. 3.1.1, with only  $T \leq N$  propagators appearing in the integral i.e.  $\nu_i > 0$  for  $i \leq T$ . Let  $Q$  denote the set of all loop momenta and independent external momenta i.e.  $Q = (k_1, \dots, k_L, p_1, \dots, p_E)$ . This integral has a total of  $Ld$  integration variables. Since the integral has no open Lorentz indices, however, it is easy to see that it depends only upon the scalar products between the loop momenta and the external momenta appearing in the propagators,  $Q_i \cdot Q_j$ , as well as masses and  $d$ . Maximum number of such scalar products is  $L(L+1)/2 + LE$  for  $i \leq j$  and  $i \leq L, j \leq L+E$ . For an integral with  $T$  distinct propagators appearing in the integral, the integral only depends on  $T$  scalar products, implying that only  $T \leq L(L+1)/2 + LE$  degrees of freedom are required to characterise the integral. The rest

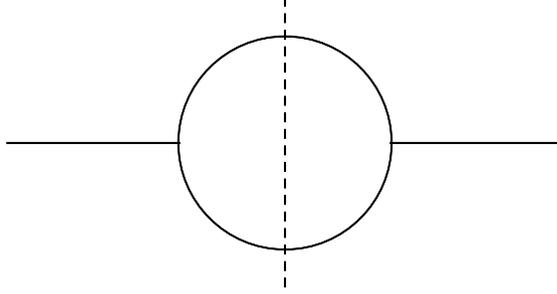


Figure 3.2.1: A two-point function with massless propagators. The dashed line corresponds to the cut propagators.

of the  $Ld - T$  variables can simply be integrated over.

This can be looked at in a slightly different way by considering the loop momentum  $k_1$ , for example, and dividing it's components into  $k_{1,\perp}$  and  $k_{1,\parallel}$  where  $k_{1,\parallel}$  is the projection of  $k_1$  onto the space spanned by  $(k_2, \dots, k_L, p_1, \dots, p_E)$  and  $k_{1,\perp}$  is orthogonal to it. This allows performing the integrations over  $k_{1,\perp}$  independently of the actual Feynman propagators.

The general  $L$ -loop  $N$  edge Feynman integral in Eq. 3.1.1 can be written in Baikov's representation as

$$I(\nu_1, \dots, \nu_N) = \mathcal{N}_0 \int dz_1 \dots dz_N \frac{1}{\prod_{i=1}^N z_i^{\nu_i}} P^{\frac{d-L-E-1}{2}}, \quad (3.2.1)$$

Here  $\mathcal{N}_0$  is a normalisation factor depending on the spacetime dimensions  $d$  and kinematics,  $P$  is the Jacobian of the transformation known as the Baikov polynomial, and  $z_1, \dots, z_N$  are the integration variables called the Baikov parameters which are just the inverse propagators  $z_i = q_i^2 - m_i^2$ . The Baikov polynomial  $P$  in general depends on the Baikov parameters and the kinematic invariants. This representation was first proposed in [253] and later clarified in [254]; the reader is referred to the above references for further details.

It was shown in [255, 256] that it is in general highly non-trivial to write down the integration domain explicitly. While it is rather challenging to perform the actual integration in this representation, a huge advantage the Baikov representation allows over the traditional momentum-space representation is in performing unitarity cuts [257]. Cutting propagators

in this representation is reduced to the simple task of taking the residue of the integrand at 0 for the corresponding Baikov parameter [258, 259].

Evaluating Feynman integrals on cuts is useful for many purposes. The discontinuity of a Feynman integral across a branch cut is given by the cut integral

$$\text{Disc}_{s_i}(I) = -\text{Cut}_{s_i}I \quad (3.2.2)$$

where  $s_i$  corresponds to some Mandelstam invariant. This relation was derived in [260, 261] and is a consequence of the optical theorem [262]:

$$\mathcal{M}(i \rightarrow f) - \mathcal{M}^*(f \rightarrow i) = i \sum_x \int d\Pi_x \mathcal{M}(i \rightarrow x) \mathcal{M}^*(f \rightarrow x). \quad (3.2.3)$$

In the above equation,  $i$  refers to the initial state,  $f$  to final state, and  $x$  to all possible intermediate states.  $d\Pi_x$  is the  $n$ -particle phase space weight for the intermediate state  $x$ . This simplifies to

$$2\text{Im} \mathcal{M}(i \rightarrow i) = \sum_x \int d\Pi_x |\mathcal{M}(i \rightarrow x)|^2. \quad (3.2.4)$$

The imaginary part of the massless bubble integral in Fig. 3.2.1, for instance, can be calculated by evaluating the integral on the  $s$ -channel cut denoted via the dashed line. The integral in Fig. 3.2.1 is known to evaluate to (for  $d = 4 - 2\epsilon$ )

$$I = \frac{1}{\epsilon} + 2 - \log(-p^2 - i0) + O(\epsilon), \quad p^2 < 0 \quad (3.2.5)$$

where  $p$  is the incoming external momentum (see Appendix B.2). This integral is analytic for  $p^2 < 0$  and can be analytically continued to the region  $p^2 > 0$  using the principal branch

of the logarithm. This results in the integral picking up an imaginary part

$$I = \frac{1}{\epsilon} + 2 - \log(p^2) - i\pi + O(\epsilon), \quad p^2 > 0. \quad (3.2.6)$$

i.e.  $\text{Im } I = -i\pi$  and the discontinuity is given by  $\text{Disc}_s(I) = -2i\pi$ .

In Baikov representation, the integral can be written as

$$I = \mathcal{N}_0 \int dz_1 dz_2 \frac{(sz_1 - \frac{1}{4}(-(p^2 + i0) - z_1 + z_2)^2)^{1/2-\epsilon}}{z_1 z_2} \quad (3.2.7)$$

with the normalisation factor

$$\mathcal{N}_0 = \frac{2i\pi^{\frac{3}{2}-\epsilon}}{\Gamma(\frac{3}{2}-\epsilon)(p^2 + i0)^{1-\epsilon}} \quad (3.2.8)$$

The  $s$ -channel cut here corresponds to setting both the propagators on-shell i.e. computing the residue at  $z_1 = z_2 = 0$  [256], which results in

$$\text{Cut}_s I = \frac{4i\pi}{p^2} \frac{p^2}{2} + O(\epsilon) = 2i\pi + O(\epsilon). \quad (3.2.9)$$

It is clear that the above results satisfy the cut-discontinuity equation (Eq. 3.2.2).

In addition to specific kinematic cuts as in the above example, the integrals can also be evaluated on maximal cuts i.e. setting all propagators on-shell. This is of particular importance in the method of evaluating Feynman integrals using differential equations since the integrals evaluated on maximal cuts correspond to the homogeneous solution of the differential equations [254, 263]. Since the homogeneous solution is expressed using the same class of functions as the full solution, it can be used to determine if the differential equations admit a polylogarithmic solution. In addition, the solution on maximal cut can be used to determine the "uniformly transcendental" basis of master integrals [264, 265] which satisfy particularly simple differential equations. The reader is referred to [255, 256] for a

detailed discussion and more complicated examples of integrals evaluated on maximal cuts, including integrals involving elliptic functions. However, the representation in Eq. 3.2.1 is not the most convenient representation for performing the integration explicitly. In fact, the number of integration variables can, in many cases, be reduced by performing the variable transformation loop-by-loop [255, 256].

Similar to the momentum-space representation, the IBP relations can also be written down in the Baikov representation [254]. For the general integral in Eq. 3.2.1, the Integration-By-Parts identities in Baikov representation are given by

$$0 = \int dz_1 \dots dz_N \sum_{i=1}^N \left( \frac{\partial f_i}{\partial z_i} + \frac{d-L-E-1}{2P} f_i \frac{\partial P}{\partial z_i} - \nu_i \frac{f_i}{z_i} \right) \frac{1}{\prod_{i=1}^N z_i^{\nu_i}} P^{\frac{d-L-E-1}{2}}. \quad (3.2.10)$$

Here  $f_i$  are arbitrary polynomials in the Baikov parameters and the kinematic invariants. The freedom in choosing the polynomials  $f_i$  reflects the freedom in choosing the vectors  $\nu_\mu$  in the momentum-space representation. Certain choices of  $f_i$  can provide advantages over the others; this is discussed in the next section (Sec. 3.3).

Integration-By-Parts relations can also be generated on cuts [266, 267] using the general formula in Eq. 3.2.10 which can be useful to simplify the system of equations by reducing them into smaller and more manageable parts as well as potentially provide significant performance improvement as argued in [267]. The IBPs generated on cuts can also be used to obtain the differential equations on cuts e.g. on maximal cuts which can be used to solve for the homogeneous solution of the differential equations.

### 3.3 Syzygies for IBP reduction

In the above general equation for IBP relations via Baikov representation (Eq. 3.2.10), the simplistic choice of the polynomials  $f_i = 1$  leads to equations that have terms with  $1/P$ . It is straightforward to see that such terms generate integrals with shifted dimensions in the IBP relations. In general it is highly desirable to avoid such integrals in the IBP systems since

they do not commonly occur in the amplitudes. In addition, the presence of such integrals leads to bloated linear systems which are relatively difficult to solve. To avoid such integrals is relatively straightforward, however, by simply imposing the constraint [266, 267, 268]

$$\left( \sum_{i=1}^N f_i \frac{\partial P}{\partial z_i} \right) + f_{N+1} P = 0 \quad (3.3.1)$$

on the polynomials  $f_i$ . This is known, in algebraic geometry, as a syzygy constraint. Note the presence of a new polynomial  $f_{N+1}$ ; together with the rest of the  $f_i$ , it forms a syzygy. The formal mathematical definition and details of syzygies is beyond the scope of this work and the reader is referred to [269] for more details regarding syzygies in the context of Feynman integral reduction. For the purposes here, a syzygy is basically a relation between polynomials in certain variables. E.g. consider a set of polynomials  $e_i = \{x, x + y, y^2\}$ . The equation

$$f_1 e_1 + f_2 e_2 + f_3 e_3 = 0 \quad (3.3.2)$$

is satisfied by  $f_i = \{\{y, -y, 1\}, \{x + y, -x, 0\}\}$ . This set of solutions forms a syzygy module. Note that any linear combination of the two syzygies, with the variables  $(x, y)$  allowed in the coefficients, also forms a syzygy. This is easy to see by replacing  $f_i$  by  $a(x, y)f_{1,i} + b(x, y)f_{2,i}$  in Eq. 3.3.2 where  $f_{1,i}$  and  $f_{2,i}$  are the two different solutions.

For the case at hand, syzygies are formed by polynomials in the Baikov parameters with the kinematic invariants appearing in the coefficients. Explicit solutions to the "no dimension-shift" constraint in Eq. 3.3.1 were pointed out in [270] in Baikov representation.

These  $f_I^{no-dim}$  are simply linear polynomials in the Baikov parameters and kinematics. Note that these  $f_i^{no-dim}$  generate integration-by-parts relations identical to those in the momentum-space representation [270]. E.g. for the bubble function with one massive

propagator in Fig. 3.1.2, the no dimension-shift syzygies are

$$f_i^{no-dim} = \{ \{-p^2 - z_1 + z_2, p^2 - z_1 + z_2, 0\}, \{2(m^2 + z_1), -p^2 - z_1 + 2(m^2 + z_1) + z_2, -2\} \}. \quad (3.3.3)$$

Substituting the above syzygies into Eq. 3.2.10, it can be seen that the IBP relations from Eqs. 3.1.10 and 3.1.11 are recovered.

As seen in Sec. 3.1, IBP relations generated this way contain integrals with doubled propagators and it is highly desirable to avoid generating such integrals. Looking at the general IBP equation in Baikov representation (Eq. 3.2.10), it is clear that such doubled propagators are generated by the term  $\nu_i f_i / z_i$ . They can be avoided by imposing the so-called "no dot" constraint

$$f_i^{no-dot} = b_i z_i \quad \forall i = 1, \dots, N' \text{ with } \nu_i \geq 1 \text{ and } N' < N. \quad (3.3.4)$$

This is a rather trivial constraint. Essentially, the  $f_i$  are required to be proportional to  $z_i$  so as to cancel the  $z_i$  in the denominator of the offending term. For the bubble function with one massive propagator in Fig. 3.1.2, the no-dot syzygies are simply

$$f_i^{no-dot} = \{ \{z_1, 0, 0\}, \{0, z_2, 0\}, \{0, 0, 1\} \}. \quad (3.3.5)$$

To impose the two constraints individually is quite straightforward. Finding  $f_i$  that satisfy simultaneously both the constraints in Eqs. 3.3.1 and 3.3.4 is, however, a highly non-trivial problem in general. The initial strategy, based on the syzygy approach using Baikov representation, proposed in [267] was to simply replace the  $f_i$  in the no dimension-shift constraint in Eq. 3.3.1 with  $b_i z_i$  and solve for  $b_i$ . This was found to be computationally expensive and impractical for complicated topologies. The approach proposed in [269] to simultaneously solve the two constraints by computing a module intersection of the two

syzygy modules was suggested to be better. Computing module intersections is a well-known problem in Algebraic geometry and several computer algebra packages exist for this task e.g. `Singular` [271].

The usual module intersection approach using `Singular` was found to be inadequate for the purpose of this calculation, however. For the planar topologies, for example, which are expected to be significantly simpler in complexity, even syzygies for sectors with 6-lines (propagators) were found to be out of reach, using `Singular` out of the box. With a few optimisations with the assistance of the author of [269], further improvement was found rendering 6-line planar sectors solvable. However, the top-level (7-line) sectors in the planar as well as non-planar topologies remained intractable. In addition, the approach was found to be computationally expensive requiring several days of run-time with large amounts of RAM.

### 3.4 Linear algebra based syzygy construction

In light of the inadequacy of the module intersection based method implemented through `Singular`, it was clear that an alternate approach was required to compute the syzygies. Specifically, it was observed that the module intersection method was computationally expensive and not feasible for the problem at hand. Instead, a custom linear algebra based syzygy solver was employed to compute the syzygies required for this work.

The basic idea behind the algorithm is as follows. As mentioned in the previous section, it is straightforward to write down the solutions to the no dimension-shift constraint as well as the no dot constraint. The important point to note here is that if certain syzygies satisfy a constraint, then so does any linear combination of those syzygies where the coefficients are allowed to depend on the variables i.e.

$$\left( \sum_{j=1}^N F_j^{no-dim} \frac{\partial P}{\partial z_j} \right) + F_{N+1}^{no-dim} P = 0 \quad (3.4.1)$$

where  $F_j^{no-dim} = \sum_i a_i(z_1, \dots, z_N, s_{12}, \dots) f_{j,i}^{no-dim}$  is the linear combination and  $a_i$  are the polynomials. The linear combination  $F_j^{no-dim}$  must now satisfy the no dot constraint  $F_j^{no-dim} = b_j z_j$  which provides constraints on the polynomials  $a_i$ .

In principle, the above construction should provide exactly the same syzygies as the module intersection method if solving for general polynomials  $a_i$ . However, solving for general  $a_i$  would require a Gröbner basis computation which is identical to the module intersection approach. Instead, this method can be used to algorithmically construct the solutions up to a certain polynomial degree by choosing an ansatz for the  $a_i$ , starting with degree 0 polynomials for  $a_i$ . Substituting an ansatz for  $a_i$  in Eq. 3.4.1 turns the problem into row reduction of a matrix. A custom linear solver based on Finite field methods [272, 273] is used to provide fast and efficient row reduction of the resulting linear system.

A brief description of the algorithm first implemented for the multivariate case in, and reproduced from, [6] is provided below.

While the module intersection method provides all possible syzygy solutions, it was observed that for amplitude calculations only syzygies up to a certain degree are needed. Indeed, in contrast with the module intersection approach based on `Singular`, the linear algebra approach presented above was able to generate the required syzygies. For the top-level planar topologies, the method was relatively quick requiring only  $O(10)$  CPU hours while the non-planar topologies turned out to be significantly more difficult requiring  $O(100)$  CPU hours per topology. This approach was also used to generate the required syzygies for the 5-point 2-loop topologies needed for the calculation of  $\gamma\gamma + jet$  amplitude in [193, 197].

While this method was able to solve the syzygies for calculations at the frontier of perturbative QCD, there are certain limitations. In particular:

1. Solutions are only produced up to a certain polynomial degree by construction. In case higher degree solutions are required, the task can be challenging in certain cases. E.g. For some of the non-planar topologies, syzygies for degree up to 5 were constructed, but these were found to be insufficient. It was estimated that computing the degree

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**Algorithm 1** Syzygies for linear relations without dimension shifts or dots [6]

---

*Input:* Syzygies of degree 1 solving Eq. 3.3.1, maximal required degree  $n_{\max}$ .

*Output:* Syzygies  $S_1, \dots, S_{n_{\max}}$  up to degree  $n_{\max}$  solving Eqs. 3.3.1 and 3.3.4.

- 1: Start with syzygies of degree  $n = 1$ . Let  $I_1$  be a complete set of solutions  $(f_i)$  to the no-dimension-shift constraint from Eq. 3.3.1, which are linear in the Baikov parameters  $z_k$ . These can directly be written down [270]. Abbreviating the momenta squared with variables  $z_{N+1}, \dots$ , the vectors in  $I_1$  are of homogeneous degree 1 in the variables  $z_k$ .
  - 2: At degree  $n$ , form a matrix  $M_n$ , where each element of  $(f_i) \in I_n$  corresponds to a row. The columns enumerate both the component  $i$  of  $(f_i)$  and the power products of  $z_i$  in them; the entries of the matrix are the coefficients. A column is called admissible, if it satisfies the no-doubled propagator constraint in Eq. 3.3.4, and non-admissible otherwise. All admissible columns are ordered to the right of the non-admissible columns.
  - 3: Perform a row reduction of  $M_n$ . In the row reduced form, select all rows, which have an admissible pivot column and form the corresponding syzygies  $S_n$  from them.  $S_n$  forms a complete set of linear combinations of the syzygies in  $I_n$ , which satisfy Eq. 3.3.4 for all of their terms, and are therefore our solutions at degree  $n$ .
  - 4: If  $n$  is the user-defined maximal degree, stop and return the solutions  $S_1, \dots, S_n$ . Otherwise, proceed.
  - 5: For each vector of polynomials  $(f_i) \in I_n$  and each  $z_k$ , form the vector of polynomials  $(z_k f_i)$ . This gives the set  $I_{n+1}$ , which are solutions of Eq. 3.3.1 of degree  $n + 1$  in the  $z_k$  but not necessarily solutions of Eq. 3.3.4.
  - 6: Replace  $n \rightarrow n + 1$  and go to step 2.
-

6 syzygies would be unfeasible with the available computing resources. Instead, some identities involving integrals with doubled propagators were indeed required for a complete and successful IBP reduction. It has been suggested that syzygies up to a certain degree, dependent on the specific problem at hand, are sufficient to obtain reductions for integrals with arbitrarily high numerators. This is not proven, however.

2. By the nature of the construction wherein the kinematic variables are treated as indeterminates of the ring similar to the Baikov parameters, often identities are generated with "true" degree less than the what it should be. In other words, the kinematic variables sometimes inflate the polynomial degree of the syzygy so that a degree "n" syzygy is effectively only "n-1". This can happen in two ways, either by having a kinematic variable as an overall factor or by having each and every individual term containing a kinematic variable. This can in principle be fixed by considering the kinematic variables in the coefficient field instead of as indeterminates like the Baikov parameters. It was suggested that in this approach the degree of syzygies required can be less than in the original formulation. This is not very well understood, however, and requires further investigation.
3. This construction may also result in syzygies that are polynomials multiplied by a Baikov variable as an overall prefactor. Such syzygies are redundant: the loop edge corresponding to the Baikov variable present as the overall prefactor is either part of the topology under consideration or appears as an inverse propagator (in the numerator). In the first case the syzygies simply generate identities from a lower topology that are presumably contained in the identities generated specifically for that topology; these are indeed dropped [274]. In the second case, the prefactor simply increases the "tensor rank" of every integral in the identity by 1 which can alternatively be achieved by increasing the rank of seed integrals in the syzygies with one degree.

Despite the above limitations, the method has been very successful in solving syzygies

for challenging problems.

### 3.5 IBP reduction using syzygies

The syzygies generated using the approach outlined in the previous section are used to generate IBP templates to generate relations between integrals, similar to templates generated through conventional IBP relations e.g. in Eqs. 3.1.10 and 3.1.11. Seed integrals with specific propagator exponents are then inserted into the templates to generate the required IBP relations, similar to Laporta's algorithm, which are subsequently solved using a row-reduction procedure. An in-house custom linear solver, `Finred`, based on Finite field methods [272, 273] is used for this. Instead of solving the symbolic linear system directly, the equations are sampled over a finite field and the symbolic dependence is reconstructed through rational reconstruction. This method allows a very high degree of parallelisation and hence can be much faster than the traditional approach.

In the case of  $ZZ$  production, top-quark and  $Z$ -boson masses are set to numbers:  $m_t = 1$  and  $m_Z^2 = 5/18m_t^2$ . The chosen ratio for  $m_Z$  corresponds to the mass of the  $Z$ -boson in terms of top-quark mass. This is to further simplify the linear relations to facilitate solving the system. Note that setting  $m_t = 1$  simply amounts to factoring out powers of  $m_t$  from the quantities of interest rendering the rest of the kinematic variables, and the form factors, dimensionless. In this way, all the Feynman integrals were successfully reduced to master integrals. Even with the above simplifications, the reductions proved to be rather challenging nevertheless and required a significant amount of computational resources. The reductions took months of run-time on the High Performance Computing Cluster (HPCC) at MSU. The resulting reduction tables occupied over 200 GB of disk space and involved extremely complicated rational functions with up to total degree 190 in the kinematic variables. As somewhat expected, the non-planar topologies proved to be the most challenging accounting for almost all of the CPU time as well as disk space.

Note that choice of basis integrals has not been mentioned yet; for  $ZZ$  the reductions were performed in terms of the conventional Laporta basis of integrals where the integrals with least numerator rank and no dots along with specific ordering prescription are simply chosen as master integrals. This is optimised for evaluation later; the details are discussed in the next section (Sec. 4). An interesting point to note is that within the planar topologies, Figs. 2.3.2a, 2.3.2c, and 2.3.2b, with adjacent gluons are significantly simpler than the topology in Fig. 2.3.2d with the gluons at the opposite vertices.

For the case of  $\gamma\gamma + \text{jet}$ , however, the situation is a lot more complicated due to the presence of an additional external particle. Indeed, while the planar topologies had been reduced, complete IBP reductions for a 5-point amplitude at 2-loops (including the non-planar topologies) had been out of reach until very recently. The solutions for the non-planar topologies were facilitated with the assistance of two key improvements. First, a canonical basis (see Sec. 4.1 for a brief description) was chosen for the reductions instead of the conventional Laporta basis. This already led to a reduction by a factor of 2 in the number of samples required for a full reconstruction, effectively reducing the required CPU time to half as well. Second, a new algorithm was used to guess the complete symbolic form of the denominators of the rational coefficients in the reduction identities [275, 276, 197]. This further led to a reduction in the number of samples by a factor of  $\sim 10$  and effectively made the reductions computationally feasible for the extremely challenging hexagon-box and double-pentagon topologies.

For both the processes, the reduction identities are generated with the rational coefficients in front of the master integrals over a common denominator. This representation is not ideal for a couple of reasons. First, reduction identities occupy a huge amount of disk space in this representation and are rather difficult to deal with for complicated processes. Second, having everything over a common denominator can often lead to numerical instabilities. Instead, multivariate partial fractioning of the rational coefficients results in a substantially simpler form of the reduction identities, in particular for the non-planar topologies. For the case of

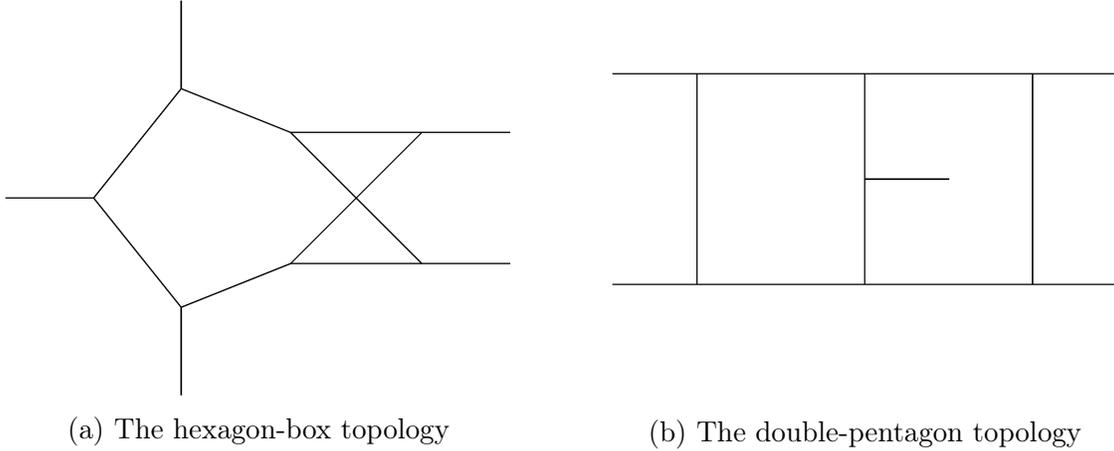


Figure 3.5.1: The non-planar topologies for 5-point 2-loop  $\gamma\gamma + \text{jet}$  production.

$ZZ$ , this resulted in a reduction from over  $\sim 10$  GB to  $< 500$  MB of disk space. Similarly, a reduction by a factor of  $\sim 100$  is achieved for the most complicated reduction identities for  $\gamma\gamma + \text{jet}$ . Multivariate partial fractioning is discussed in more detail in Sec. 5.1.

While such drastic simplifications are certainly impressive, a deeper understanding behind this would be highly desirable. It has been suggested that the difference in complexity is due to combining structures with different singular behaviours and limits. Ideally, it would be extremely useful to generate reduction identities directly in the partial fractioned form. While an efficient algorithm to do this is not yet known, recent progress in understanding this partial fractioning procedure as well as general structure of the singularities holds promise for the future.

# Chapter 4

## Finite basis integrals

### 4.1 Evaluation of Feynman integrals

Once the Feynman integrals are reduced to a set of master integrals, the next step is to evaluate them. There are a number of methods used to evaluate master integrals, both analytically and numerically. This section will focus on the methods of differential equations [277, 278, 279, 280, 281, 282] and sector decomposition [76, 283, 284, 285], used for  $\gamma\gamma + jet$  and  $ZZ$  production respectively.

#### 4.1.1 Differential equations

Consider the tadpole integral from Eq. 3.1.3. This integral is dependent only on the mass  $m$ . Taking the derivative under the integral sign with respect to  $m^2$ ,

$$\begin{aligned}\frac{d}{dm^2}I(m^2) &= \int \frac{d^d k}{i\pi^{d/2}} \frac{d}{dm^2} \frac{1}{k^2 - m^2} \\ &= - \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 - m^2)^2}.\end{aligned}\tag{4.1.1}$$

The integral in the second line of the above equation, resulting from the derivative, can be reduced to  $I(m^2)$  using IBP relation from Eq. 3.1.6. This results in the following differential

equation for  $I(m^2)$ :

$$\frac{d}{dm^2}I(m^2) = -\frac{d-2}{2m^2}I(m^2). \quad (4.1.2)$$

The above differential equation can be solved given appropriate boundary conditions to solve for  $I(m^2)$ .

Consider now a more general system with  $m$  master integrals denoted by  $I_1, \dots, I_m$ . Taking the derivative with respect to the dimensionless kinematic variable  $y_\alpha$  and using IBP reduction to reduce the right hand side integrals to master integrals, the derivatives can be written as

$$\frac{\partial}{\partial y_\alpha} \vec{I} = A_\alpha \vec{I}, \quad (4.1.3)$$

where  $\vec{I}$  is the vector of master integrals  $(I_1, \dots, I_m)$  and  $A_\alpha$  is an  $m \times m$  matrix whose entries are rational functions in the invariants and the dimensional regulator  $\epsilon$ . Repeating the procedure for all the kinematic variables results in the following form

$$d\vec{I} = \mathbf{A} \vec{I}, \quad (4.1.4)$$

where  $\mathbf{A}$  is an  $m \times m$  matrix of one-forms

$$\mathbf{A} = \sum_{\alpha} A_\alpha(\vec{y}, \epsilon) dy_\alpha \quad (4.1.5)$$

It turns out that in many cases, the above system can be drastically simplified following an *algebraic* basis change  $\vec{I} = T(\vec{y}, \epsilon) \vec{I}'$  [286, 281] such that the differential equation in Eq. 4.1.4 becomes

$$d\vec{I}' = \epsilon \mathbf{A}'(\vec{y}) \vec{I}'. \quad (4.1.6)$$

The above form is colloquially referred to as the *canonical* or  $\epsilon$ -form. Note that the entries of the transformation matrix  $T$  are rational functions in the kinematic variables  $\vec{y}$  and  $\epsilon$ . In the canonical form, the differential equations in most known cases can be easily solved order by order in  $\epsilon$  in terms of iterated integrals that evaluate to multiple polylogarithms (MPLs) [287, 288, 289, 290, 291, 292, 293]. Efficient numerical codes are available for evaluation of such multiple polylogarithms [294, 295, 296, 297, 298, 299, 300, 301] allowing a fast and convenient numerical implementation of Feynman amplitudes expressed in terms of such functions.

The method of differential equations is very powerful and has been used to evaluate a large number of Feynman integrals for many different processes. E.g. this method was used to calculate the master integrals for the massless  $ZZ$  amplitudes [302, 303, 304, 305] as well as the master integrals for massless 5-point scattering [306, 307, 308, 309] required for the  $\gamma\gamma + \text{jet}$  calculation.

Despite the large-scale applicability and success of this method, however, there are some limitations. In particular, reducing the system to the canonical form through a algebraic transformation matrix  $T(\vec{y}, \epsilon)$  is impossible in many cases; such systems evaluate to a class of functions beyond MPLs. This is expected to be the case for  $gg \rightarrow ZZ$  with massive internal quarks. There has been considerable progress over the recent years in understanding such functions beyond polylogarithms, mostly concerning their elliptic generalisations [310, 311, 312, 313, 314, 315]. However, most of the progress has been achieved for simple topologies with very few mass scales; for processes such as  $gg \rightarrow ZZ$  with many different mass scales, evaluation in terms of such functions remains extremely challenging. Furthermore, it isn't clear if the so-called *elliptic multiple polylogarithms* (eMPLs) are enough to represent the master integrals for this process and functions beyond even eMPLs could appear. On the other hand, for certain cases, even if the  $\epsilon$ -form is achievable, the functions cannot be evaluated in terms of multiple polylogarithms [316].

An alternative approach that has been recently used for many phenomenological applications is the solution of differential equations by expansions [317, 318, 319, 320, 171, 172, 321].

Instead of solving the differential equations analytically, a series expansion is constructed around a point and the equations are solved numerically term-by-term in the expansion parameters. This method is extremely useful for problems involving elliptic functions or beyond given the lack of understanding concerning these functions. Another advantage is that any given system of differential equations can be solved, in principle, regardless of whether a canonical form can be achieved or not. This is, again, especially beneficial for elliptic integrals since it can be challenging to obtain the canonical form in many cases with only a few cases where this has been successfully done [322, 323].

### 4.1.2 Feynman parametric representation

Consider the general  $L$ -loop integral in  $d$  dimensions with  $N$  distinct propagators in the momentum space representation from Eq. 3.1.1

$$I(\nu_1, \dots, \nu_N) = \int \left( \prod_{l=1}^L \frac{d^d k_l}{i\pi^{d/2}} \right) \prod_{j=1}^N \frac{1}{(q_j^2 - m_j^2 + i0)^{\nu_j}} \quad (4.1.7)$$

with integer exponents  $\nu_j \in \mathbb{Z}$ . Note the  $i0$  for the correct Feynman propagator prescription. If all indices  $\nu_j$  are positive, then the identity (see e.g. [324, 325])

$$\frac{1}{D^{\nu_j}} = \frac{(-1)^{\nu_j}}{\Gamma(\nu_j)} \int_0^\infty dx_j x_j^{\nu_j-1} e^{x_j D} \quad \text{for } \nu_j > 0, D < 0 \quad (4.1.8)$$

can be used to write down the integral as

$$I(\nu_1, \dots, \nu_N) = (-1)^\nu \int \left( \prod_{j=1}^N \frac{dx_j x_j^{\nu_j-1}}{\Gamma(\nu_j)} \right) \int \left( \prod_{l=1}^L \frac{d^d k_l}{\pi^{d/2}} \right) e^{\sum_{j=1}^N x_j (q_j^2 + m_j^2 - i0)} \quad (4.1.9)$$

where  $\nu = \sum_j \nu_j$  is the sum of all propagator exponents. The integral over loop momenta is just a Gaussian integral and can be performed easily after combining the propagators and performing Wick rotation; see e.g. [324, 326] for more details. The resulting expression is

the Feynman parametric representation of this integral:

$$I(\nu_1, \dots, \nu_N) = (-1)^\nu \Gamma(\nu - L d/2) \int \left( \prod_{j=1}^N \frac{dx_j x_j^{\nu_j-1}}{\Gamma(\nu_j)} \right) \delta \left( 1 - \sum_{j=1}^N x_j \right) \frac{\mathcal{U}^{\nu-(L+1)d/2}}{\mathcal{F}^{\nu-Ld/2}} \quad (\nu_j > 0). \quad (4.1.10)$$

The  $\mathcal{U}$  and  $\mathcal{F}$  polynomials in the above equation are the Symanzik polynomials of the first and second kind respectively.  $\mathcal{U}$  and  $\mathcal{F}$  are both homogeneous polynomials in the Feynman parameters  $x_1, \dots, x_N$  of degrees  $L$  and  $L + 1$  respectively. The  $\mathcal{U}$  polynomial is strictly non-negative while the  $\mathcal{F}$  polynomial vanished for certain kinematic configurations which, in fact, correspond to the branch cuts of the integral.

The Feynman parametric representation can be used to directly evaluate integrals in many cases. E.g. the calculation of the massless bubble integral in Fig. 3.2.1 (without the  $s$ -channel cut) is shown in Appendix B. In particular, the work on linear reducibility [327, 328] has expanded the applicability of the method to much more complex examples. A maple implementation of the integration algorithm, `HyperInt` [329], is also available that performs direct integration for integrals that evaluate to multiple polylogarithms. The integrals in Feynman parametric representation can also be evaluated numerically using e.g. Monte-Carlo methods; this is the method of choice for evaluating the master integrals for  $ZZ$  and is discussed later in the chapter. A crucial roadblock, before evaluating integrals in the Feynman parametric representation numerically, is the presence of divergences. It is discussed in the next section on how to deal with the divergences.

## 4.2 Finite integrals

As discussed in Sec 1.4.4, Feynman integrals often have divergences. Those originating from regions of large values of loop momentum are called Ultraviolet (UV) poles and those originating from regions of small loop momentum are called Infrared (IR) poles. It is instructive

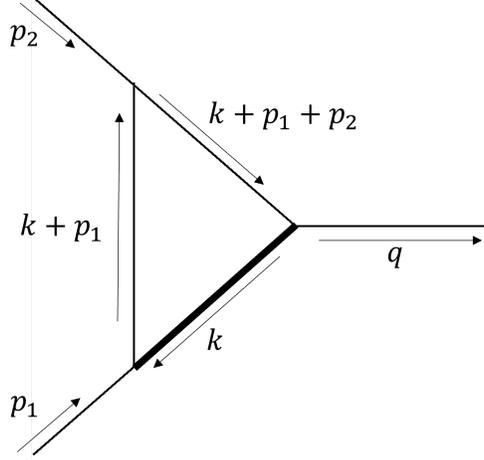


Figure 4.2.1: 3-point integral with 1 massive propagator.

to study the structure of integrands as well as origin of divergences in the Feynman parametric representation.

#### 4.2.1 Divergences in Feynman parametric representation

Consider the 3-propagator massless integral with 1-mass in  $d$ -dimensions in Fig. 4.2.1:

$$I = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 - m^2)(k + p_1)^2(k + p_1 + p_2)^2}. \quad (4.2.1)$$

This integral can be written in Feynman parametric representation as

$$I = -\Gamma(1 + \epsilon) \int_0^\infty dx_1 dx_2 dx_3 \delta\left(1 - \sum_{j=1}^3 x_j\right) \frac{\mathcal{U}^{-1+2\epsilon}}{\mathcal{F}^{1+\epsilon}} \quad (4.2.2)$$

where the Symanzik polynomials are given by

$$\begin{aligned} \mathcal{U} &= x_1 + x_2 + x_3, \\ \mathcal{F} &= (m^2 x_1 + Q^2 x_3)(x_1 + x_2 + x_3) \end{aligned} \quad (4.2.3)$$

where  $Q^2 = -q^2 = -(p_1 + p_2)^2$ . This integral is known to be IR divergent and hence

the integrand cannot simply be expanded in  $\epsilon$ . Naively expanding the integrand around  $\epsilon = 0$  leads to a finite result which is clearly incorrect; the divergence isn't explicit in  $\epsilon$  but rather hidden in the integrand. To see this, consider the region  $Q^2 > 0$  where the  $\mathcal{U}$  and  $\mathcal{F}$  polynomials for the above integral are both strictly non-negative. The only possible divergences then occur at the endpoints of the integration region.

To simplify the expression, integration in  $x_2$  can be trivially performed using the delta distribution resulting in

$$I = -\Gamma(1 + \epsilon) \int_0^\infty dx_1 dx_3 \frac{1}{(m^2 x_1 + Q^2 x_3)^{1+\epsilon}}. \quad (4.2.4)$$

Under the scaling  $\{x_1, x_3\} \rightarrow \{x_1/\lambda, x_3/\lambda\}$  and the limit  $\lambda \rightarrow 0$ , the integrand behaves as  $1/\lambda$  indicating a non-integrable divergence near the boundary. This is the origin of the IR divergence.

For UV divergences, consider a different example, a 2-loop "double" tadpole integral:

$$I = \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{(k_1^2 - m^2)(k_2^2 - m^2)}. \quad (4.2.5)$$

The above integral clearly has a UV divergence in both loops since this is simply a product of two 1-loop tadpole integrals. In Feynman parametric representation, the integral becomes

$$I = -\Gamma(-2 + 2\epsilon) \int_0^\infty dx_1 dx_2 \delta(1 - x_1 - x_2) \frac{\mathcal{U}^{-4+3\epsilon}}{\mathcal{F}^{-2+2\epsilon}} \quad (4.2.6)$$

with  $\mathcal{U} = x_1 x_2$  and  $\mathcal{F} = m^2(x_1 + x_2)\mathcal{U}$ . This simplifies to

$$I = -\Gamma(-2 + 2\epsilon) (m^2)^{2-2\epsilon} \int_0^\infty dx_1 dx_2 \delta(1 - x_1 - x_2) \frac{(x_1 + x_2)^{2-2\epsilon}}{(x_1 x_2)^{2-\epsilon}}. \quad (4.2.7)$$

In this case, the Gamma function prefactor is divergent in the limit  $\epsilon \rightarrow 0$ :

$$\Gamma(-2 + 2\epsilon) = \frac{1}{4\epsilon} + O(\epsilon). \quad (4.2.8)$$

Using Cheng-Wu theorem [330], the variable  $x_1$  can be taken out of the delta distribution. The integrand then diverges in the limit  $\lambda \rightarrow 0$  for the scaling  $x_1 \rightarrow x_1/\lambda$ ; this is the other UV divergence.

The above approach of determining the scaling behaviour of integrands near boundaries was presented in [331, 332]. These boundary divergences prevent a naive Monte-Carlo implementation since the integrand cannot be expanded around  $\epsilon = 0$ . A universal approach to resolve these singularities and enable an expansion in  $\epsilon$  is sector decomposition [76, 283, 284, 285, 333]. After performing sector decomposition, the resulting expressions can be integrated numerically. Public codes that implement sector decomposition and perform numerical integration for arbitrary loop integrals in physical kinematical region are available e.g. `Fiesta4` [334] and `pySecDec` [3].

## 4.2.2 Finite integrals with dimension shifts

The usual approach to Integration-By-Parts reduction using Laporta's algorithm leads to a conventional basis of master integrals prefers master integrals with the lowest number of propagators and irreducible scalar products in the numerator. This basis is, however, rather difficult to evaluate numerically using sector decomposition due to the presence of boundary singularities which can cause severe numerical instability during monte-carlo evaluation. A natural solution would be to use integrals that are finite in  $d = 4$  to avoid such issues. Indeed it was observed in [335, 336] that the use of finite integrals in the basis leads to significantly improved numerical stability. In addition, finite integrals often require fewer orders in their  $\epsilon$  expansion which makes them easier to evaluate numerically requiring less computing resources to achieve similar precision. Another advantage is that the poles in the

regulator  $\epsilon$  drop out of the Feynman integrals and become explicit. The resulting analytic expressions for the amplitude are a lot more convenient and easier to simplify (see Sec. 5.1). These factors together improve the overall numerical performance significantly.

To construct such finite integrals, one approach is to use integrals in dimensions different than  $d = 4$ . This was explored in [337] by considering finite integrals in  $d = 6$ . This approach is often used in conjunction with increasing propagator exponents, with finite integrals often requiring both dimension-shifts and higher powers of propagators. Consider the massless scalar integral with 3-propagators in Eq. 1.4.26 in  $d$ -dimensions:

$$I = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)(k - p_1)^2(k + p_2)^2}. \quad (4.2.9)$$

In the limit of small loop-momentum,

$$I \sim \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)(2k \cdot p_1)(2k \cdot p_2)} \sim \int_{k \rightarrow 0} \frac{d^d k}{k^4}. \quad (4.2.10)$$

This integral has an IR divergence for  $k \rightarrow 0$  in  $d = 4$ . It was shown in Sec. 1.4.4 that this integral will converge for  $d > 4$ . This is the general idea behind introducing dimensionally-shifted integrals to cure IR divergences; for large enough values of  $d$ , the singular behaviour in soft or collinear regions is cured. The opposite approach works for UV divergences. E.g. the bubble integral with 1-mass,

$$I = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 - m^2)(k + p)^2} \sim \int_{k \rightarrow \infty} \frac{d^d k}{k^4}, \quad (4.2.11)$$

is finite for  $d < 4$ . Alternatively, the exponent of one of the propagators can be increased to cure the singular behaviour for large values of loop-momentum. For example,

$$I = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 - m^2)^2(k + p)^2} \sim \int_{k \rightarrow \infty} \frac{d^d k}{k^6} \quad (4.2.12)$$

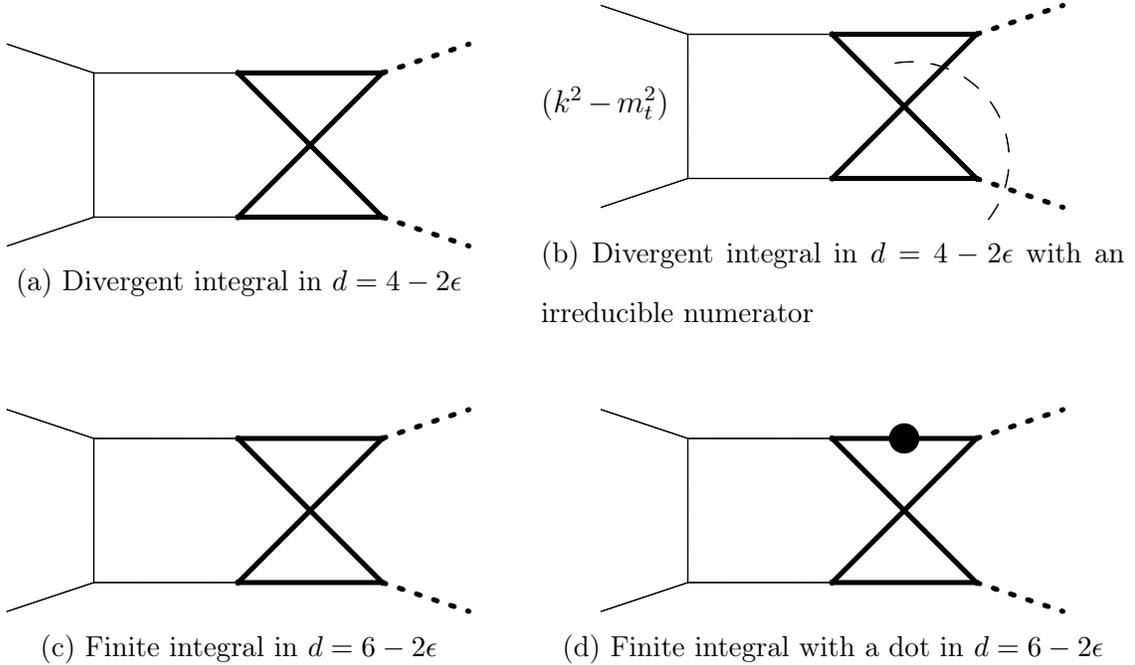


Figure 4.2.2: Examples of divergent and finite integrals in the limit  $\epsilon \rightarrow 0$  for a non-planar topology. Thick solid lines represent the top-quark while thick dashed lines represent Z-bosons. Topology (b) contains an irreducible numerator, where  $k$  is the difference of the momenta of the edges marked by the thin dash lines. Reproduced from [6].

is now finite for  $d < 6$ . It is straightforward to see that increasing propagators tames the behaviour for large loop-momentum.

It was shown in [338, 332] that a basis of such finite integrals constructed using dimension-shifts and higher propagator powers can always be found. From the above examples it is clear that the dimension shifts act in opposite way to IR and UV poles. Consequently, finite integrals often require a combination of both dimension shifts and higher propagator powers.

Examples of such finite integrals for a non-planar topology are shown in Fig. 4.2.2. Fig. 4.2.2a shows the corner integral for the topology while 4.2.2b shows an integral with 1 irreducible numerator; both are divergent in  $d = 4 - 2\epsilon$ . The corner integral, as shown in Fig. 4.2.2c, is in fact finite in  $d = 6 - 2\epsilon$ . Another finite integral can be constructed by putting a dot on one of the propagators as shown in Fig. 4.2.2d.

### 4.2.3 Finite integrals with numerators

As described in the previous subsection, it is rather straightforward to find finite integrals of such kind involving dimension shifts and dots, especially using e.g. `Reduze 2`. However, integrals with dots typically do not appear in amplitudes which is in fact the primary motivation for syzygy based IBP reduction as discussed in Sec. 3; this is especially true for the top-level topologies. Furthermore, dimension-shifted integrals require reductions for  $L$  additional dots for topologies with  $L$ -loops which can also be quite challenging. It is therefore desirable to consider alternative approaches to construct finite integrals.

Here, a different approach to construct finite integrals by considering linear combinations of divergent integrals is explored. Feynman parametric representation of a general linear combination is considered and analyse the boundary regions to put constraints on the coefficients [339]. This ensures that the non-integrable divergences of the individual integrals cancel for the combined integrand resulting in a single Feynman parametric integral that is finite. Before the algorithm to construct such integrals with numerators, first described in [6], is presented, it is useful to define some relevant quantities and formulas.

Similar to Eq. 4.1.8, propagators with negative exponents i.e. numerators with  $\nu_j < 0$  can be included by employing the identity [326, 340]

$$\frac{1}{D^{\nu_j}} = \left[ \frac{\partial^{-\nu_j}}{\partial x_j^{-\nu_j}} e^{x_j D} \right]_{x_j=0} \quad \text{for } \nu_j \leq 0. \quad (4.2.13)$$

Consider an integral  $I(\nu_1, \dots, \nu_N)$ . Let  $\mathcal{N}_+$  be the set of all positive  $\nu_j$ ,  $\mathcal{N}_-$  the set of all negative  $\nu_j$ , and  $r = \sum_{j \in \mathcal{N}_+} \nu_j$ . Then an integral with positive *or* negative indices, using

Eqs. 4.1.8 and 4.2.13 respectively, can be written as

$$I(\nu_1, \dots, \nu_N) = (-1)^r \Gamma(\nu - L d/2) \int \left( \prod_{j \in \mathcal{N}_+} \frac{dx_j x_j^{\nu_j-1}}{\Gamma(\nu_j)} \right) \delta \left( 1 - \sum_{j \in \mathcal{N}_+} x_j \right) \left[ \left( \prod_{j \in \mathcal{N}_-} \frac{\partial^{|\nu_j|}}{\partial x_j^{|\nu_j|}} \right) \frac{\mathcal{U}^{\nu-(L+1)d/2}}{\mathcal{F}^{\nu-Ld/2}} \right]_{x_j=0 \forall j \in \mathcal{N}_-} \quad (\nu_j \neq 0). \quad (4.2.14)$$

Using the above formula, an integral with an arbitrary combination of numerators and dots can be expressed in Feynman parametric representation.

The ultimate objective here is to combine different integrals sharing a common parent topology into one merged parametric representation. To that end, it would be desirable to include integrals from subtopologies as well to write down the most general ansatz for a finite linear combination. However, integrals from subtopologies, if their Feynman parametric representations are generated naively using the above formula, have  $\mathcal{U}$  and  $\mathcal{F}$  polynomials different from the parent topology, and hence the linear combination cannot be combined over a common "denominator" easily. It would be useful, then, to be able to express the subsector integrals in a similar way to the numerator integrals using the same Symanzik polynomials as the parent topology. This can be achieved by taking derivatives with respect to the Feynman parameters corresponding to the pinched lines without setting them to zero,

$$\frac{1}{D^{\nu_j}} = - \int_0^\infty dx_j \frac{\partial^{-\nu_j+1}}{\partial x_j^{-\nu_j+1}} e^{x_j D} \quad \text{for } \nu_j \leq 0, D < 0. \quad (4.2.15)$$

Here line corresponds to a propagator with a positive index i.e. a propagator present in the topology. Before writing down the general formula, it is necessary to specify some notation.

Let  $\mathcal{N} = \{1, \dots, N\}$  be the set of all indices,  $\mathcal{N}_T$  the set of positive indices of the parent topology (parent lines),  $\mathcal{N}_t$  the set of positive indices  $\nu_j$  of the current topology which could either be the parent topology or a subtopology,  $\mathcal{N}_{\Delta t} = \mathcal{N}_T \setminus \mathcal{N}_t$  i.e. the set of pinched lines,  $\mathcal{N}_{\setminus T} = \mathcal{N} \setminus \mathcal{N}_T$  be the set of negative indices of the parent topology (parent numerators),

$r = \sum_{j \in \mathcal{N}_t} \nu_j$  the sum of positive indices of the current integral, and  $\Delta t = |\mathcal{N}_{\Delta T}|$  the number of pinched lines. The general integral can then be written in the Feynman parametric representation as

$$\begin{aligned}
I(\nu_1, \dots, \nu_N) = & (-1)^{r+\Delta t} \Gamma(\nu - L d/2) \int \left( \prod_{j \in \mathcal{N}_T} dx_j \right) \left( \prod_{j \in \mathcal{N}_t} \frac{x^{\nu_j-1}}{\Gamma(\nu_j)} \right) \delta \left( 1 - \sum_{j \in \mathcal{N}_T} x_j \right) \\
& \left[ \left( \prod_{j \in \mathcal{N}_{\setminus T}} \frac{\partial^{|\nu_j|}}{\partial x_j^{|\nu_j|}} \right) \left( \prod_{j \in \mathcal{N}_{\Delta t}} \frac{\partial^{|\nu_j|+1}}{\partial x_j^{|\nu_j|+1}} \right) \frac{\mathcal{U}^{\nu-(L+1)d/2}}{\mathcal{F}^{\nu-Ld/2}} \right]_{x_j=0 \forall j \in \mathcal{N}_{\setminus T}} \quad (\nu_j \in \mathbb{Z}).
\end{aligned} \tag{4.2.16}$$

This expression for a completely general integral was first presented in [6]. Note that the pinched lines are allowed to appear as numerators i.e.  $\nu_j \leq 0$  for  $j \in \mathcal{N}_{\Delta t}$ . Also, the Symanzik polynomials  $\mathcal{U}$  and  $\mathcal{F}$  are calculated by taking all indices  $\mathcal{N}$  into account.

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**Algorithm 2** Finite Feynman integrals [6]

*Input:* Dimensionally regularized multiloop integrals with a common parent sector, possibly involving higher powers of propagators, irreducible numerators, or dimension shifts.

*Output:* Linear combinations of the input integrals which are finite, i.e. they have a convergent Feynman parametric representation for  $\epsilon = 0$ .

- 1: From the  $n_s$  input or “seed” integrals, form a general linear combination

$$I = \sum_{i=1}^{n_s} a_i I_i, \quad (4.2.17)$$

where  $I_i$  are the seed integrals and  $a_i$  are the unknown coefficients. The  $a_i$  are assumed to depend on the kinematic invariants and the dimensional regulator  $\epsilon$ .

- 2: Using Eq. 4.2.16, write the Feynman parametric representation for each seed integral and bring their linear combination over a common denominator such that

$$I = (-1)^{\nu_0} \int \left( \prod_{j \in \mathcal{N}_T} dx_j \right) \delta\left(1 - \sum_{j \in \mathcal{N}_T} x_j\right) \mathcal{P} \frac{\mathcal{U}^{\nu_0 - (L+1)(d_0 - 2\epsilon)/2}}{\mathcal{F}^{\nu_0 - L(d_0 - 2\epsilon)/2}} \quad (4.2.18)$$

where  $\mathcal{N}_T$  is the set of distinct propagators in the parent sector,  $\nu_0$  is the effective number of propagators, and  $d_0 \in \mathbb{Z}$  the effective number of space-time dimensions to be expanded around. The numerator  $\mathcal{P}$  is a homogeneous polynomial in the Feynman parameters,

$$\mathcal{P} = \sum_j c_j M_j(x_1, \dots, x_{N_T}), \quad (4.2.19)$$

where the coefficients  $c_j$  are polynomials in  $a_i$ , the kinematic variables, and  $\epsilon$ , and  $M_j(x_1, \dots, x_{N_T})$  are monomials in Feynman parameters. Note that the numerator polynomial  $\mathcal{P}$  in general depends on  $\epsilon$  and it is crucial to keep this dependence to produce correct results. It is sufficient, however, to set  $\epsilon = 0$  in the exponents of the  $\mathcal{U}$  and  $\mathcal{F}$  polynomials for the convergence analysis in the following two steps.

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- 
- 3: Check the scaling behaviour of the integrand near an integration boundary using the prescription outlined in [331, 332].
  - 4: Make sure a convergent integration of Eq. 4.2.18 is not prevented by a rapid growth of the integrand near the boundary. This can be achieved by requiring the coefficients of the offending monomials in the numerator to vanish, which provides constraints on the  $a_i$ .
  - 5: Repeat 3-4 until all boundaries are checked.

At the end of this exercise, we are left with  $I = \sum_{i=1}^{n_{\text{fin}}} a_i \left( \sum_{j=1}^{n_s} b_{ij} I_j \right)$ , where  $n_{\text{fin}} \geq 0$  is the number of finite integrals found, and  $\sum b_{ij} I_j$  are the finite combinations.

---

With the definitions above and the formula for a general Feynman integral, the algorithm 2 to construct general linear combinations that are finite in  $d = 4$  can be presented. This algorithm was first described in [6].

Applying this algorithm to a set of seed integrals from topology in Fig. 4.2.2a with the constraint of integrals with only up to 1 numerator rank, the following linear combination is obtained:

$$I_{fin,1} = s(m_z^2 - s - t) I_{1,1} + s I_{2,1} + s I_{3,1} - s I_{4,1} - s I_{5,1} - (m_z^2 - s - t) I_{6,1} - (m_z^2 - t) I_{7,1}. \quad (4.2.20)$$

The integrals  $I_{1,1}$  etc. are as defined in Fig. 4.2.3. Allowing for integrals with up to numerator rank 2 in the list of seed integrals generates several new linear combinations. Amongst them, the linear combination

$$I_{fin,2} = s(m_z^2 - s - t) I_{1,2} + s I_{2,2} + s I_{3,2} - s I_{4,2} - s I_{5,2} - (m_z^2 - s - t) I_{6,2} - (m_z^2 - t) I_{7,2}, \quad (4.2.21)$$

with the constituent integrals given in Fig. 4.2.4, is quite similar to the linear combination in Eq. 4.2.20. In fact, this linear combination can be generated simply by appending an additional inverse propagator  $k^2 - m_t^2$  to all the constituent integrals. Indeed it is straightforward

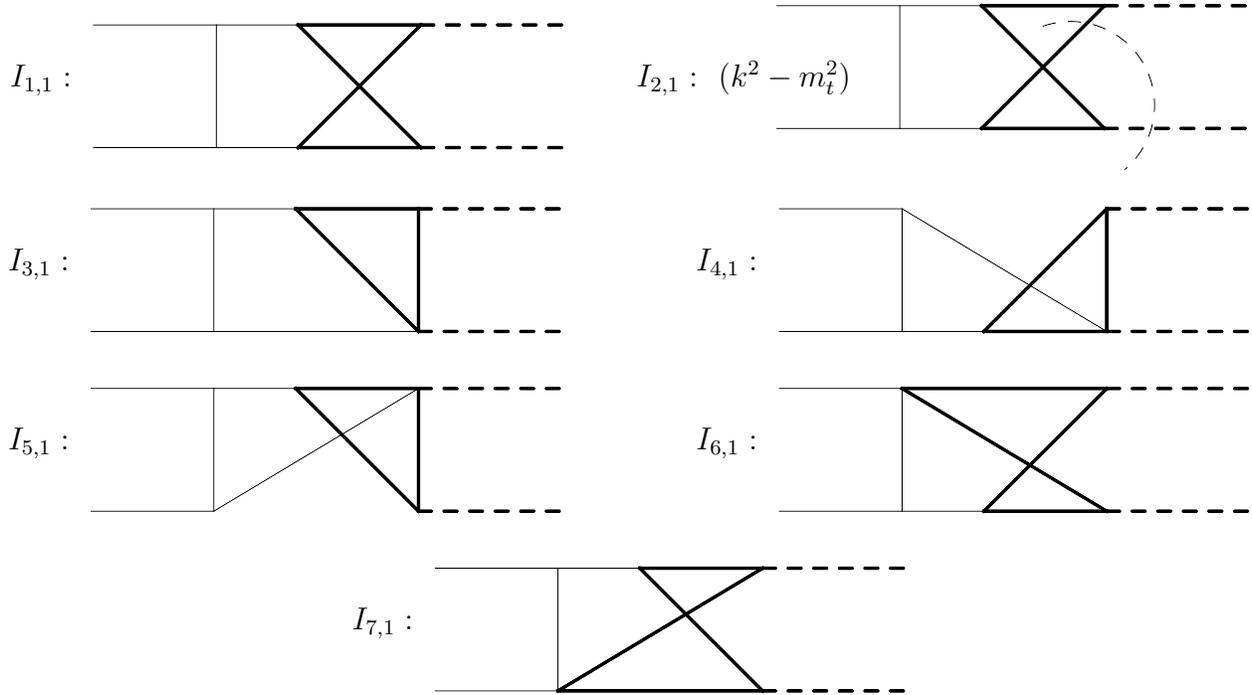


Figure 4.2.3: Integrals appearing in the linear combination in Eq. 4.2.20.  $I_{1,1}$  is the corner integral of the topology under consideration.  $I_{2,1}$  is a second integral in the topology, but with a numerator  $(k^2 - m_t^2)$ , where  $k$  is equal to the difference of the momenta of the edges marked by the thin dashed lines. Integrals  $I_{3,1}, I_{4,1}, I_{5,1}, I_{6,1}, I_{7,1}$  belong to subtopologies. All integrals are defined in  $d = 4 - 2\epsilon$  dimensions. Reproduced from [6].

to see that additional numerators can be added to any finite linear combination keeping them finite in the IR limit. However, adding numerators can lead to UV divergences; it is clear in the current example that this is not the case though through power counting, but adding additional numerators might lead to UV poles. One way to ensure UV convergence is to consider integrals with dots. In fact another linear combination can be generated by just adding a dot (on a specific propagator) in all integrals of the combination in Eq. 4.2.20. It must be pointed out, though, that such linear combinations obtained by adding numerators or dots to other finite linear combinations aren't the only possibilities. It was observed that the number of finite linear combinations increases, in general, with the numerator rank.

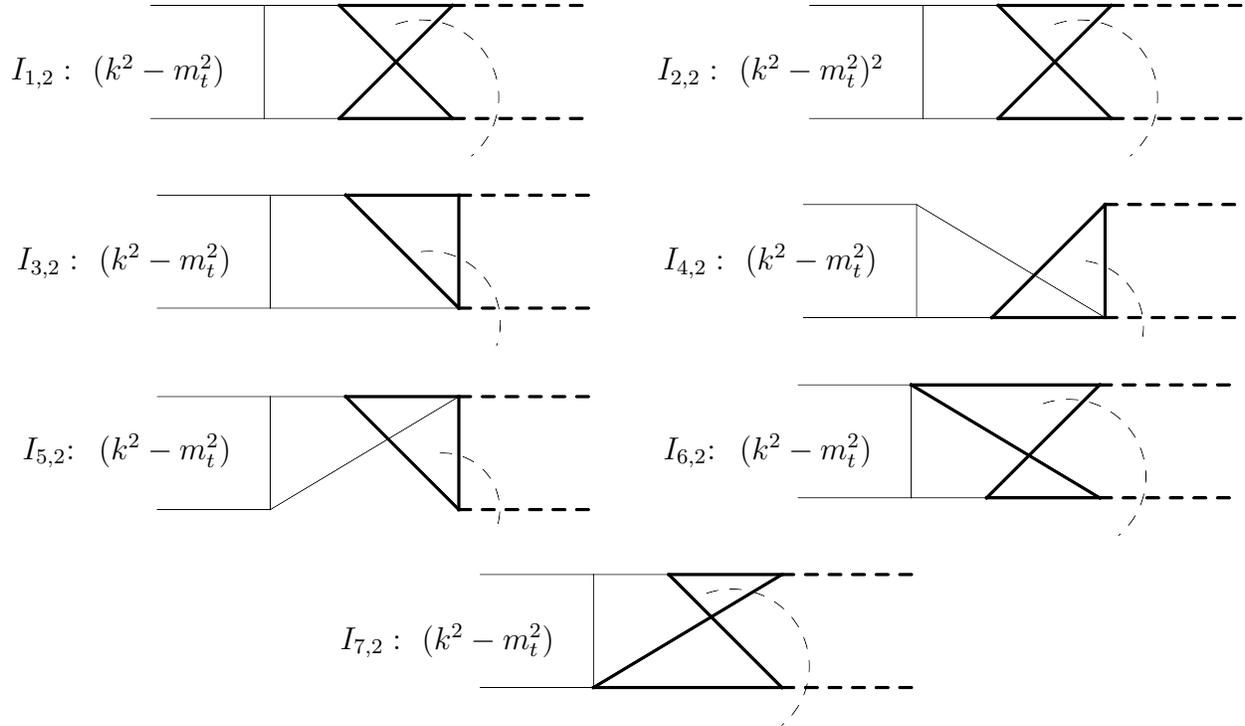


Figure 4.2.4: Integrals appearing in the linear combination in Eq. 4.2.21.  $I_{1,2}$  is the corner integral of the topology under consideration but with a numerator  $(k^2 - m_t^2)$ , identical to  $I_{2,1}$  from Eq. 4.2.20.  $I_{2,2}$  is  $I_{1,2}$  but with an extra numerator  $(k^2 - m_t^2)$  where  $k$  is equal to the difference of the momenta of the edges marked by the thin dashed lines. Integrals  $I_{3,2}, I_{4,2}, I_{5,2}, I_{6,2}, I_{7,2}$  are the same as  $I_{3,1}, I_{4,1}, I_{5,1}, I_{6,1}, I_{7,1}$  but with an extra numerator  $(k^2 - m_t^2)$ . All integrals are defined in  $d = 4 - 2\epsilon$  dimensions. Reproduced from [6].

#### 4.2.4 Choice of finite integrals for $gg \rightarrow ZZ$

The algorithm presented above expands significantly the choice of integrals available for a basis of finite integrals. In principle, the amplitude can be expressed only in terms of such finite linear combinations defined in  $d = 4 - 2\epsilon$  containing integrals with only additional numerators. In fact, such a basis can be constructed for  $gg \rightarrow ZZ$  where only finite linear combinations are used to replace IR divergent integrals. There are, however, certain limitations to this approach and in practice it was found to be useful to consider also integrals with dimension-shifts and dots. This is largely due to the following reasons:

1. In certain cases, the corner integral of a topology already has a UV divergence. This cannot be cured using a subtopology/numerator subtraction. In particular, UV divergences arising from the  $\Gamma$  prefactor in Eq. 4.2.16 cannot be cured through a subtraction since they only affect the endpoint divergences. Additionally, integrals from subtopologies or with numerators will in general have worse UV behaviour, unless an integral with many dots is considered. It is possible to consider an integral from a super-topology instead but this often leads to increase in complexity both analytically and numerically, and as such is not desirable.
  
2. While finite linear combinations exist for higher numerator ranks and, in fact, the number of available finite integrals grows with numerator rank, choosing such integrals leads to extreme proliferation in the number of terms in the numerator polynomial from Eq. 4.2.19. E.g. starting at numerator rank 1, the number of terms is  $O(10)$  e.g. for the linear combination in Eq. 4.2.20. For the linear combinations with numerator rank 2 this increases to  $O(1000)$  e.g. for the integral in Eq. 4.2.21 while for rank 3 this becomes  $O(100000)$ . Note that these numbers depend on the topology under consideration and in general the numerator polynomials would be simpler for lower topologies. A significant disadvantage of this is that the `pySecDec` libraries required to compute such integrals can reach sizes of  $\sim 1GB$  on the disk and are, hence, difficult to compile on GPUs. Trying to condense the numerator polynomials result in the appearance of spurious poles which worsen the numerical stability further.
  
3. As mentioned previously, finite linear combinations containing both numerators and dots can also be constructed. This approach was explored to avoid the linear combinations with high numerator rank. While these integrals, as long as numerator ranks are small, have more manageable numerator polynomials, their numerical performance in the physical region of kinematics, i.e. the region where contour deformation is required, suffers drastically. This is largely due to the higher power of the  $\mathcal{F}$  polynomial

Integral	Max. order in $\epsilon$	Rel. error	Time(s)
Divergent corner integral (Fig. 4.2.2a)	0	$\sim 2 \cdot 10^{-3}$	45
Divergent numerator integral (Fig. 4.2.2b)	0	$\sim 4 \cdot 10^{-2}$	63
Finite integral in $d = 6 - 2\epsilon$ , (Fig. 4.2.2c)	1	$\sim 8 \cdot 10^{-6}$	60
Finite integral in $d = 6 - 2\epsilon$ with dot (Fig. 4.2.2d)	1	$\sim 8 \cdot 10^{-4}$	55
Finite linear combination in Eq. 4.2.20	1	$\sim 1 \cdot 10^{-4}$	18
Finite linear combination in Eq. 4.2.21	0	$\sim 5 \cdot 10^{-4}$	150

Table 4.2.1: Numerical performance of different non-planar integrals for a physical phase-space point. Timings generated with pySecDec [3] using the QMC algorithm [4, 5] on a single Nvidia Tesla V100S GPU, with number of evaluations  $neval = 10^7$ . Note that the divergent integrals are only evaluated to  $O(\epsilon^0)$  since they start at  $\epsilon^{-1}$ . Reproduced from [6].

in the denominator. In general, integrals with lower exponent of the  $\mathcal{F}$  polynomial are preferred.

Table 4.2.1 shows a comparison for the numerical performance for different divergent and finite integrals from the non-planar topology in Fig. 4.2.2a. The first two orders in the  $\epsilon$  expansion are evaluated for most integrals except the finite linear combination in Eq. 4.2.21. From the table it is clear that finite integrals show significantly improved numerical performance compared to divergent integrals. The dimensionally-shifted finite integral in Fig. 4.2.2c has the lowest exponent for the  $\mathcal{F}$  polynomial, specifically the integrand  $\sim 1/\mathcal{F}$ ; unsurprisingly, it shows the best numerical performance. What is indeed surprising is that the finite linear combination in Eq. 4.2.20, which has the integrand  $\sim 1/\mathcal{F}^3$  has a numerical performance on par with the dimension-shifted integral without dots and better than the dimension-shifted integral with a dot (which has the integrand  $\sim 1/\mathcal{F}^2$ ). The linear combination with an additional numerator (Eq. 4.2.21) also has the integrand  $\sim 1/\mathcal{F}^2$ . However, the numerical performance is worse than the "simpler" finite linear combination due largely to significantly more complicated numerator polynomial and consequently much larger pySecDec libraries which are harder to evaluate on GPUs.

At the end, the best numerical performance for  $gg \rightarrow ZZ$  is achieved using a combination of both finite linear combinations and dimensionally-shifted integrals. Finite linear combina-

tions with low numerator ranks are supplemented with dimensionally-shifted integrals with low number of dots. To cure the UV poles, however, especially the ones appearing in lower topologies such as the double tadpole, integrals with dots are used. This does not affect the numerical performance though since the integrals are rather trivial to evaluate numerically as well as calculating IBP reductions for them is rather easy. The full list of finite master integrals is given in the ancillary files provided with [6].

It has been mentioned previously that choosing a canonical basis of master integrals leads to a simpler form of the IBP relations, in general, over the generic choice of basis. For  $gg \rightarrow ZZ$ , the canonical basis is not known and is expected to be rather challenging to find. In the absence of a canonical basis, a  $d$ -factoring basis of master integrals is used to simplify the IBP reductions. That is, in addition to choosing the basis integrals to be finite, they are chosen in such a way that the  $d$ -dependence of the denominators appearing in the reduction identities factors out. Simply speaking, there are no irreducible denominator factors appearing in the IBP identities that are simultaneously polynomials in both the kinematic variables and  $d$  for the basis of integrals used for final numerical evaluation. This was done using the code provided in [341] (see also [342]). It is also worth mentioning that provided that a canonical basis is found, the structure of IBP reductions as well as the amplitude can be simplified further by the use of a canonical basis that is also finite, referred to as a uniform weight finite basis [343].

# Chapter 5

## Compiling the 2-loop amplitude

### 5.1 Inserting reductions into the amplitude

Even with the  $d$ -factoring basis of finite master integrals, the IBP identities are extremely complicated. For  $gg \rightarrow ZZ$ , the IBP identities occupy over 200 GB of disk space before any processing. This would be a nightmare to deal with, and hence several techniques are used to simplify them.

#### 5.1.1 Multivariate partial fractioning

The reduction identities are first processed through `Fermat` [245] which performs a GCD (Greatest Common Divisor) calculation to simplify the coefficients. This already results in a reduction in size by about a factor of 10-20 depending on the topology. The form obtained from `Fermat` where all the terms are put over a common denominator, however, is not ideal for numerical evaluation. In addition, further simplifications can be obtained by employing multivariate partial fractioning on the reduction coefficients. Partial fraction decomposition is simply the expansion of a rational function to a sum of simpler rational functions, or rather

rational decomposition of poles. E.g.

$$\frac{x}{x^2 - 1} = \frac{1}{2(x - 1)} + \frac{1}{2(x + 1)}. \quad (5.1.1)$$

The simple example above demonstrates how partial fraction decomposition can be used to simplify expressions. However, naive approach to partial fractioning, for instance using `Mathematica`, can lead to appearance of new denominators in the multivariate case. E.g.

$$\frac{x^2 y}{(x^2 + y - 1)(y + 1)} = \frac{-x^2}{(x^2 - 2)(y + 1)} + \frac{x^2(x^2 - 1)}{(x^2 - 2)(x^2 + y - 1)}. \quad (5.1.2)$$

In the above example, a new denominator factor  $x^2 - 2$  appears. For scattering amplitudes, such spurious denominators that weren't present before can cause problems. They can cause severe numerical instabilities especially if they have poles in the physical kinematical region. They can also make the final expressions more complicated. Instead, a multivariate partial fractioning procedure based on polynomial reductions with respect to a Gröbner basis [344, 275, 345, 276] is used. A Gröbner basis is essentially just relations between polynomials in an ideal. These relations can then be used to relate more complicated polynomials to simpler polynomials where "simpler" is defined according to a polynomial ordering. E.g. for the above example, assuming a ring ordering where variable  $y$  is preferred over  $x$ , and the denominator  $1/(1 + y)$  is preferred over  $1/(-1 + y + x^2)$ , a polynomial reduction with respect to the Gröbner basis using `Singular` [271] yields

$$\begin{aligned} \frac{x^2 y}{(x^2 + y - 1)(y + 1)} = & -2 \left( \frac{1}{-1 + y + x^2} \right) \left( \frac{1}{1 + y} \right) \\ & - \left( \frac{1}{-1 + y + x^2} \right) y + 2 \left( \frac{1}{-1 + y + x^2} \right) - \left( \frac{1}{1 + y} \right) + 1. \end{aligned} \quad (5.1.3)$$

which has a maximum degree of 1 in the numerator compared to 3 for the left-hand-side expression. The result is also much simpler than the naive partial fraction result in Eq. 5.1.2 where even degree 4 terms are present in the numerator.

This approach to multivariate partial fractioning is used to simplify the reduction identities for both  $gg \rightarrow ZZ$  and  $\gamma\gamma + jet$  production. In both cases, a reduction in disk size by a factor of 100 is observed.

### 5.1.2 Backsubstitution of IBPs

For  $gg \rightarrow ZZ$ , the partial fractioning is implemented using `Singular` with a polynomial ordering that prefers polynomials with lower degrees in kinematic variables and smaller coefficients [276]. As mentioned in Sec. 3.5, the reduction identities are first calculated in the traditional Laporta basis with a generic ordering preferring master integrals with lower numerator ranks. Partial fractioning these IBP identities reduces their size drastically and makes the backsubstitution procedure a lot more manageable. Custom `FORM` scripts are then used to backsubstitute the reduction identities into the form factors. Partial fractioning is again performed on the resulting expressions which simplify drastically at this step; the total size of amplitudes reduces from  $\sim 300$  GB to  $\sim 600$  MB. Note that for this step, first partial fractioning in only the denominators dependent on  $d$  (including the denominators that depend on both  $d$  and kinematics) is performed. In a second step, partial fractioning is performed in denominators that depend only on the kinematic variables. This is simply to facilitate the calculation of the Gröbner basis since the traditional Laporta basis is not  $d$ -factoring i.e. there are irreducible denominator factors that are polynomials in  $d$  as well as kinematic variables, e.g.

$$-75 + 25d + 540t - 180td - 972t^2 + 324t^2d + 450s - 90sd - 972st + 324std,$$

where  $s, t$  are the Mandelstam variables defined in Eq. 2.1.9. Such denominators make the Gröbner basis computation extremely challenging.

In the next step, a basis change is performed to express the form factors in terms of the basis of finite integrals as chosen in Sec. 4.2.4; this basis *is*  $d$ -factoring. The basis

change identities are themselves partial fractioned and then backsubstituted into the reduced amplitude in terms of the traditional Laporta basis. The form factors, which are now in the  $d$ -factoring finite basis, are then partial fractioned, again in two steps. First, a partial fractioning in only  $d$ -dependent denominators is performed. However, this step is simpler than for the traditional Laporta basis since there are no complicated denominators depending on kinematic variables involved. Since the amplitude is now expressed in a basis of finite integrals and the 2-loop amplitude is required to be expanded only to  $O(\epsilon^0)$ , the expressions can be simplified drastically by setting  $d = 4$  everywhere except the poles  $1/(d - 4)$ . Note that this is possible only in the case of a  $d$ -factoring finite basis since after partial fractioning in  $d$ , the poles get isolated. Finally, partial fractioning in kinematic variables is performed on the resulting expressions to produce the final reduced amplitude. In the final representation, the size of the worst coefficients is brought down to less than 1 MB. The list of all surviving denominator factors are given in Appendix E. A C++ library is created for quick and efficient numerical evaluation of the coefficients. All the coefficients for a generic point in phase space can be evaluated within half a minute using rational arithmetic or within 3s using floating point arithmetic with a target precision of 15 digits on a single CPU core.

A slightly different approach is used for  $\gamma\gamma + jet$ . Since the integrals are already reduced to a canonical basis, there are no  $d$ -dependent denominators that also depend on kinematic variables. However, the basis is not finite, so it is not possible to set  $d = 4$  after partial fractioning in  $d$ . In addition, the amplitude in this case involves many crossings of the integrals. Performing a partial fraction decomposition for each crossing would be quite cumbersome and computationally wasteful. Instead, the uncrossed IBP identities, after simplifying through `Fermat`, are partial fractioned using `MultivariateApart` [276] and then the crossings are applied. The crossed IBPs are then inserted into the amplitude and the resulting expressions are partial fractioned again. At the end, the canonical master integrals are expanded in terms of the "Pentagon functions" defined in [309] and the resulting coefficients of the pentagon functions are partial fractioned again. The final expressions, as expected,

are drastically simpler with the largest helicity coefficient for the most complicated colour factor only 4.5 MB in size.

## 5.2 Renormalisation, IR subtraction and checks

### 5.2.1 UV renormalisation and IR subtraction

For  $gg \rightarrow ZZ$ , the bare form factors  $A_i$  can be expanded perturbatively according to

$$A_i = \frac{\alpha_{s,0}}{2\pi} A_i^{(1)} + \left(\frac{\alpha_{s,0}}{2\pi}\right)^2 A_i^{(2)} + O(\alpha_s^3), \quad (5.2.1)$$

where  $\alpha_{s,0}$  is the bare QCD coupling. Since the LO process for  $gg \rightarrow ZZ$  already starts at one loop, the two-loop process is effectively an NLO correction.

First, UV renormalisation of  $\alpha_s$  is performed in the 5-flavour  $\overline{\text{MS}}$  scheme i.e.  $n_f = 5$  using

$$\alpha_{s,0} = \alpha_s S_\epsilon^{-1} Z_{\alpha_s} \left(\frac{\mu_R^2}{\mu_0^2}\right)^\epsilon, \quad (5.2.2)$$

where  $S_\epsilon = (4\pi)^\epsilon e^{-\gamma_E \epsilon}$ ,  $\gamma_E \approx 0.577$  is Euler's constant,  $\mu_R$  is the renormalisation scale, and  $\mu_0$  is the 't Hooft scale introduced in the bare amplitude through conventional dimensional regularisation. Note that the top-quark contribution to the gluon self energy is subtracted at zero momentum [346].

The renormalisation constant  $Z_{\alpha_s}$  in the above equation is given by

$$Z_{\alpha_s} = 1 + \frac{\alpha_s}{2\pi} \delta Z_{\alpha_s} + \mathcal{O}(\alpha_s^2), \quad (5.2.3)$$

where

$$\delta Z_{\alpha_s} = -\frac{1}{\epsilon} \beta_0 + \frac{1}{\epsilon} \left( \frac{2}{3} T_F \left( \frac{\mu_R^2}{m_t^2} \right)^\epsilon \right). \quad (5.2.4)$$

In the above equation,  $\beta_0$  is the first coefficient of the beta function expanded in  $\alpha_s$ , given in Eq. 1.4.38,

$$\beta_0 = \frac{11 C_A - 4 T_F n_f}{6}, \quad (5.2.5)$$

and  $C_A, C_F$  are the quadratic Casimir invariants

$$C_A = N, \quad C_F = \frac{N^2 - 1}{2N}, \quad (5.2.6)$$

with  $T_F = \frac{1}{2}$ .

The top-quark mass is renormalised in the on-shell scheme. The renormalised top-quark mass is related to its bare mass according to

$$m_{t,0}^2 = m_t^2 Z_m, \quad (5.2.7)$$

with the renormalisation constant defined as

$$Z_m = 1 + \frac{\alpha_s}{2\pi} \delta Z_m, \quad \delta Z_m = C_F \left( -\frac{3}{\epsilon} - 4 \right) \left( \frac{\mu_R^2}{m_t^2} \right)^\epsilon. \quad (5.2.8)$$

In principle, the top-quark mass in the amplitude can simply be replaced by the renormalised mass in Eq. 5.2.7. However, in practice it is a lot more convenient to instead explicitly subtract 1-loop counterterm diagrams (See Fig.).

Lastly, the gluon wave function is renormalised by multiplying the amplitude with  $Z_G^{1/2}$

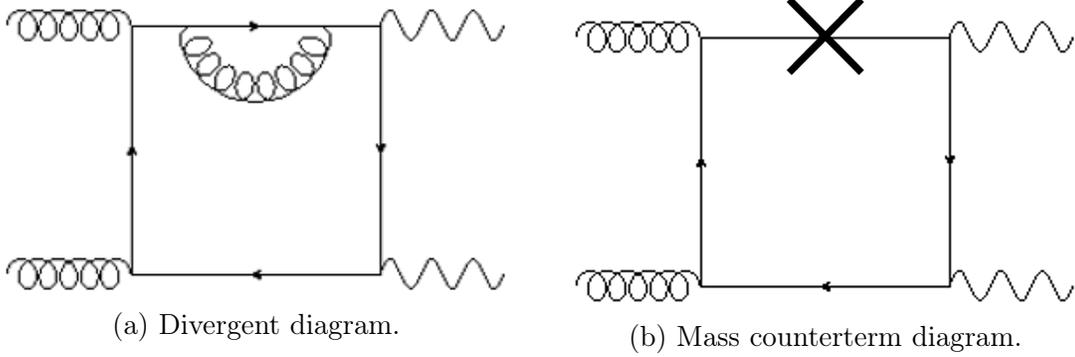


Figure 5.2.1: Mass counterterm diagrams required at 2-loops. The big dark cross in Fig. 5.2.1b corresponds to the counterterm vertex insertion.

for each external gluon, where the gluon renormalisation constant  $Z_G$  is defined as

$$Z_G = 1 + \frac{\alpha_s}{2\pi} \left( -\frac{2}{3} T_F \left( \frac{\mu_R^2}{m_t^2} \right)^\epsilon \right) + \mathcal{O}(\alpha_s^2). \quad (5.2.9)$$

This results in renormalised form factors given by

$$A_i^{\text{ren}} = \frac{\alpha_s}{2\pi} A_i^{(1),\text{ren}} + \left( \frac{\alpha_s}{2\pi} \right)^2 A_i^{(2),\text{ren}} + \mathcal{O}(\alpha_s^3). \quad (5.2.10)$$

The renormalisation constants in the above equations are derived in Appendix D.

After UV renormalisation, IR subtraction must be performed to obtain finite results. The IR structure of NLO amplitudes was first predicted by Catani in [13]. IR subtraction for this calculation is performed using the “ $q_T$  scheme” described in [347]. The finite remainders resulting from IR subtraction are given by

$$A_i^{(2),\text{fin}} = A_i^{(2),\text{ren}} - A_i^{(1),\text{ren}} I_{(1)}(\epsilon). \quad (5.2.11)$$

The  $I$ -operators in the “ $q_T$  scheme” are given by

$$I_{(1)}(\epsilon) = I_{(1)}^{\text{soft}}(\epsilon) + I_{(1)}^{\text{collinear}}(\epsilon), \quad (5.2.12)$$

$$I_{(1)}^{\text{soft}}(\epsilon) = -\frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)} \left(\frac{\mu_R^2}{s}\right)^\epsilon \left(\frac{1}{\epsilon^2} + \frac{i\pi}{\epsilon} + \delta_{q_T}^{(0)}\right) C_A, \quad (5.2.13)$$

$$I_{(1)}^{\text{collinear}}(\epsilon) = -\left(\frac{\mu_R^2}{s}\right)^\epsilon \frac{\beta_0}{\epsilon}, \quad (5.2.14)$$

where  $\delta_{q_T}^{(0)} = 0$ . UV and IR finite form factors are then given by

$$A_i^{\text{fin}} = \frac{\alpha_s}{2\pi} A_i^{(1),\text{fin}} + \left(\frac{\alpha_s}{2\pi}\right)^2 A_i^{(2),\text{fin}} + \mathcal{O}(\alpha_s^3). \quad (5.2.15)$$

Note that the renormalisation scale  $\mu_R^2 = s$  for all the results presented in this work.

## 5.2.2 Checks of the calculation

Multiple checks are performed to establish the correctness of this calculation and results, as described below:

1. The 2-loop form factors are explicitly checked to satisfy the identities in Eq. 2.1.16. These identities hold at the level of reduced amplitude with fully symbolic kinematic dependence.
2. The crossing relations in Eq. 2.1.17, on the other hand, are highly non-trivial to check analytically. Instead, these relations are verified numerically for a phase-space point within numerical precision.
3. All the finite linear combinations used in the basis are checked by numerically evaluating them and comparing them against their explicit definitions in terms of divergent integrals for a phase space point.
4. Algebraic cancellation of the spurious  $1/\epsilon^4$  pole is seen when the amplitude is expressed in the chosen basis of finite integrals; the spurious  $1/\epsilon^3$  pole remains, however. This

is verified to vanish to 15 digits by calculating the amplitude numerically for a phase-space point in the Euclidean region.

5. The  $1/\epsilon^2$  and  $1/\epsilon$  poles are verified to match Catani's IR formula [13] numerically, for a phase-space point in the Euclidean region, to 9 digits for the double pole and 7 digits for the single pole. Numbers are shown explicitly in Appendix F.
6. The  $1/\epsilon^2$  and  $1/\epsilon$  poles are also verified to match Catani's IR formula [13] for a point in the physical region with explicit numbers shown in Appendix F.
7. The amplitude is also evaluated using an alternate finite basis constructed along the same lines as the primary basis. The numerical results from the two bases are compared and found to agree within expected numerical error. It must be pointed out that this is a very stringent check on the correctness of the calculation since it simultaneously checks the IBP reductions, the basis change from the primary to the alternate basis, definitions of the finite integrals chosen, and their numerical evaluation. An error in any one of these steps would imply disagreement between the two bases. Note that the alternate basis, while composed of finite integrals, is numerically not as stable as the primary basis and although it is useful for validating the calculation, it is generally unsuitable for evaluation at a large number of phase-space points.
8. The axial-axial ( $a_i^2$ ) piece of the 2-loop amplitude is evaluated using the Kreimer's anti-commuting  $\gamma_5$  scheme [210, 211]. In addition, a separate numerical calculation is performed using Larin's  $\gamma_5$  scheme [348, 349]. Agreement is found at the level of helicity amplitudes defined using Eq. 2.1.19 between the two schemes for a physical as well as Euclidean phase space point within numerical precision. This provides another strong check of the calculation since only physical gauge-invariant observables are expected to agree for two different schemes.
9. Lastly, the calculation is compared against several known approximations in the lit-

erature, specifically the heavy top approximation [163, 164, 165] below the top-quark pair production threshold and the small top-mass expansion [12] above the threshold (at high center of mass energies). Agreement is found with both approximations in the relevant regions of phase-space with the results of the comparison presented in Sec. 6.

# Chapter 6

## Results

### 6.1 Results for the 2-loop $gg \rightarrow ZZ$ amplitude

In this section, the results for the 2-loop amplitude for  $gg \rightarrow ZZ$  are presented and compared against several approximations. Before presenting the results however, some quantities relevant for presentation of the results need to be defined. For most of the results presented below, the helicity basis defined in Eq. 2.1.19 is used. Similar to the form factors, UV renormalised and IR subtracted helicity amplitudes can be written as

$$\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{\text{fin}} = \mathcal{M}_{\mu\nu\rho\sigma}^{\text{fin}} \epsilon_{\lambda_1}^{\mu}(p_1) \epsilon_{\lambda_2}^{\nu}(p_2) \epsilon_{\lambda_3}^{*\rho}(p_3) \epsilon_{\lambda_4}^{*\sigma}(p_4). \quad (6.1.1)$$

where  $\lambda_1, \lambda_2$  are the helicities of the incoming gluons and  $\lambda_3, \lambda_4$  are the helicities of the outgoing  $Z$ -bosons. The helicity amplitudes can be expanded in  $\left(\frac{\alpha_s}{2\pi}\right)$  similar to form factors in Eq. 5.2.15 as

$$\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{\text{fin}} = \left(\frac{\alpha_s}{2\pi}\right) \mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(2)} + \mathcal{O}(\alpha_s^3). \quad (6.1.2)$$

It is useful to consider the squared 1-loop helicity amplitudes  $\mathcal{V}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(1)}$  as well as the interference between the 1-loop and 2-loop helicity amplitudes  $\mathcal{V}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(2)}$ , which can be defined

as

$$\mathcal{V}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^{(1)} = \mathcal{M}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^{*(1)} \mathcal{M}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^{(1)}, \quad (6.1.3)$$

$$\mathcal{V}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^{(2)} = 2 \operatorname{Re} \left( \mathcal{M}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^{*(1)} \mathcal{M}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^{(2)} \right). \quad (6.1.4)$$

Most of the results shown below are obtained by averaging over the helicities of the incoming gluons as

$$\mathcal{V}_{\lambda_3 \lambda_4}^{(i)} = \frac{1}{4} \sum_{\lambda_1, \lambda_2} \mathcal{V}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^{(i)}, \quad (i = 1, 2), \quad (6.1.5)$$

with  $\lambda_1, \lambda_2 \in \{+, -\}$ , and by summing over the helicities of the outgoing  $Z$ -bosons as

$$\mathcal{V}^{(i)} = \sum_{\lambda_3, \lambda_4} \mathcal{V}_{\lambda_3 \lambda_4}^{(i)}, \quad (i = 1, 2), \quad (6.1.6)$$

with  $\lambda_3, \lambda_4 \in \{+, -, 0\}$ . Note that the 1-loop amplitudes used in calculating  $\mathcal{V}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^{(i)}$  are just the top-quark contributions. Contributions from massless quarks are ignored for the purpose of presentation of the numerical results below.

For the numerical results shown, the electroweak couplings are chosen as

$$\begin{aligned} G_F &= 1.1663787 \cdot 10^{-5} \text{ GeV}^{-2}, \\ m_Z &= 91.1876 \text{ GeV}, \end{aligned} \quad (6.1.7)$$

where the Fermi constant  $G_F$  and  $Z$  boson mass  $m_Z$  are fixed according to [1]. Since the ratio of top-mass and  $Z$ -boson mass is fixed to  $m_Z^2/m_t^2 = 5/18$ , inserting the above value of  $m_Z = 91.1876 \text{ GeV}$  implies the top-mass value  $m_t = 173.016 \text{ GeV}$ . The  $W$ -boson mass is fixed according to the ratio

$$m_W^2/m_t^2 = 14/65. \quad (6.1.8)$$

This fixes the mass of the  $W$ -boson to  $m_W = 80.296$  GeV. For this calculation, the so-called  $G_\mu$  scheme is used i.e.  $G_F$ ,  $m_Z$ , and  $m_W$  are chosen as input quantities for the electroweak parameters. The weak mixing angle is then fixed according to  $\sin(\theta_W) = \sqrt{1 - m_W^2/m_Z^2}$ .

To numerically evaluate the master integrals, `pySecDec` [3, 5] is used. Sector decomposition is applied and the integration is performed using the quasi-Monte Carlo (QMC) algorithm [350, 4]. The QMC algorithm has been observed to show a much better convergence compared to the traditional Monte-Carlo algorithms [335]. The different colour structures appearing in the form factors ( $C_F$  and  $C_A$ ) as well as the vector-vector ( $v_t^2$ ) and axial-axial ( $a_t^2$ ) pieces are all evaluated separately. The target precision for each form factor, for each colour structure and coupling type ( $v_t^2$  or  $a_t^2$ ), is set to percent level. This is achieved using a variant of the optimisation algorithm presented in [335]. The helicity amplitudes are then calculated from the form factors.

Using the optimisation algorithm, requiring percent level precision on all form factors, the time taken for numerical evaluation varies between 1.5-24 hours on 2 Nvidia Tesla v100 GPUs; this usually results in most form factors being calculated to per mille or better. Almost the entire runtime is spent on calculating master integrals; the time taken to evaluate the coefficients is negligible in comparison (see Sec. 5.1). The independent helicity amplitudes for a phase-space point in the physical region are shown in Tab. 6.1.1

Fig. 6.1.1 shows a comparison of the interference term  $\mathcal{V}^{(2)}$  calculated in this work against the large top-mass and the small top-mass expansions (using the analytic expressions for the expansions provided in [12]) for varying  $\sqrt{s}/m_t$  and fixed value of  $\cos(\theta) = -0.1286$ , with the scattering angle  $\theta$  defined in Eq. 2.1.18. Also compared is the Padé improved small top-mass expansion. The plot shows excellent agreement for this calculation with the expansion results in the relevant regions of phase-space. For the smallest value of  $\sqrt{s} = 235$  GeV shown on the plot, the large top-mass expansion agrees within 0.1% with our result. Similar agreement to sub-per mille level is observed for the point with the largest  $\sqrt{s} = 878$  GeV when compared to the small top-mass expansion as well as the Padé improved result. Also observed is the

$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	$\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(1)}$ (1-loop)	$\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(2)}$ (2-loop)
+++	$0.1337854(1) - 0.0286060(1) i$	$3.15549(8) + 0.47235(8) i$
+++-	$0.0015573(1) + 0.0052282(1) i$	$0.15950(7) + 0.14052(8) i$
+--+	$-0.01512820(8) - 0.01060416(8) i$	$-0.38609(7) + 0.10539(7) i$
-+++	$-0.0291599(1) - 0.0062178(1) i$	$-0.46990(8) + 0.40207(8) i$
++ +0	$0.0292668(5) + 0.0212966(5) i$	$1.1248(2) - 0.0805(2) i$
+ - +0	$-0.0643073(5) - 0.0459584(5) i$	$-1.4803(2) + 0.4940(2) i$
++ 00	$0.910006(2) + 1.132536(2) i$	$17.2585(6) + 29.5669(6) i$
+ - 00	$0.355092(2) + 0.404469(2) i$	$10.2869(5) - 1.0571(6) i$

Table 6.1.1: 1-loop and 2-loop helicity amplitudes for  $gg \rightarrow ZZ$  for the phase-space point  $s/m_t^2 = 142/17$ ,  $t/m_t^2 = -125/22$ ,  $m_Z^2/m_t^2 = 5/18$ , and  $m_t = 1$ , with  $\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(1)}$  and  $\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(2)}$  defined in Eq. 6.1.2. Only the 8 independent helicity amplitudes (See Eqs. 2.1.20, 2.1.21, and 2.1.22) are shown here. Note that these include only the top-quark contributions from class A diagrams defined in Sec. 2.3.2. The numbers in parentheses denote the uncertainty in the last digit. Reproduced from [6].

fact that the Padé approximation improves drastically the agreement with our exact result compared to the small top-mass expansion. Note that the expansion result is only visible for the two highest values of  $\sqrt{s}$  and diverges rapidly from the exact result for smaller values. The Padé improved result, on the other hand, agrees to much lower energies closer to the  $\sqrt{s} = 2m_t$  threshold. In fact, this plot also shows that the expansions considered here can reproduce the exact result within a few percent for most values of  $\sqrt{s}$  except for the region near the top-quark pair production threshold  $\sqrt{s} = 2m_t$ , for the value of  $\cos\theta$  considered here.

In Fig. 6.1.2, a comparison of the interference term  $\mathcal{V}^{(2)}$  is shown for varying  $\cos\theta$  for several fixed values of  $\sqrt{s}$ . The large top-mass expansion changes very little with variation in  $\cos\theta$ , as is clear from the top-left panel. On the other hand, the small top-mass expansion diverges rapidly away from central scattering angles ( $\cos\theta \sim 0$ ) as seen in the bottom panel. In fact, for the points far away from the centre, the small top-mass expansion is way off the plotting range. This can be understood from the way the expansion is performed i.e. in the limit  $m_Z^2 \ll m_t^2 \ll s, |t|, |u|$ . In this limit,  $|t| \sim |u| \sim s/2$  for central scattering which

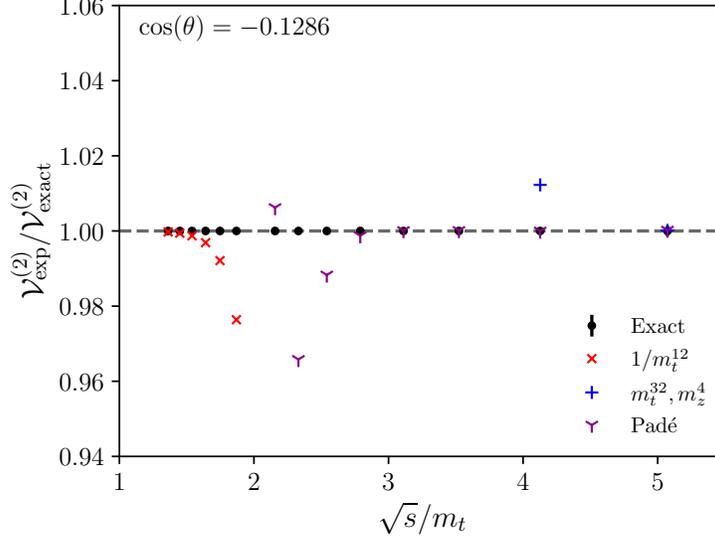


Figure 6.1.1: Comparison of the  $\sqrt{s}$  dependence of the unpolarised interference  $\mathcal{V}^{(2)}$  with expansion for large and small top-quark mass [12] at fixed  $\cos(\theta) = -0.1286$ . The large top-mass expansion is shown in colour red, the small top-mass expansion in blue, and the Padé improved small top-mass expansion in purple. The exact result is shown in black. Note that the error bars have been plotted for the exact result, they are too small to be visible on the plot, however. Reproduced from [6].

justifies the expansion. However for back-to-back scattering ( $|\cos\theta| \sim 1$ ),  $|t|$  and  $|u|$  are no longer guaranteed to be large compared to  $m_t^2$  and hence the approximation breaks down. It must be pointed out that Padé approximation significantly improves the convergence of the expansion. For the high energy point ( $\sqrt{s} = 814 \text{ GeV}$ ) in the bottom panel, the agreement of our result with the Padé improved result is almost perfect. Even for the intermediate energy point in the top-right panel ( $\sqrt{s} = 403 \text{ GeV}$ ), the agreement is generally within a few percent close to the central scattering region.

Fig. 6.1.3 shows comparisons for specific outgoing helicities averaged over the incoming gluon helicities as defined in Eq. 6.1.5 against the expansions. Both the 1-loop amplitude squared  $\mathcal{V}_{\lambda_3\lambda_4}^{(1)}$  and the 2-loop interference with 1-loop  $\mathcal{V}_{\lambda_3\lambda_4}^{(2)}$  is shown for varying  $\sqrt{s}$  with a fixed scattering angle  $\cos(\theta) = -0.1286$ . Note that the expansions are only plotted for the 2-

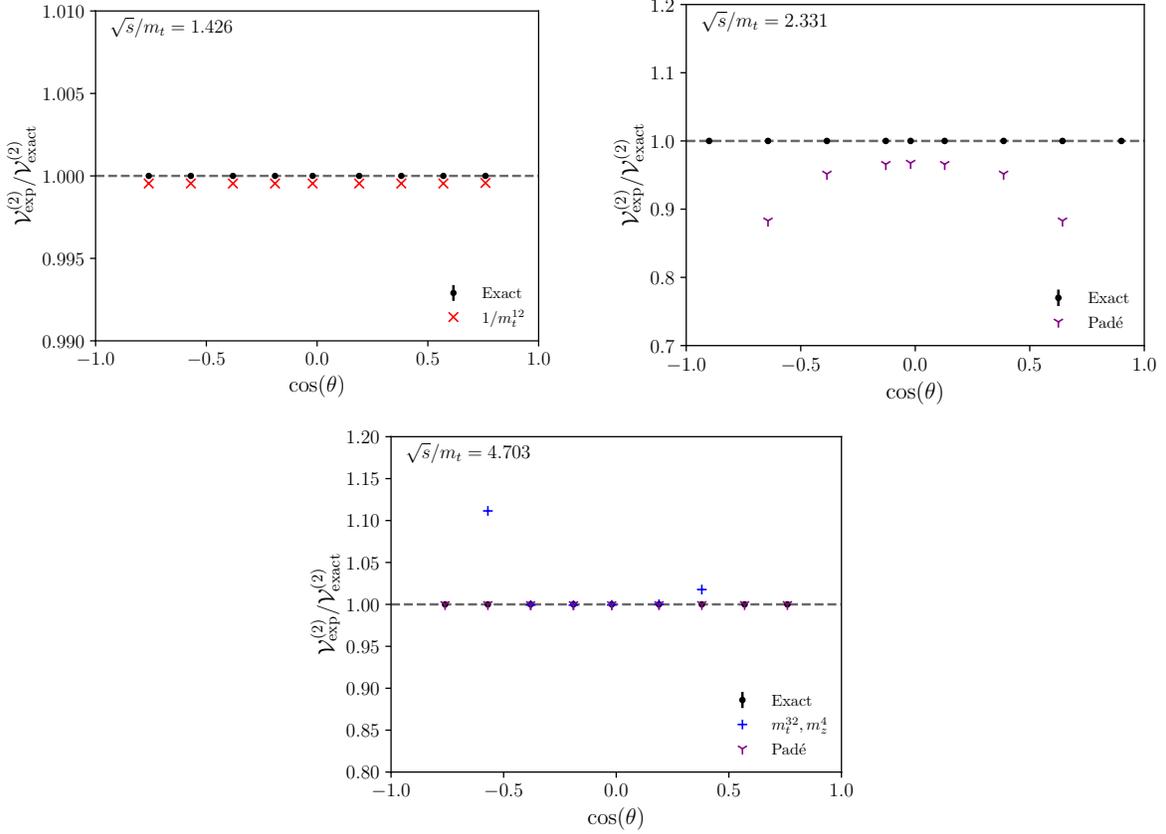


Figure 6.1.2: Comparison of the  $\cos(\theta)$  dependence of the unpolarised interference  $\mathcal{V}^{(2)}$  with the results expanded in the limit of large top-quark mass for  $\sqrt{s} = 247$  GeV (Top Left Panel) and small top-quark mass for  $\sqrt{s} = 403$  GeV (Top Right Panel) and  $\sqrt{s} = 814$  GeV (Bottom Panel). Reproduced from [6].

loop interference term. Looking at the plots, there is good agreement with the best available expansions in the relevant regions. Unsurprisingly, the small top-mass expansion lies far outside the plotting range for most points. The Padé improved expansion, however, agrees well all the way down to lower  $\sqrt{s}$  near the  $\sqrt{s} = 2m_t$  threshold. The level of agreement of our result with the expansions greatly depends on the specific helicity configuration under consideration. E.g. good agreement is seen for the  $\mathcal{V}_{00}^{(2)}$  configuration while agreement for  $\mathcal{V}_{+-}^{(2)}$  and  $\mathcal{V}_{+0}^{(2)}$  clearly starts to appear worse closer to the threshold. Note that the longitudinal configuration  $\mathcal{V}_{00}^{(2)}$  is dominant over the others even at the 2-loop level, similar to what was observed for 1-loop in Sec. 2.2.3 as well as in the above plots. A rapid increase in  $\mathcal{V}_{00}^{(1)}$  and

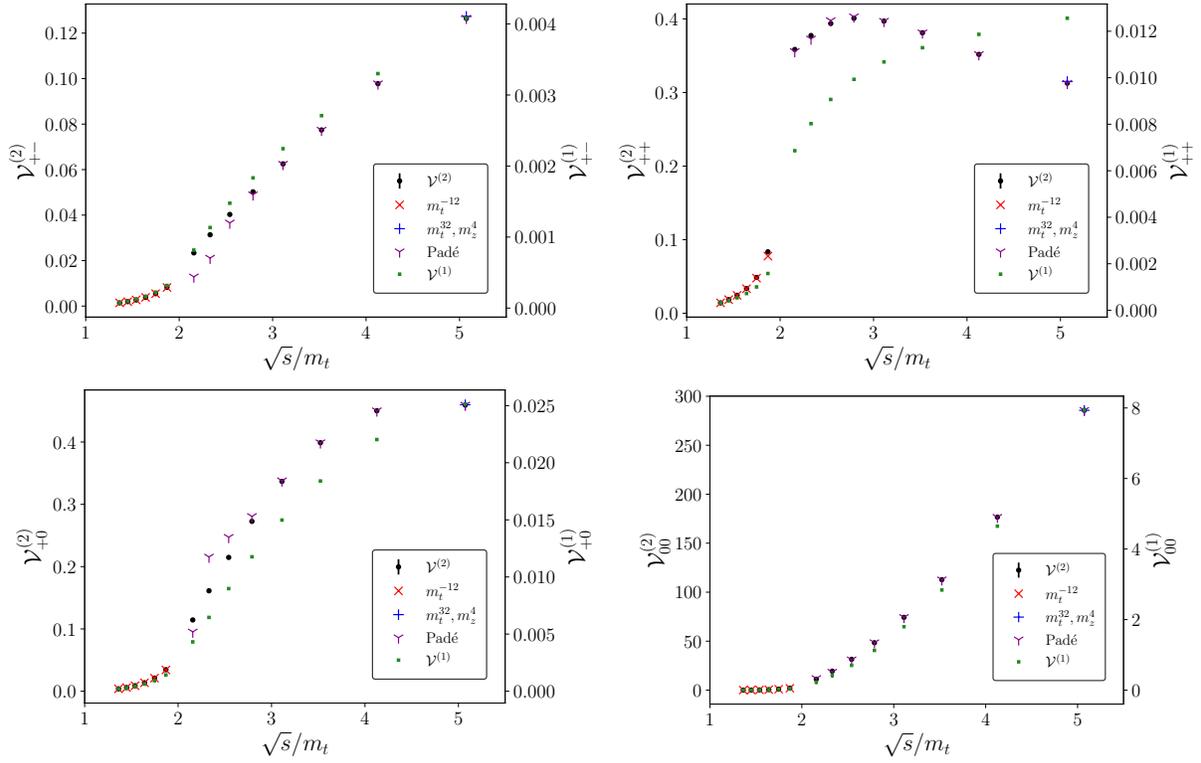


Figure 6.1.3: The  $\sqrt{s}$  dependence of 1-loop and 2-loop interferences for polarised  $ZZ$  production in gluon fusion at  $\cos(\theta) = -0.1286$ . Reproduced from [6].

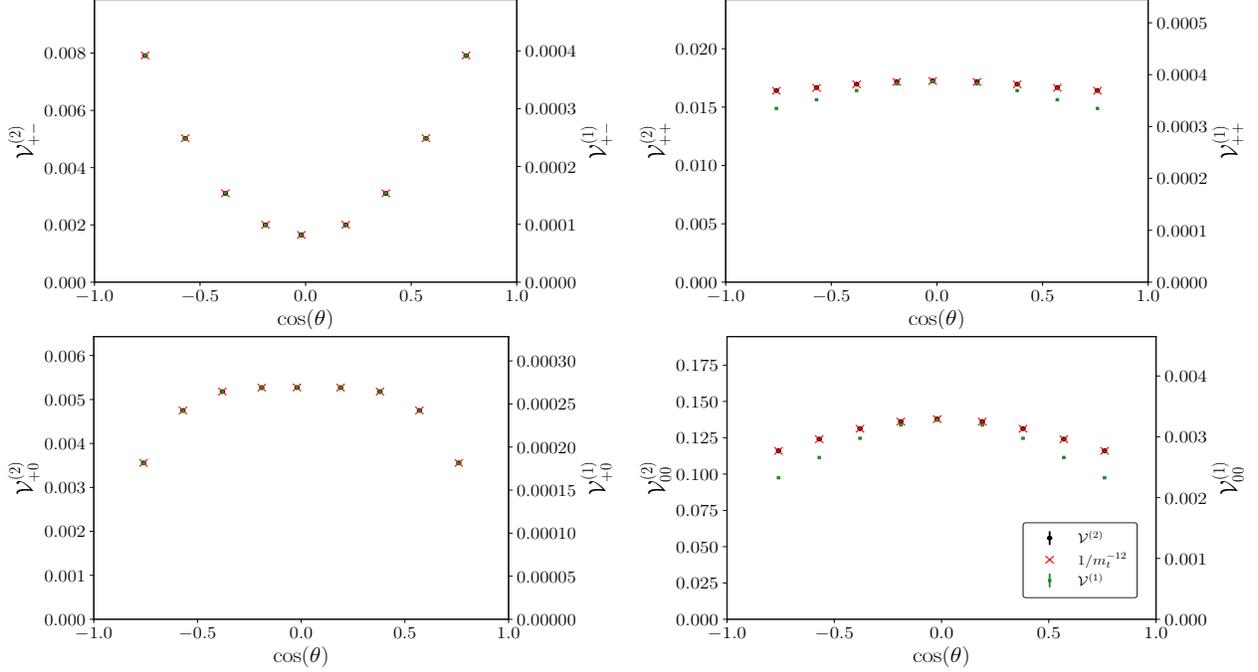


Figure 6.1.4: The  $\cos(\theta)$  dependence of 1-loop and 2-loop interferences for polarised  $ZZ$  production in gluon fusion at  $\sqrt{s}/m_t = 1.426$ . The large top-quark mass expansion [12] (to order  $1/m_t^{12}$ ) is shown for comparison. Reproduced from [6].

$\mathcal{V}_{00}^{(2)}$  past the top-quark pair production threshold  $\sqrt{s} = 2m_t$  is also seen, similar to what was observed for 1-loop in Sec. 2.2.3.

Figs. 6.1.4, 6.1.5 and 6.1.6 show the comparisons for interference terms with specific helicity configurations for varying  $\cos\theta$  and 3 different fixed values of  $\sqrt{s}$ . Fig. 6.1.4 shows the comparison against large top-mass expansion. It is clear from the plot that there is very good agreement for all helicity configurations. Note that in the chosen scaling, the 1-loop points almost lie on top of the 2-loop point with an almost complete overlap for  $\mathcal{V}_{+-}^{(2)}$  and  $\mathcal{V}_{+0}^{(2)}$ .

Fig. 6.1.5 shows the comparison against the Padé improved small top-mass expansion for intermediate energy. While in general the Padé approximation agrees well with our result, there are some deviations, notably for  $\cos\theta$  away from 0 and sub-dominant helicities.

Fig. 6.1.6 shows the comparison plots for a high energy point with  $\sqrt{s} = 814 \text{ GeV}$ . The Padé improved small top-mass expansion again shows excellent agreement with our result

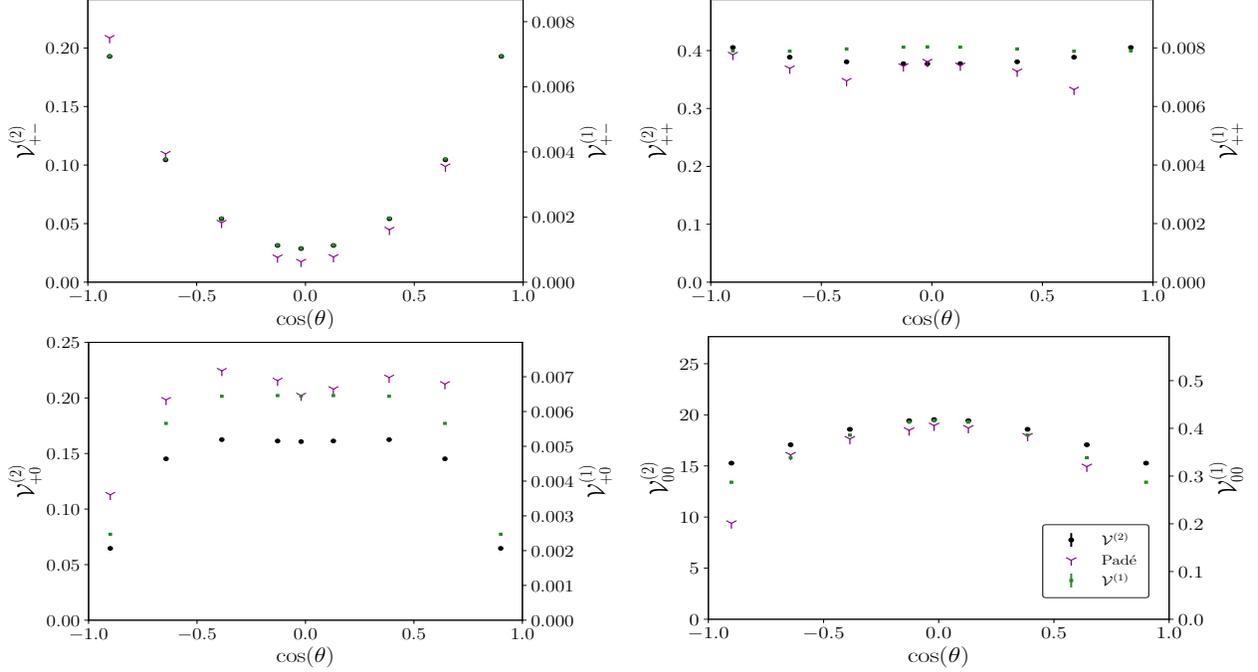


Figure 6.1.5: The  $\cos(\theta)$  dependence of 1-loop and 2-loop interferences for polarised  $ZZ$  production in gluon fusion at  $\sqrt{s}/m_t = 2.331$ . The Padé improved small top-quark mass expansion [12] is shown for comparison. Reproduced from [6].

for even larger values of  $|\cos\theta|$ , away from the center. This is a drastic improvement over the "regular" small top-mass expansion which is highly divergent away from the center.

It must be pointed out that the *relative* level of agreement shown in the above plots between our exact result and the various expansions depends significantly on the exact scheme used for IR subtraction. The comparisons shown in the above plots are done in the “ $q_T$  scheme” [347]. Conversion to Catani’s original subtraction scheme [13] can be performed using

$$A_i^{(1),\text{fin,Catani}} = A_i^{(1),\text{fin}}, \quad (6.1.9)$$

$$A_i^{(2),\text{fin,Catani}} = A_i^{(2),\text{fin}} + \Delta I_1 A_i^{(1),\text{fin}}, \quad (6.1.10)$$

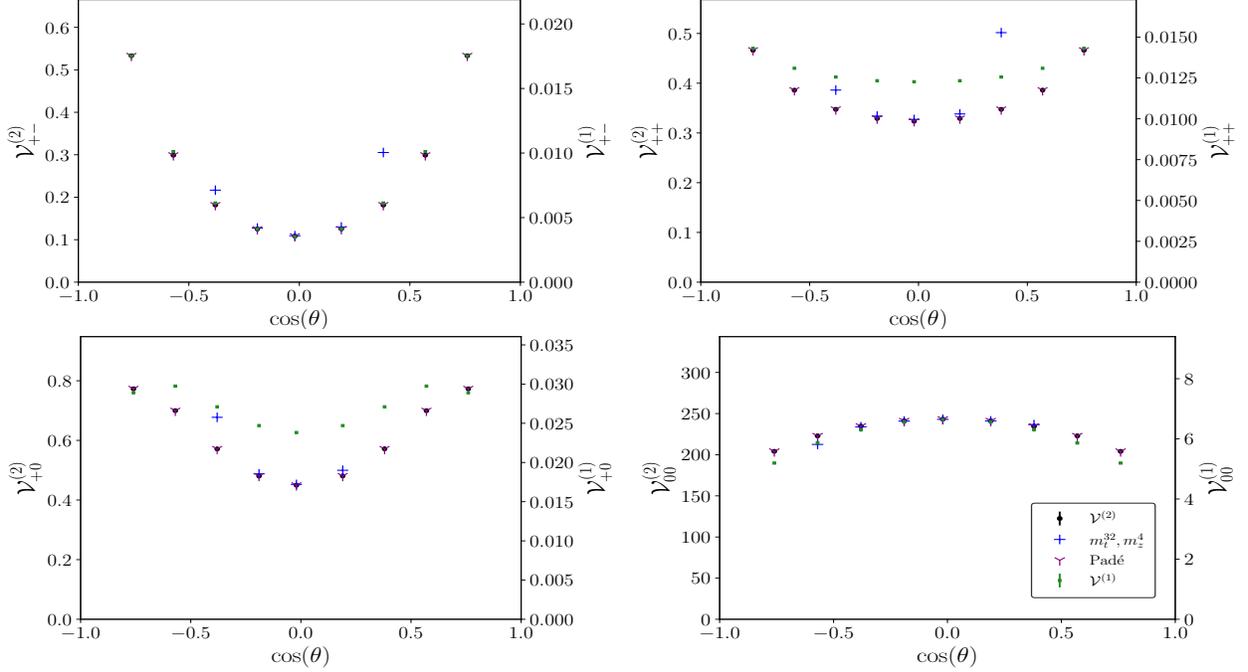


Figure 6.1.6: The  $\cos(\theta)$  dependence of 1-loop and 2-loop interferences for polarised  $ZZ$  production in gluon fusion at  $\sqrt{s}/m_t = 4.703$ . The small top-quark mass expansion (to order  $m_t^{32}$ ) and Padé improved expansion [12] are shown for comparison. Reproduced from [6].

where

$$\Delta I_1 = -\frac{1}{2}\pi^2 C_A + i\pi\beta_0, \quad (6.1.11)$$

Similar transformation rules apply for the helicity amplitudes. This implies that conversion from the “ $q_T$  scheme” to Catani’s original scheme requires subtraction of  $\pi^2 C_A \mathcal{V}^{(1)}$  where the factor of 2 compared to Eq. 6.1.11 comes from interference (see Eq. 6.1.4) and the  $i\pi\beta_0$  term is ignored since it does not contribute to the interference. This corresponds to a difference of  $\sim 30\mathcal{V}^{(1)}$  which can be as large as  $\mathcal{V}^{(2)}$ . Consequently, the 2-loop results can show a very different qualitative behaviour compared to the 1-loop results when using Catani’s original scheme as can be seen in Figs. 6.1.7, 6.1.8, 6.1.9, and 6.1.10. Note that the relative agreement between our results and the expansions is visibly better in the “ $q_T$  scheme” than in Catani’s original scheme.

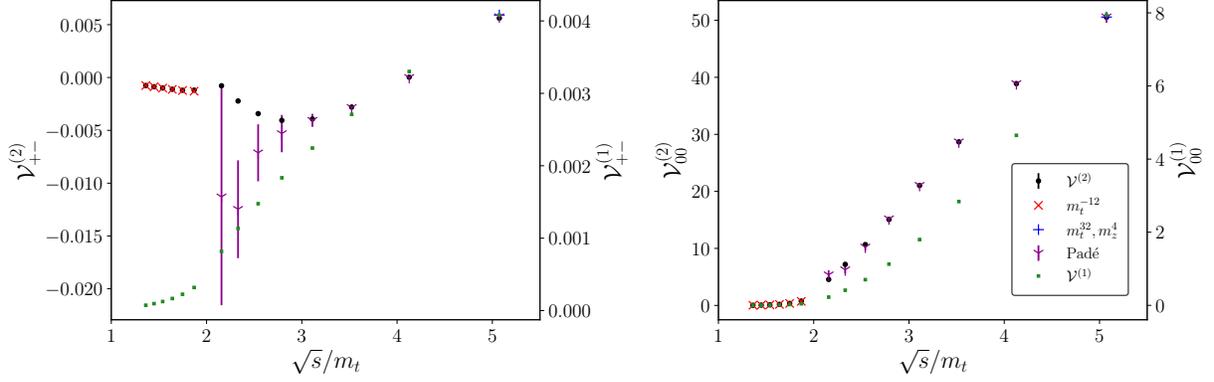


Figure 6.1.7: The  $\sqrt{s}$  dependence of 1-loop and 2-loop interferences for polarised  $ZZ$  production in gluon fusion at  $\cos(\theta) = -0.1286$ . Here, the top left and bottom right panels of Fig. 6.1.3 are reproduced using Catani’s original subtraction scheme [13]. Reproduced from [6].

The above discussion implies that the comparisons shown above are largely scheme dependent, at least looking at relative agreement. As such, to better understand the deviation of the expansions from the exact result and estimate their impact on the physically relevant observables, real radiation contributions also need to be taken into account.

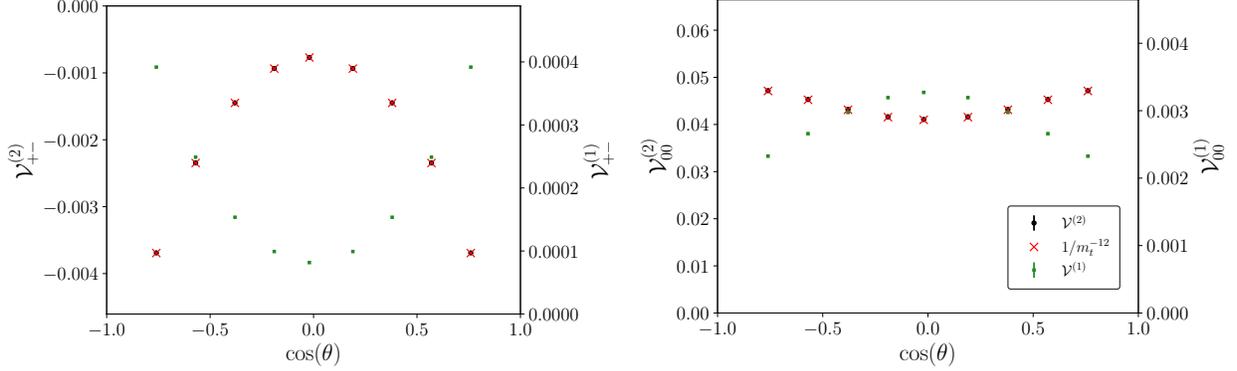


Figure 6.1.8: The  $\cos(\theta)$  dependence of 1-loop and 2-loop interferences for polarised  $ZZ$  production in gluon fusion at  $\sqrt{s}/m_t = 1.426$ . The large top-quark mass expansion [12] (to order  $1/m_t^{12}$ ) is shown for comparison. Here, the top left and bottom right panels of Fig. 6.1.4 are reproduced using Catani’s original subtraction scheme [13]. Reproduced from [6].

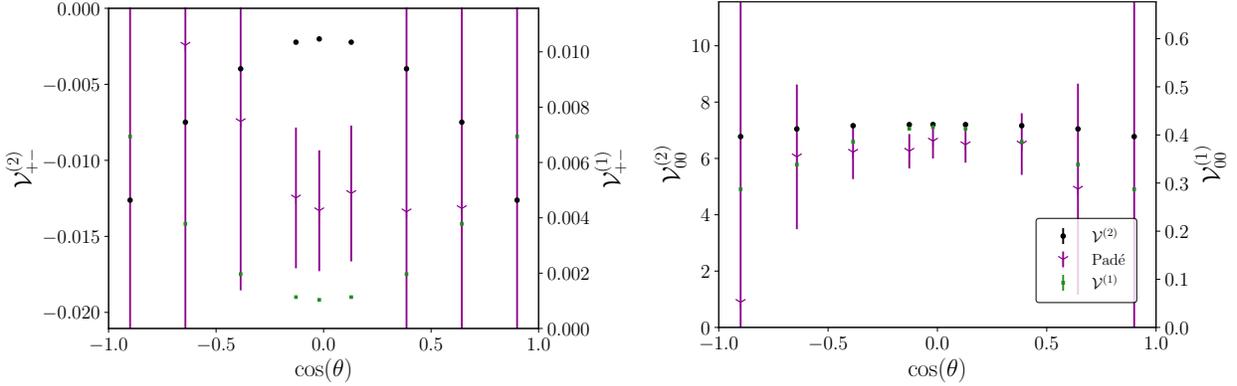


Figure 6.1.9: The  $\cos(\theta)$  dependence of 1-loop and 2-loop interferences for polarised  $ZZ$  production in gluon fusion at  $\sqrt{s}/m_t = 2.331$ . The Padé improved small top-quark mass expansion [12] is shown for comparison. Here, the top left and bottom right panels of Fig. 6.1.5 are reproduced using Catani’s original subtraction scheme [13]. Reproduced from [6].

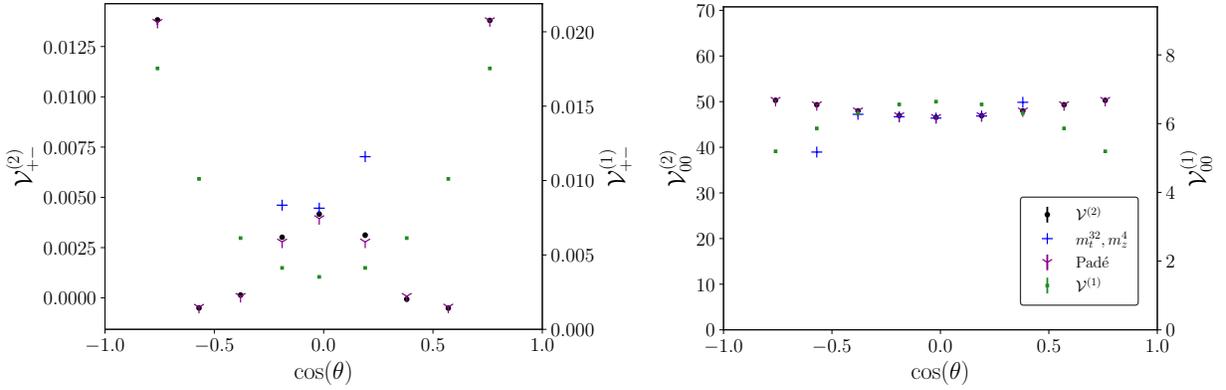


Figure 6.1.10: The  $\cos(\theta)$  dependence of 1-loop and 2-loop interferences for polarised  $ZZ$  production in gluon fusion at  $\sqrt{s}/m_t = 4.703$ . The small top-quark mass expansion (to order  $m_t^{32}$ ) and Padé improved expansion [12] are shown for comparison. Here, the top left and bottom right panels of Fig. 6.1.6 are reproduced using Catani's original subtraction scheme [13]. Reproduced from [6].

# Chapter 7

## Conclusions

In this work, the calculation of the 2-loop corrections to the process  $gg \rightarrow ZZ$  with internal top quarks has been presented. Exact dependence on the mass of the top quark has been kept. The amplitude is represented in terms of a basis of finite master integrals which are evaluated numerically.

To perform the very challenging IBP reductions, the method of syzygies is used to avoid introduction of integrals with dots in the linear systems; a new algorithm to construct such syzygies using linear algebra is employed. The resulting system of equations is then reduced using finite field techniques.

A new algorithm is presented to construct finite integrals as linear combinations with the building blocks being integrals with numerators, higher powers of propagators, dimensionally shifted integrals, and subsector integrals. The resulting parametric integrand for such integrals is integrable for  $\epsilon = 0$  and admits an expansion in  $\epsilon$  allowing for numerical integration. This approach allows the construction of finite integrals considered more natural since the constituent integrals in the linear combinations appear in the amplitudes already. To evaluate the master integrals numerically, `pySecDec`, based on the method of sector decomposition, is employed. Choice of such finite integrals is observed to significantly improve the numerical performance.

Results are provided for our 2-loop amplitudes along with comparisons against various approximations. Good agreement is observed with the large top-mass and small top-mass expansions in the regions where the approximations are expected to perform well. In comparison to the regular small top-mass expansion, the Padé improved small top-mass expansion is found to work over a significantly larger region of phase space, in particular for moderate energies and/or non-central scattering. The amplitudes presented in this paper provide the building block required to include the full top-quark mass effects in the next-to-leading order cross section for  $ZZ$  production in gluon fusion.

# APPENDICES

# APPENDIX A

## QCD Feynman Rules

### A.1 QCD Lagrangian

QCD Lagrangian:

$$\mathcal{L} = \bar{\psi}^i (i\gamma^\mu D_\mu^{ij} - \delta^{ij} m) \psi^j - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} - \frac{1}{2\xi} (\partial^\mu G_\mu^a)^2 + (\partial^\mu c^{*a}) D_\mu^{ab} c^b. \quad (\text{A.1})$$

- $\psi^i$  : Quark field with colour index  $i$  in fundamental representation
- $G_{\mu\nu}^a$  : Gluon field strength tensor with colour index  $a$  in the adjoint representation
- $c^a$  : Faddeev-Popov ghost field with colour index  $a$  in the adjoint representation
- $D_\mu^{ij} = \delta^{ij} \partial_\mu - ig_s T^{a,ij} G_\mu^a$  : Covariant derivative in the fundamental representation
- $D_\mu^{ab} = \delta^{ab} \partial_\mu - g_s f^{abc} G_\mu^c$  : Covariant derivative in the adjoint representation
- $T^{a,ij}$  : SU(N) generator
- $f^{abc}$  : Structure functions

Split the Lagrangian into free and interacting part:

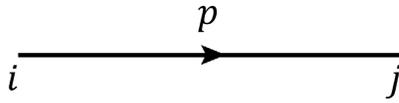
$$\begin{aligned} \mathcal{L}_{free} = & \bar{\psi}^i (i\gamma^\mu \partial_\mu - m) \psi^i - \frac{1}{4} (\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) - \underbrace{\frac{1}{2\xi} (\partial^\mu G_\mu^a) (\partial^\nu G_\nu^a)}_{\text{Gauge fixing term}} \\ & + \underbrace{(\partial^\mu c^{*a}) (\partial_\mu c^a)}_{\text{Ghost field}}, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned}
\mathcal{L}_{int} = & \underbrace{g_s T^{a,ij} \bar{\psi}^i G^a \psi^j}_{\text{Fermion-Gauge term}} - \underbrace{\frac{g_s}{2} f^{abc} (\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) G_\mu^b G_\nu^c}_{\text{3-point Gauge interaction}} + \underbrace{\frac{g_s^2}{4} f^{abe} f^{cde} G^{a,\mu} G^{b,\nu} G_\mu^c G_\nu^d}_{\text{4-point Gauge interaction}} \\
& - \underbrace{g_s f^{abc} (\partial^\mu c^{*a}) c^b G_\mu^c}_{\text{Gauge-Ghost interaction}}.
\end{aligned} \tag{A.3}$$

## A.2 Feynman rules

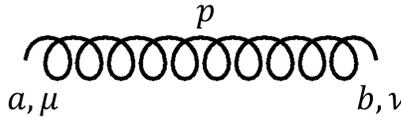
### QCD Propagators:

1. Quark



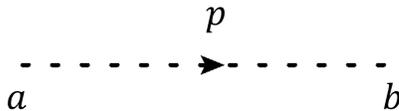
$$i \frac{\delta_{ij}}{\not{p} - m}$$

2. Gluon



$$-i \frac{\delta^{ab}}{p^2} \left( g^{\mu\nu} + (\xi - 1) \frac{p^\mu p^\nu}{p^2} \right)$$

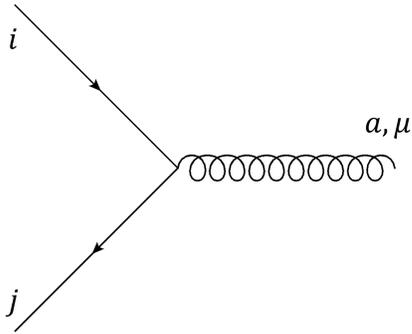
3. Faddeev-Popov Ghost



$$i \frac{\delta_{ab}}{p^2}$$

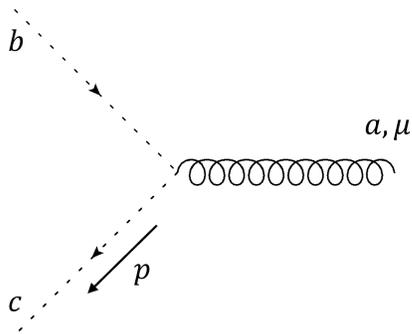
### QCD Vertices:

1. Quark-gluon vertex



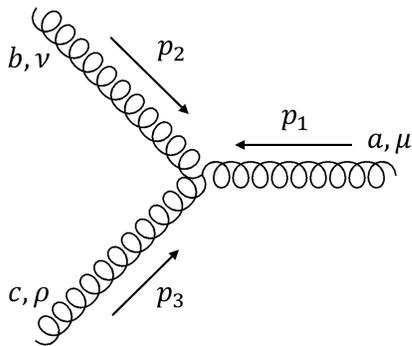
$$i g_s T_{ij}^a \gamma^\mu$$

2. Ghost-gluon vertex



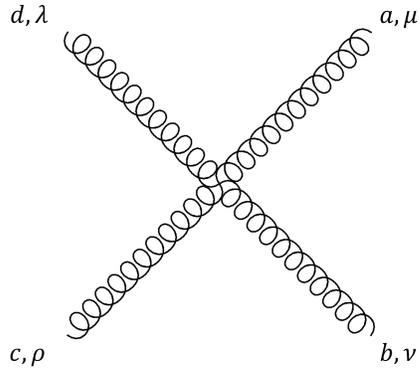
$$g_s f^{abc} p^\mu$$

3. 3-gluon vertex



$$g_s f^{abc} (g^{\mu\nu} (p_1 - p_2)^\rho + g^{\nu\rho} (p_2 - p_3)^\mu + g^{\mu\rho} (p_3 - p_1)^\nu)$$

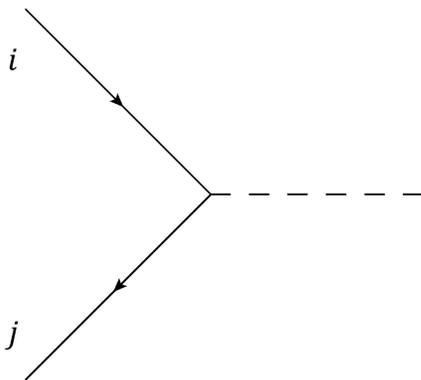
4. 4-gluon vertex



$$\begin{aligned}
 & -ig_s^2 \left[ f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho}) \right. \\
 & \quad + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\lambda} - g^{\mu\lambda} g^{\nu\rho}) \\
 & \quad \left. + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\lambda} - g^{\mu\rho} g^{\nu\lambda}) \right]
 \end{aligned}$$

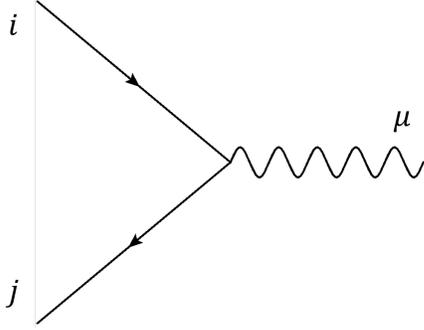
**Electroweak vertices relevant for  $gg \rightarrow ZZ$ :**

1. Higgs-top vertex



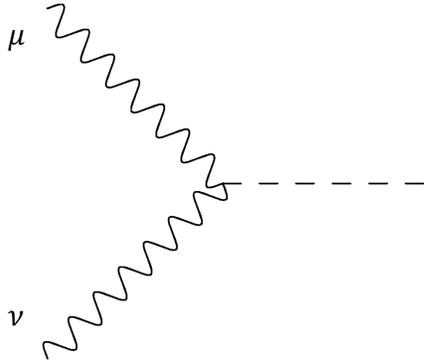
$$i\delta_{ij} \frac{m}{v}$$

2. Z-quark vertex



$$i \frac{e}{2 \sin \theta_W \cos \theta_W} \gamma_\mu (v_t + a_t \gamma_5)$$

### 3. Higgs-Z vertex



$$i \frac{e}{\sin \theta_W \cos \theta_W} m_Z g^{\mu\nu}$$

## A.3 SU(N) algebra

For the  $SU(N)$  group, the Lie generators are defined using the commutation relation

$$[T^a, T^b] = -F_{bc}^a T^c, \quad (\text{A.4})$$

where the structure constants  $F_{bc}^a$  themselves satisfy the commutation relation

$$[F_{de}^a, F_{ef}^b] = -F_{bc}^a F_{df}^c. \quad (\text{A.5})$$

$T^a$  are the generators in the fundamental representation while  $F_{bc}^a$  are the generators in the adjoint representation. The generators  $T^a$  are normalised according to

$$\text{Tr} (T^a T^b) = T_F \delta^{ab} \quad (\text{A.6})$$

where  $T_F = 1/2$  by convention. The quadratic Casimir  $C_r$  invariant for the representation  $r$  is defined as

$$\mathbf{T}^a \mathbf{T}^a = C_r \mathbf{1}_n \quad (\text{A.7})$$

with  $n$  being the dimensions of the representation  $r$ . For the fundamental representation,

$$C_F = \frac{N^2 - 1}{2N}. \quad (\text{A.8})$$

This is easily derived by taking the trace of Eq. A.7.

The generators in the adjoint representation can be written in terms of the generators in fundamental representation as

$$F_{bc}^a = -2 \text{Tr} ([T^a, T^b] T^c). \quad (\text{A.9})$$

This can be used, along with the Fierz identity

$$T_{ij}^a T_{kl}^a = \frac{1}{2} \left( \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right), \quad (\text{A.10})$$

to write down trace relation for adjoint representation similar to Eq. A.6:

$$\text{Tr} (F^a F^b) = N \delta^{ab}. \quad (\text{A.11})$$

The Casimir invariant in the adjoint representation is

$$C_A = N \tag{A.12}$$

derived simply by taking the trace of  $F^a F^a = C_A \mathbf{1}_n$  with  $n = N^2 - 1$  and Eq. A.11.

An explicit representation for the generators  $T^a$  can be written in terms of the Gell-Mann matrices  $T^a = \lambda_a/2$  where the Gell-Mann matrices are

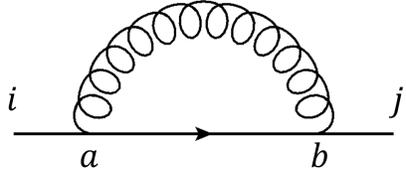
$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

The adjoint generators  $F^a$  can be written in terms of the totally antisymmetric structure constants  $f^{abc}$

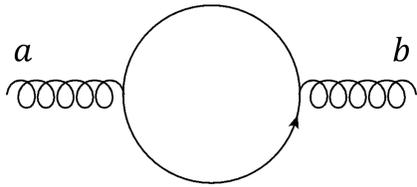
$$F_{bc}^a = -i f^{abc}. \tag{A.13}$$

Explicit values of  $f^{abc}$  for a given representation can be determined using Eq. A.9.

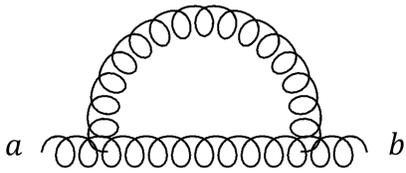
## A.4 Colour factors of some simple diagrams



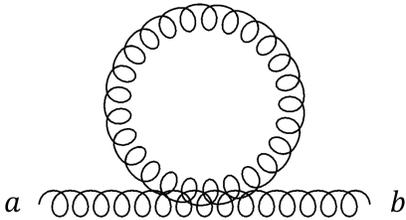
$$\mathcal{C} = T_{ik}^a \delta_{kl} T_{lj}^b \delta^{ab} = T_{il}^a T_{lj}^a = C_F \delta_{ij}$$



$$\mathcal{C} = T_{ik}^a \delta_{kl} T_{lj}^b \delta_{ji} = T_{il}^a T_{li}^b = \text{Tr}(T^a T^b) = T_F \delta^{ab}$$

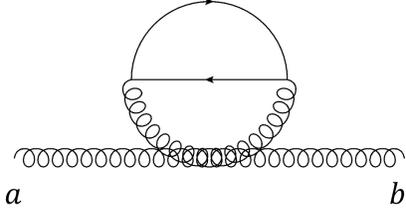


$$\mathcal{C} = f^{acd} f^{bcd} = \text{Tr}(F^a F^b) = C_A \delta^{ab}$$

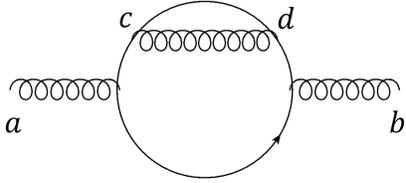


$$\mathcal{C} = f^{ace} f^{bce} = \text{Tr}(F^a F^b) = C_A \delta^{ab}$$

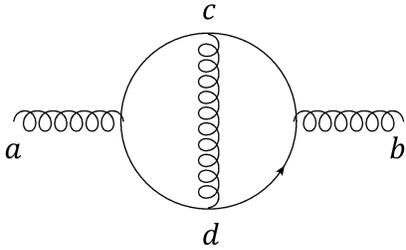
At 2-loops, considering diagrams with 1 closed fermion loop:



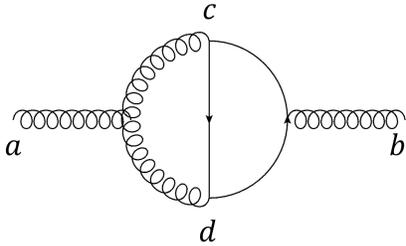
$$\mathcal{C} = f^{acd} f^{bcd} \times T_F = \text{Tr} (F^a F^b) T_F = T_F C_A \delta^{ab}$$



$$\begin{aligned} \mathcal{C} &= \text{Tr} (T^a T^c T^d T^b) \delta^{cd} = \text{Tr} (T^a T^c T^c T^b) \\ &= \text{Tr} (T^a T^b) C_F = T_F C_F \delta^{ab} \end{aligned}$$



$$\begin{aligned} \mathcal{C} &= \text{Tr} (T^a T^c T^b T^d) \delta^{cd} = \text{Tr} (T^a T^c T^b T^c) \\ &= -\frac{1}{2N} T_F \delta^{ab} = T_F \left( C_F - \frac{1}{2} C_A \right) \delta^{ab} \end{aligned}$$



$$\begin{aligned} \mathcal{C} &= \text{Tr} (T^b T^c T^d) f^{acd} = i \frac{1}{4N} \delta^{ab} \\ &= -i T_F \left( C_F - \frac{1}{2} C_A \right) \delta^{ab} \end{aligned}$$

# APPENDIX B

## Evaluation using Feynman parameters

### B.1 Massive tadpole

Consider the integral in Minkowski space

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 - m^2 + i0}. \quad (\text{B.1})$$

This integral is dependent on  $k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$ . To perform the integration in, Wick rotation (Fig. B.1) can be used to transform Minkowski space into Euclidean space using  $k_0 \rightarrow ik_{0,E}$  with spatial components remaining unchanged. Thus, the integral over  $k_0$  can be replaced by the integral over  $k_{0,E}$  as

$$\int_{-\infty}^{\infty} dk_0 F(k_0) \rightarrow i \int_{-\infty}^{\infty} dk_{0,E} F(ik_{0,E}). \quad (\text{B.2})$$

This is easy to see using Cauchy's theorem; the integrals over the arcs in Fig. B.1 can be shown to vanish near infinity leaving the above relation. Note that the  $i0$  prescription is essential to ensure that the poles lie outside the integration contour. The integral over loop momentum then becomes

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 - m^2 + i0} = i \int \frac{d^d k_E}{i\pi^{d/2}} \frac{1}{-k_E^2 - m^2 + i0} \quad (\text{B.3})$$

where  $k_E^2 = k_{0,E}^2 + k_{1,E}^2 + k_{2,E}^2 + k_{3,E}^2$ . The  $d$  dimensional Euclidean integral above can be written in spherical coordinates using

$$d^d k_E = k_E^{d-1} dk_E d\Omega_d. \quad (\text{B.4})$$

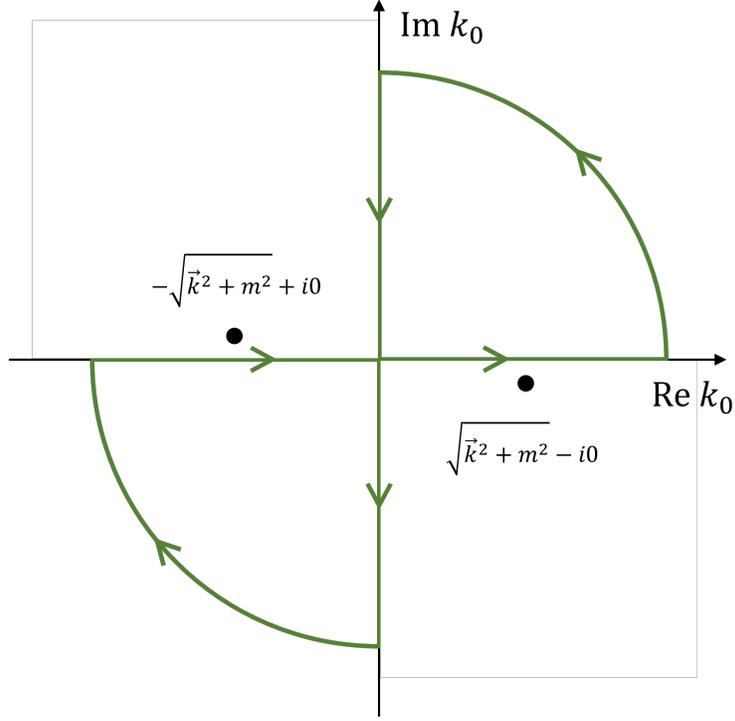


Figure B.1: The integration contour to perform Wick rotation. Note that the poles lie outside the contour.

Integral over the  $d$  dimensional solid angle gives

$$\int d\Omega_d = \left( \prod_{i=1}^{d-2} \int_0^\pi d\theta_i \sin^{d-1-i} \theta_i \right) \int_0^{2\pi} d\theta_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (\text{B.5})$$

The remaining integral in  $k_E$  is straightforward:

$$\int_0^\infty k_E^{d-1} dk_E \frac{1}{k_E^2 + m^2} = \frac{1}{2} m^{d-2} \Gamma\left(1 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right). \quad (\text{B.6})$$

Combining the two, the integral gives

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 - m^2 + i0} = -m^{d-2} \Gamma\left(1 - \frac{d}{2}\right). \quad (\text{B.7})$$

Note that the integral converges only for  $\text{Re}(d) < 2$ . Using  $d = 4 - 2\epsilon$  and expanding around  $\epsilon = 0$ ,

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 - m^2 + i0} = \frac{1}{\epsilon} m^2 + m^2(1 - \gamma_E - \log m^2) + O(\epsilon) \quad (\text{B.8})$$

where  $\gamma_E$  is the Euler-Mascheroni constant. It is common to remove these factors of  $\gamma_E$  appearing in  $\epsilon$  expansions of Feynman integrals by multiplying the integrals with  $e^{\epsilon\gamma_E}$ . Note that as expected, the divergence shows up as the  $1/\epsilon$  pole.

Evaluating the tadpole integral with the propagator raised to power 3 instead gives

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 - m^2 + i0)^3} = -\frac{1}{8}(d-4)(d-2)m^{d-6}\Gamma\left(1 - \frac{d}{2}\right) \quad (\text{B.9})$$

which when expanded gives

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 - m^2 + i0)^3} = -\frac{1}{2m^2} + \frac{1}{2m^2}(\gamma_E + \log m^2)\epsilon + O(\epsilon^2). \quad (\text{B.10})$$

Simple power counting shows that this integral should be finite in  $d = 4$  as seen above.

## B.2 Massless bubble

The massless bubble integral from Fig. 3.2.1 (without cuts) can be written as

$$I = e^{\epsilon\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2 + i0)((k-p)^2 + i0)}. \quad (\text{B.11})$$

Applying Feynman's trick, this becomes

$$I = \int_0^1 dx \int \frac{d^d k}{i\pi^{d/2}} \frac{e^{\epsilon\gamma_E}\Gamma(2)}{((1-x)k^2 + x(k-p))^2} = \int_0^1 dx \int \frac{d^d k}{i\pi^{d/2}} \frac{e^{\epsilon\gamma_E}\Gamma(2)}{(k^2 + 2xk \cdot p + xp^2 + i0)^2}. \quad (\text{B.12})$$

Completing the square  $(k + xp)^2$  gives

$$I = \int_0^1 dx \int \frac{d^d k}{i\pi^{d/2}} \frac{e^{\epsilon\gamma_E} \Gamma(2)}{((k + xp)^2 - (\Delta - i0))^2} = \int_0^1 dx \int \frac{d^d k}{i\pi^{d/2}} \frac{e^{\epsilon\gamma_E} \Gamma(2)}{(k^2 - \Delta + i0)^2} \quad (\text{B.13})$$

where  $\Delta = -p^2 x(1 - x)$ . Integrating over the loop momentum, we get

$$\begin{aligned} I &= e^{\epsilon\gamma_E} \Gamma(2) \Gamma(2 - d/2) \int_0^1 dx \frac{1}{(\Delta - i0)^{\frac{4-d}{2}}} \\ &= e^{\epsilon\gamma_E} \Gamma(2) \Gamma(2 - d/2) \int_0^1 dx \frac{1}{(-p^2 - i0)^{\frac{4-d}{2}} (x - x^2)^{\frac{4-d}{2}}}. \end{aligned} \quad (\text{B.14})$$

The integral over  $x$  converges for  $\text{Re}(d) > 2$ . Integrating, and expanding in  $\epsilon$ , we get

$$I = \frac{1}{\epsilon} + (2 - \log(-p^2 - i0)) + O(\epsilon). \quad (\text{B.15})$$

# APPENDIX C

## Dirac algebra and $\gamma^5$ schemes

### C.1 Gamma matrices and identities

Fermions in a free theory satisfy the Dirac equation

$$\bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi = 0. \quad (\text{C.1})$$

$\gamma^\mu$  are referred to as the Dirac or Gamma matrices. They generate a Clifford algebra and satisfy the following anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\}_+ = 2\eta^{\mu\nu} \mathbb{1}_d. \quad (\text{C.2})$$

where  $\mathbb{1}_d$  is the  $d$ -dimensional identity matrix and  $\eta^{\mu\nu}$  is the metric tensor for Minkowski space. An explicit representation can be written for the Gamma matrices in 4 dimensions in the Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \quad (\text{C.3})$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma_i & 0 \end{pmatrix}, \quad (\text{C.4})$$

for  $i = 1, 2, 3$ . Here  $\sigma^i$  are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{C.5})$$

The Gamma matrices satisfy the following identities in  $d$ -dimensions following from the anti-commutation rule in Eq. C.2

$$\gamma^\mu \gamma_\mu = d \mathbf{1}_d, \quad (\text{C.6})$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = (2 - d) \gamma^\nu, \quad (\text{C.7})$$

$$\gamma^\mu \gamma^\nu \gamma_\rho \gamma_\mu = 2 \eta^{\nu\rho} - (2 - d) \gamma^\rho \gamma^\nu, \quad (\text{C.8})$$

as well as the following trace identities

$$\text{Tr}(\gamma^\mu) = 0, \quad (\text{C.9})$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4 \eta^{\mu\nu}, \quad (\text{C.10})$$

$$\text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_m}) = \text{Tr}(\gamma^{\mu_m} \dots \gamma^{\mu_1}), \quad (\text{C.11})$$

$$\text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_{2m+1}}) = 0, \quad (\text{C.12})$$

$$\text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_{2m}}) = 4 \sum_{perms} \mathbf{Sgn}(\sigma) g_{\mu_{i_1} \mu_{j_1}} \dots g_{\mu_{i_n} \mu_{j_n}} \quad (\text{C.13})$$

with  $1 = i_1 < \dots < i_n$  and  $i_k < j_k$ .  $\sigma$  refers to permutation of the indices  $i_1, j_1, \dots, i_n, j_n$  and  $\text{Sgn}(\sigma)$  is the sign of the permutation.

## C.2 $\gamma^5$ in $d$ dimensions

In  $d = 4$  dimensions, we can define a fifth Gamma matrix  $\gamma^5$  as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \quad (\text{C.14})$$

An explicit representation in the Dirac basis is given by

$$\gamma^5 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}. \quad (\text{C.15})$$

$\gamma^5$  can also be defined using the totally anti-symmetric Levi-Civita tensor  $\epsilon^{\mu\nu\rho\sigma}$

$$\gamma^5 = \frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma, \quad (\text{C.16})$$

which leads to the trace identity

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma} \quad (\text{C.17})$$

in  $d = 4$ . An anti-commutation relation follows from Eq. C.2 and C.14,

$$\{\gamma^\mu, \gamma^5\}_+ = 0, \quad (\text{C.18})$$

as well as the identity

$$(\gamma^5)^2 = \mathbf{1}_4. \quad (\text{C.19})$$

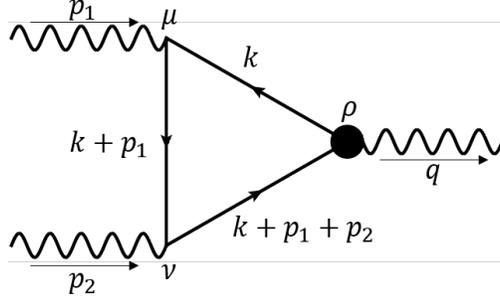


Figure C.1: The triangle anomaly graph. Here the dark blob vertex represents the axial-vector coupling.

Since the anti-symmetric tensor  $\epsilon^{\mu\nu\rho\sigma}$  is a purely 4-dimensional object,  $\gamma^5$  and, as a consequence, the trace identity in Eq. C.17 are not defined for  $d \neq 4$ . Dimensional regularisation is one of the most popular methods to perform higher-order calculations. It preserves gauge invariance which makes it highly attractive and simpler to implement. However, for processes involving parity violating interactions, the naive regularisation scheme with the 4-dimensional  $\gamma^5$  cannot be used.

There are several methods to "extend" the trace relation of Eq. C.17 to  $d$  dimensions. For a consistent definition of  $\gamma^5$  in  $d$ -dimensions, both the cyclicity of the trace and Eq. C.18 cannot be preserved [68, 69]. This can be seen most easily through the triangle anomaly graph in Fig. C.1. The sum of the above diagram and its crossed diagram, where the incoming legs with momenta  $p_1, p_2$  are exchanged, can be written as

$$I^{\mu\nu\rho} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\text{Tr}(\gamma^\rho \gamma^5 \not{k} \gamma^\mu (\not{k} + \not{p}_1) \gamma^\nu (\not{k} + \not{p}_1 + \not{p}_2))}{k^2 (k + p_1)^2 (k + p_1 + p_2)^2} + [p_1 \leftrightarrow p_2, \mu \leftrightarrow \nu]. \quad (\text{C.20})$$

It is straightforward to show that assuming cyclicity of trace along with the anti-commuting  $\gamma^5$  (Eq. C.18) implies that the above integral, vanishes. This is of course incorrect and demonstrates the erroneous approach to evaluating the anomaly. To appropriately deal with this issue, there are two major classes of schemes:

1. Schemes where the anti-commutation relation in Eq. C.18 is violated in favour

of preserving cyclicity of trace e.g. the HVBM scheme [213, 214, 215, 68] and the Larin scheme [348, 349].

2. Schemes where the anti-commutation relation in Eq. C.18 is preserved while cyclicity of the trace is violated [210, 211, 212]

### C.3 Anti-commuting $\gamma^5$ scheme

In Kreimer's anti-commuting  $\gamma^5$  scheme, the cyclicity of traces is not preserved. This also implies that the traces need to be read from a specific point and the result in general depends on the "reading point", which is chosen to be the axial-vector vertex so as to conserve vector currents. All diagrams, and their traces, must be read from the same reading point and in case of multiple  $\gamma^5$ 's, the traces need to be symmetrised.

Assuming that trace is not cyclic, the triangle anomaly above can be calculated using the following trace identities concerning  $\gamma^5$  in addition to the trace identities in the previous section for the "regular"  $\gamma$ -matrices:

$$\text{Tr}(\gamma^5) = 0, \tag{C.21}$$

$$\text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_{2m+1}} \gamma^5) = 0, \tag{C.22}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^5) = 0, \tag{C.23}$$

$$\text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_4} \gamma^5) = 4i \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4}, \tag{C.24}$$

$$\text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_m} \gamma^5) = \text{Tr}(\gamma^{\mu_m} \dots \gamma^{\mu_1} \gamma^5), \tag{C.25}$$

$$\text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_{2m}} \gamma^5) = 4i \sum_{\sigma} \mathbf{Sgn}(\sigma) \epsilon^{\mu_{i_{n+1}} \mu_{i_{n+2}} \mu_{j_{n+1}} \mu_{j_{n+2}}} g_{\mu_{i_1} \mu_{j_1}} \dots g_{\mu_{i_{n+2}} \mu_{j_{n+2}}} \tag{C.26}$$

with  $1 = i_1 < \dots < i_{n+2}$  and  $i_k < j_k$ .  $\sigma$  refers to permutation of the indices  $i_1, j_1, \dots, i_{n+2}, j_{n+2}$  and  $\mathbf{Sgn}(\sigma)$  is the sign of the permutation. Note that any contractions with the  $\epsilon$ -tensor are treated in 4-dimensions since the  $\epsilon$ -tensor is a 4-dimensional object.

Above trace rules along with the requirement of reading traces from the same vertex in each diagram. Applying the above rules, the correct result for the anomalous triangle

diagram can be recovered.

# APPENDIX D

## UV renormalisation

### D.1 Renormalised Lagrangian

The bare Lagrangian can be written as

$$\begin{aligned}
\mathcal{L}_{bare} = & \bar{\psi}_0^i (i\gamma^\mu \partial_\mu - m_0) \psi_0^i - \frac{1}{4} (\partial^\mu G_0^{a,\nu} - \partial^\nu G_0^{a,\mu}) (\partial_\mu G_{0,\nu}^a - \partial_\nu G_{0,\mu}^a) - \frac{1}{2\xi} (\partial^\mu G_{\mu,0}^a) (\partial^\nu G_{\nu,0}^a) \\
& + (\partial^\mu c_0^{*a}) (\partial_\mu c_0^a) + g_{S,0} T^{a,ij} \bar{\psi}_0^i \gamma^\mu G_{0,\mu}^a \psi_0^j - \frac{g_{S,0}}{2} f^{abc} (\partial^\mu G_0^{a,\nu} - \partial^\nu G_0^{a,\mu}) G_{0,\mu}^b G_{0,\nu}^c \\
& + \frac{g_{S,0}^2}{4} f^{abe} f^{cde} G_0^{a,\mu} G_0^{b,\nu} G_{0,\mu}^c G_{0,\nu}^d - g_{S,0} f^{abc} (\partial^\mu c_0^{*a}) c_0^b G_{0,\mu}^c
\end{aligned} \tag{D.1}$$

where all the bare parameters and fields are denoted by the subscript "0" e.g.  $G_{0,\mu}^a$ . The bare parameters can be replaced by the *renormalised* parameters:

$$\begin{aligned}
\psi_0 &= \sqrt{Z_\psi} \psi_R \\
G_0^{a,\mu} &= \sqrt{Z_G} G_R^{a,\mu} \\
c_0^a &= \sqrt{Z_c} c_R^a \\
m_0 &= Z_m m_R \\
g_{S,0} &= Z_g g_{S,R} \\
\alpha_{S,0} &= Z_{\alpha_S} \alpha_{S,R} \\
\xi_0 &= Z_\xi \xi
\end{aligned} \tag{D.2}$$

where  $Z_\psi$  etc. are referred to as renormalisation constants. In what follows, the  $R$  subscript has been dropped for the sake of brevity and any quantity without the subscript 0 is understood to be renormalised.

The renormalisation constants can be expanded order-by-order in  $\alpha_S/(2\pi)$  e.g.

$$Z_\psi = 1 + \frac{\alpha_S}{2\pi} \delta Z_\psi + O(\alpha_S^2) \quad (\text{D.3})$$

keeping terms only up to  $O(\alpha_S)$ . For brevity, it is useful to define

$$\begin{aligned} \delta_\psi &= \alpha_S/(2\pi) \delta Z_\psi, & \delta_G &= \alpha_S/(2\pi) \delta Z_G, & \delta_c &= \alpha_S/(2\pi) \delta Z_c \\ \delta_m &= \alpha_S/(2\pi) \delta Z_m, & \delta_g &= \alpha_S/(2\pi) \delta Z_g, & \delta_{\alpha_S} &= \alpha_S/(2\pi) \delta Z_{\alpha_S}. \end{aligned} \quad (\text{D.4})$$

Replacing the bare parameters in the Lagrangian with the renormalised parameters and expanding the renormalisation constants gives

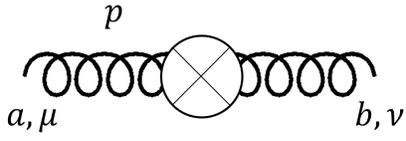
$$\mathcal{L} = \mathcal{L}_{ren} + \mathcal{L}_{ct} \quad (\text{D.5})$$

where

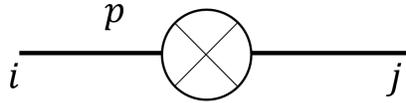
$$\begin{aligned} \mathcal{L}_{ren} &= \bar{\psi}^i (i\gamma^\mu \partial_\mu - m) \psi^i - \frac{1}{4} (\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) - \frac{1}{2\xi} (\partial^\mu G_\mu^a) (\partial^\nu G_\nu^a) \\ &\quad + (\partial^\mu c^{*a}) (\partial_\mu c^a) + g_S T^{a,ij} \bar{\psi}^i \gamma^\mu G_\mu^a \psi^j - \frac{g_S}{2} f^{abc} (\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) G_\mu^b G_\nu^c \\ &\quad + \frac{g_S^2}{4} f^{abe} f^{cde} G^{a,\mu} G^{b,\nu} G_\mu^c G_\nu^d - g_S f^{abc} (\partial^\mu c^{*a}) c^b G_\mu^c, \\ \mathcal{L}_{ct} &= \bar{\psi}^i (i(\delta_\psi) \gamma^\mu \partial_\mu - (\delta_\psi + \delta_m) m) \psi^i - \frac{1}{4} \delta_G (\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) \\ &\quad - \frac{1}{2\xi} \delta_G (\partial^\mu G_\mu^a) (\partial^\nu G_\nu^a) + \delta_c (\partial^\mu c^{*a}) (\partial_\mu c^a) + (\delta_\psi + \delta_g + 1/2\delta_G) g_S T^{a,ij} \bar{\psi}^i \gamma^\mu G_\mu^a \psi^j \\ &\quad - (\delta_g + 3/2\delta_G) \frac{g_S}{2} f^{abc} (\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) G_\mu^b G_\nu^c + 2(\delta_g + \delta_G) \frac{g_S^2}{4} f^{abe} f^{cde} G^{a,\mu} G^{b,\nu} G_\mu^c G_\nu^d \\ &\quad - (\delta_g + \delta_c + 1/2\delta_G) g_S f^{abc} (\partial^\mu c^{*a}) c^b G_\mu^c. \end{aligned} \quad (\text{D.6})$$

## D.2 1-loop counterterms

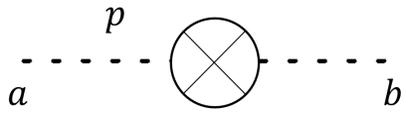
From the above Lagrangian, counterterm diagrams can be written as follows:



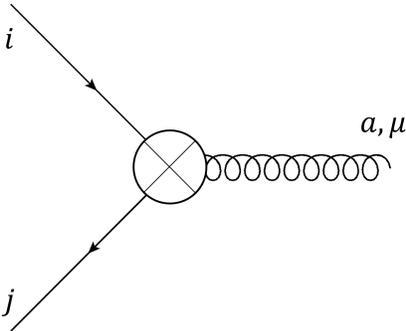
$$i\delta_G \delta^{ab} (p^\mu p^\nu - g^{\mu\nu} p^2)$$



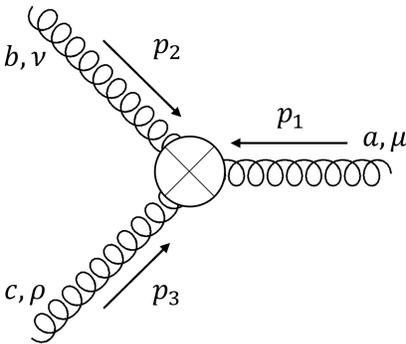
$$i\delta_{ij} (\delta_\psi \not{p} - (\delta_\psi + \delta_m) m)$$



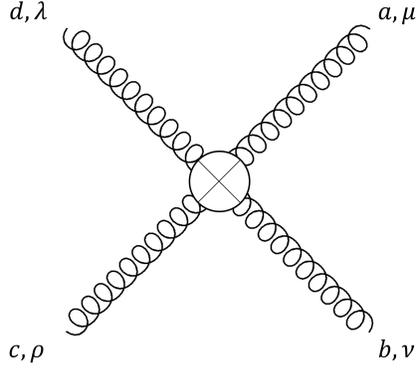
$$i\delta_c \delta^{ab} p^2$$



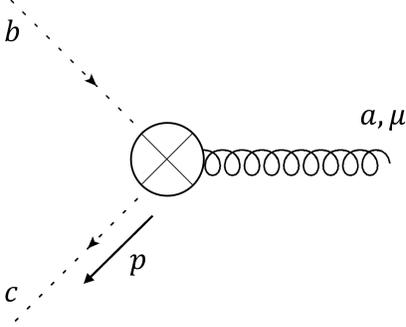
$$i(\delta_\psi + \delta_g + \frac{1}{2}\delta_G) g_s T_{ij}^a \gamma_\mu$$



$$(\delta_g + \frac{3}{2}\delta_G) g_s f^{abc} \left[ g^{\mu\nu} (p_1 - p_2)^\rho + g^{\nu\rho} (p_2 - p_3)^\mu + g^{\mu\rho} (p_3 - p_1)^\nu \right]$$



$$-i(2\delta_g + 2\delta_G) g_s^2 \left[ f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho}) \right. \\ \left. + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\lambda} - g^{\mu\lambda} g^{\nu\rho}) \right. \\ \left. + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\lambda} - g^{\mu\rho} g^{\nu\lambda}) \right]$$



$$(\delta_c + \delta_g + \frac{1}{2}\delta_G) g_s f^{abc} p^\mu \gamma_\mu$$

The counterterms can be determined by requiring that the divergences in the 1-loop contributions are cancelled by the counterterm vertices. E.g. for gluon field renormalisation, all the diagrams contributing to the 2-point function at 1-loop and the 1-loop counterterm must sum up to be finite as shown in Fig. D.1. Note that in principle, any arbitrary finite term can be added to the counterterm without affecting the divergent piece. This allows a lot of freedom in choosing the exact form of the counterterms. There are multiple renormalisation "schemes" that differ in the choice of these finite terms. The most popular is the so-called modified minimal subtraction ( $\overline{MS}$ ) scheme. In this scheme, an additional  $-\gamma_E + \ln 4\pi$  is added to the counterterm.

Consider the 1-loop corrections to the gluon propagator in Fig. D.1. The sum of all the

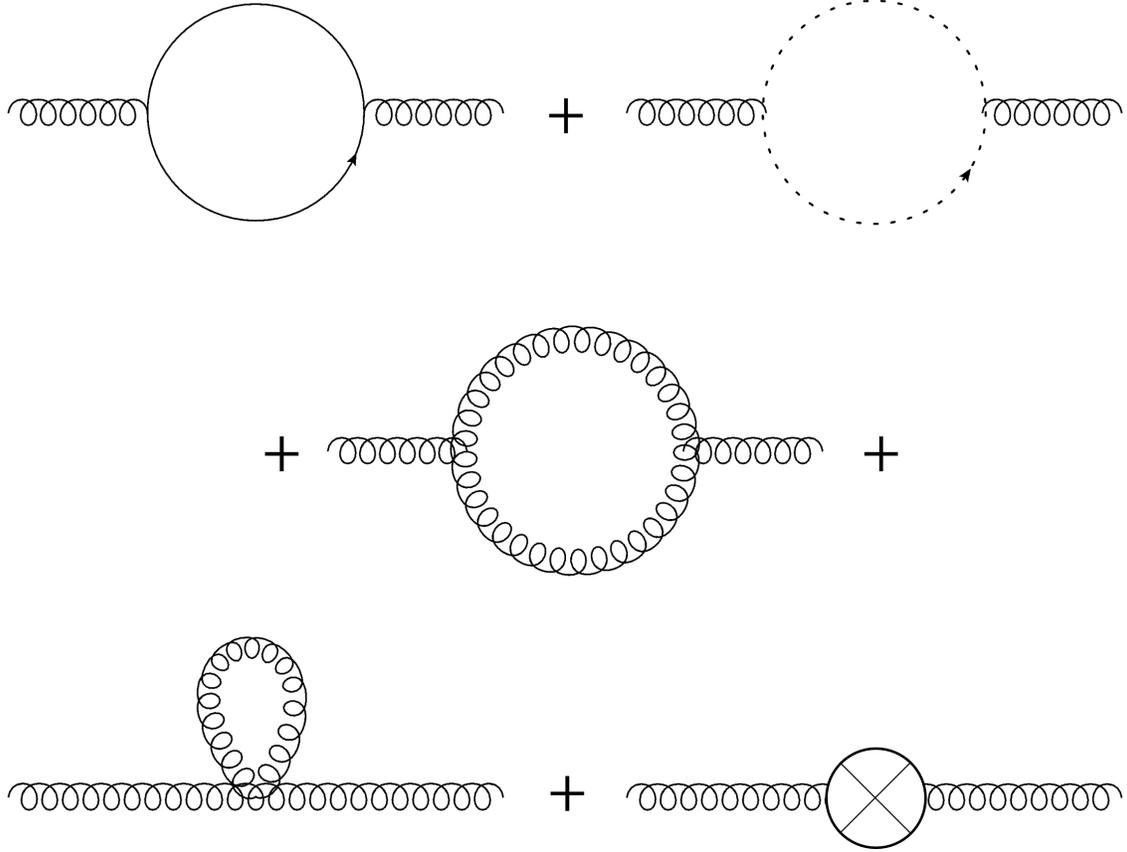


Figure D.1: All the diagrams contributing to 1-loop correction to the gluon propagator, including the counterterm diagram. Requiring that the sum is finite allows the calculation of the counterterm  $\delta_G$ .

diagrams can be written as

$$\Sigma(p) = n_f \Sigma_f(p) + n_h \Sigma_h(p) + \Sigma_c(p) + \Sigma_G(p) + \Sigma_{ct}(p) \quad (\text{D.7})$$

where  $\Sigma_f(p)$  is the contribution from light quarks,  $\Sigma_h(p)$  from heavy quarks,  $\Sigma_c(p)$  from ghost fields,  $\Sigma_G(p)$  from gluon self-interactions, and  $\Sigma_{ct}(p)$  from the counterterm. For example, the contribution from massless quarks can be written as

$$\Sigma_f(p) = \frac{1}{2} (4\pi\mu^2)^{2-d/2} \frac{g_s^2}{16\pi^2} \delta^{ab} \int \frac{d^d k}{(i\pi^{d/2})} \frac{\text{Tr}(\not{k}\gamma^\mu(\not{k} + \not{p})\gamma^\nu)}{k^2(k+p)^2}, \quad (\text{D.8})$$

with a factor of  $\mu^{4-d}$  added to render it dimensionless. The integral above is straightforward to evaluate and yields

$$\Sigma_f(p) = \frac{\alpha_S}{6\pi} \delta^{ab} (p^\mu p^\nu - g^{\mu\nu} p^2) \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi + O(\epsilon^0) \right). \quad (\text{D.9})$$

Rest of the integrals can be evaluated similarly yielding, for the counterterm,

$$\delta_G = \left[ C_A \left( \frac{13 - 3\xi}{12} \right) - \frac{2n_f + 2n_h}{3} T_F \right] \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right) - \frac{2}{3} T_F \sum_i^{n_h} \ln \frac{\mu_R^2}{m_i^2} \quad (\text{D.10})$$

with the heavy quark diagrams subtracted at zero momentum. Note that the longitudinal part of the gluon propagator does not receive any corrections. Ghost propagator counterterm can be calculated similarly:

$$\delta_c = -C_A \left( \frac{3 - \xi}{8} \right) \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right). \quad (\text{D.11})$$

Above counterterms were calculated using the  $\overline{MS}$  scheme. For massive quarks, the on-shell renormalisation scheme is commonly used. In this scheme, the renormalisation conditions are set as

$$\begin{aligned} \delta_\psi &= \left[ - \frac{d}{d\cancel{p}} \Sigma(p) \right]_{\cancel{p}=m_p} \\ \delta_m &= \frac{1}{m_p} \Sigma(m_p). \end{aligned} \quad (\text{D.12})$$

Evaluating  $\Sigma(p)$  from the 1-loop quark self-energy diagram yields

$$\delta_\psi = \delta_m = -\frac{1}{2} C_F \left( \frac{3}{\epsilon} + 4 + 3 \ln \left( 4\pi e^{-\gamma_E} \frac{\mu_R^2}{m_p^2} \right) \right). \quad (\text{D.13})$$

With  $\delta_\psi, \delta_G, \delta_c, \delta_m$  known,  $\delta_g$  can be calculated using the 3-point vertex. In Feynman

gauge

$$\delta_\psi + \delta_g + \frac{1}{2}\delta_G = -\frac{1}{2}(C_F + C_A) \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right). \quad (\text{D.14})$$

In  $\overline{MS}$  scheme,  $\delta_g$  is

$$\delta_g = - \left( \frac{11C_A - 4(n_f + n_h)T_F}{12} \right) \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right) + \frac{1}{3}T_F \sum_i^{n_h} \ln \frac{\mu_R^2}{m_i^2}. \quad (\text{D.15})$$

### D.3 QCD $\beta$ -function

From the above result, considering only the top-quark as the heavy particle,

$$Z_{\alpha_S} = 1 + 2\delta_g = 1 + \frac{\alpha_S}{2\pi} \left[ -\beta_0 \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \right) + \frac{1}{\epsilon} \left( \frac{2}{3}T_F + \epsilon \ln \left( 4\pi e^{-\gamma_E} \frac{\mu_R^2}{m_t^2} \right) \right) \right] \quad (\text{D.16})$$

where

$$\beta_0 = \frac{11C_A - 4n_f T_F}{6} \quad (\text{D.17})$$

is the beta function at 1-loop.

# APPENDIX E

## List of denominators

In this appendix, the denominator factors which occur in the coefficients of the master integrals in the helicity amplitudes for  $gg \rightarrow ZZ$  are listed. In the calculation, the masses of the top quark and the  $Z$  boson,  $m_t$  and  $m_Z$ , have been set to  $m_t = 1$  and  $(m_z/m_t)^2 = 5/18$ . The exact set of denominator factors depends on the choice of master integrals. In this work, the basis is chosen such that the dependence on the space-time dimension  $d$  factorises from that of the kinematic invariants  $s = (p_1 + p_2)^2$  and  $t = (p_1 - p_3)^2$ .

We find 9  $d$ -dependent denominators

$$d - 5, d - 4, d - 3, d - 2, d, 2d - 7, 3d - 10, 3d - 8, 5d + 52$$

and the 48  $d$ -independent denominators sorted according to the polynomial ordering used for partial fractioning.

8 polynomials dependent only on  $s$ :

$$s, \quad -4 + s, \quad -5 + 9s, \quad -10 + 9s, \quad -5 + 18s, \quad 134 + 9s, \quad -335 + 324s, \\ 90 - 245s + 324s^2.$$

6 polynomials dependent only on  $t$ :

$$t, \quad -4 + t, \quad 10 + 13t, \quad -5 + 18t, \quad 5 + 18t, \quad 5 + 31t.$$

9 degree 1 polynomials in  $s, t$ :

$$\begin{aligned} & -5 + 6s + 6t, \quad -5 + 9s + 9t, \quad 31 + 9s + 9t, \quad -5 + 9s + 18t, \quad -5 - 18s + 18t, \\ & -5 + 18s + 18t, \quad -5 + 36s + 18t, \quad -155 + 117s + 117t, \quad -200 + 279s + 279t. \end{aligned}$$

13 degree 2 polynomials in  $s, t$ :

$$\begin{aligned} & 25 - 90t + 324st, \quad 25 + 180s - 90t + 558st, \quad 5s - 5t + 18st + 18t^2 \\ & 25 - 135t + 162st + 162t^2, \quad 25 - 180t + 324st + 324t^2, \\ & 25 - 180t + 648st + 324t^2, \quad 25 - 270s - 90t + 324s^2 + 324st, \\ & 25 - 580s - 90t + 558s^2 + 558st, \quad 25 + 180s - 180t - 324s^2 + 324t^2, \\ & 25 - 1296s - 180t + 324st + 324t^2, \quad 25 - 25s - 180t + 90s^2 + 90st + 324t^2 \\ & 25 - 155s - 180t + 324s^2 + 558st + 324t^2, \\ & 3350 - 6030s - 1675t + 2916s^2 + 3015st + 3015t^2. \end{aligned}$$

2 degree 3 polynomials in  $s, t$ :

$$\begin{aligned} & -25 + 180t - 324st - 324t^2 + 81st^2, \\ & -25 + 25s + 180t - 90s^2 - 414st - 324t^2 + 81s^3 + 162s^2t + 81st^2. \end{aligned}$$

8 degree 4 polynomials in  $s, t$ :

$$400 - 3080s - 2880t + 4489s^2 + 11808st + 5184t^2 - 2772s^2t - 2592st^2 + 324s^2t^2,$$

$$400 - 200s - 2880t - 695s^2 + 1440st + 5184t^2 - 180s^3 - 2772s^2t - 2592st^2 + 324s^4 \\ + 648s^3t + 324s^2t^2,$$

$$112225 - 997920s - 808020t + 1679616s^2 + 3841992st + 1454436t^2 - 839808s^2t \\ - 898128st^2 + 104976s^2t^2,$$

$$112225 + 51300s - 808020t - 144180s^2 + 64800st + 1454436t^2 - 174960s^3 - 1073088s^2t \\ - 898128st^2 + 104976s^4 + 209952s^3t + 104976s^2t^2,$$

$$3125 + 173250s - 45000t + 145800s^2 - 1206900st + 243000t^2 + 3149280s^2t \\ + 1953720st^2 - 583200t^3 + 1889568s^2t^2 + 524880st^3 + 524880t^4,$$

$$- 8375 - 23400s + 88200t - 32400s^2 + 68040st - 301320t^2 + 29160s^3 - 244944s^2t \\ + 29160st^2 + 303264t^3 - 104976s^3t - 104976s^2t^2 + 104976st^3 + 104976t^4,$$

$$+ 9625 - 64800s - 106200t + 599400st + 398520t^2 - 839808s^2t - 1405512st^2 \\ - 536544t^3 + 209952s^2t^2 + 314928st^3 + 104976t^4,$$

$$- 2750 + 1800s + 22725t + 2025s^2 - 52650st - 56700t^2 + 26973s^2t + 64881st^2 \\ + 37908t^3 + 13122s^3t + 26244s^2t^2 + 13122st^3.$$

2 degree 6 polynomials in  $s, t$ :

$$\begin{aligned}
& + 2500 - 18000 t + 64800 s t + 27900 t^2 - 249480 s t^2 + 32400 t^3 + 419904 s^2 t^2 \\
& + 116640 s t^3 - 56295 t^4 - 209952 s^2 t^3 - 224532 s t^4 - 14580 t^5 + 26244 s^2 t^4 \\
& + 52488 s t^5 + 26244 t^6, \\
& + 105625 - 3676500 s - 468000 t + 25891650 s^2 + 17309700 s t - 797850 t^2 - 50490540 s^3 \\
& - 73614420 s^2 t - 19260180 s t^2 + 3863700 t^3 + 29452329 s^4 + 80752788 s^3 t \\
& + 75149694 s^2 t^2 + 25850340 s t^3 + 2001105 t^4 - 18187092 s^4 t - 60466176 s^3 t^2 \\
& - 72275976 s^2 t^3 - 35901792 s t^4 - 5904900 t^5 + 2125764 s^4 t^2 + 8503056 s^3 t^3 \\
& + 12754584 s^2 t^4 + 8503056 s t^5 + 2125764 t^6.
\end{aligned}$$

Note that during the initial IBP reduction to the traditional Laporta basis, spurious denominators with mixed dependence on  $d$  and kinematic invariants were indeed introduced. Rotation to the current basis followed by the partial fractioning approach described in Sec. 5.1 allowed us to systematically eliminate those denominators.

# APPENDIX F

## Numerical checks

In this appendix, details of the pole cancellation are provided. The amplitudes are evaluated in the chosen basis of finite integrals using Kreimer's anti-commuting  $\gamma_5$  scheme. For an  $L$  loop amplitude, at worst  $1/\epsilon^{2L}$  could appear. However, since this process is only an NLO correction, only the  $1/\epsilon^2$  and  $1/\epsilon$  poles should remain before UV renormalisation and IR subtraction. Indeed, for the spurious  $1/\epsilon^4$  and  $1/\epsilon^3$  poles, analytical and high precision numerical cancellations are seen, respectively. For the Euclidean point with  $s/m_t^2 = -191$ ,  $t/m_t^2 = -337$ ,  $m_Z^2/m_t^2 = -853$ ,  $m_t = 1$ , 15 digits of cancellation is seen for the  $1/\epsilon^3$  pole while 8 digit cancellation is seen for the point in the physical region:  $s/m_t^2 = 142/17$ ,  $t/m_t^2 = -125/22$ ,  $m_Z^2/m_t^2 = 5/18$ ,  $m_t = 1$ .

The tables below show our results for the  $1/\epsilon^2$  and  $1/\epsilon$  poles following UV renormalisation, as well as the  $\epsilon^0$  term, and compare them against the predicted IR poles as in Eq. 5.2.13 and 5.2.14, for the Euclidean (Tab. F.1) and the physical (Tab. F.2) point. The digits in parentheses for the  $\epsilon^0$  term denote the uncertainty in the last digit. It is clear from the tables below that the calculated poles show the structure predicted in [347] with good numerical precision for both the Euclidean and the physical point which serves as a strong check for the calculation.

FF	$1/\epsilon^2$	$1/\epsilon$	$\epsilon^0$
$A_1$	$+2.436734851 \cdot 10^{-1}$	$+8.212518984 \cdot 10^{-1} + 1.531045661 i$	$-2.806661(2) + 4.18190980(3) i$
Pred.	$+2.436734852 \cdot 10^{-1}$	$+8.212518977 \cdot 10^{-1} + 1.531045662 i$	
$A_2$	$-1.760872097 \cdot 10^{-1}$	$-6.021429768 \cdot 10^{-1} - 1.106388569 i$	$+2.509969(1) - 3.07651654(4) i$
Pred.	$-1.760872097 \cdot 10^{-1}$	$-6.021429781 \cdot 10^{-1} - 1.106388569 i$	
$A_3$	$-3.815946068 \cdot 10^{-2}$	$-7.236587884 \cdot 10^{-2} - 2.397629627 \cdot 10^{-1} i$	$+1.2102(3) \cdot 10^{-2} - 3.015063(4) \cdot 10^{-1} i$
Pred.	$-3.815946069 \cdot 10^{-2}$	$-7.236587838 \cdot 10^{-2} - 2.397629627 \cdot 10^{-1} i$	
$A_4$	$-1.565000574 \cdot 10^{-4}$	$-5.374251500 \cdot 10^{-4} - 9.833188615 \cdot 10^{-4} i$	$+2.18538(3) \cdot 10^{-3} - 2.748510(3) \cdot 10^{-3} i$
Pred.	$-1.565000575 \cdot 10^{-4}$	$-5.374251489 \cdot 10^{-4} - 9.833188622 \cdot 10^{-4} i$	
$A_5$	$+7.608919171 \cdot 10^{-4}$	$+1.926944077 \cdot 10^{-3} + 4.780824914 \cdot 10^{-3} i$	$-1.051486(4) \cdot 10^{-2} + 9.052930(4) \cdot 10^{-3} i$
Pred.	$+7.608919168 \cdot 10^{-4}$	$+1.926944068 \cdot 10^{-3} + 4.780824912 \cdot 10^{-3} i$	
$A_6$	$+7.576619247 \cdot 10^{-4}$	$+2.735071357 \cdot 10^{-3} + 4.760530273 \cdot 10^{-3} i$	$-7.02484(5) \cdot 10^{-3} + 1.41435102(3) \cdot 10^{-2} i$
Pred.	$+7.576619247 \cdot 10^{-4}$	$+2.735071351 \cdot 10^{-3} + 4.760530273 \cdot 10^{-3} i$	
$A_7$	$-1.565000574 \cdot 10^{-4}$	$-5.374251500 \cdot 10^{-4} - 9.833188615 \cdot 10^{-4} i$	$+2.18538(3) \cdot 10^{-3} - 2.748510(3) \cdot 10^{-3} i$
Pred.	$-1.565000575 \cdot 10^{-4}$	$-5.374251489 \cdot 10^{-4} - 9.833188622 \cdot 10^{-4} i$	
$A_8$	$-3.055600405 \cdot 10^{-4}$	$-1.158849558 \cdot 10^{-3} - 1.919890357 \cdot 10^{-3} i$	$+4.35036(1) \cdot 10^{-3} - 6.0546699(5) \cdot 10^{-3} i$
Pred.	$-3.055600405 \cdot 10^{-4}$	$-1.158849559 \cdot 10^{-3} - 1.919890357 \cdot 10^{-3} i$	
$A_9$	$+2.001982671 \cdot 10^{-4}$	$+7.482078266 \cdot 10^{-4} + 1.257882810 \cdot 10^{-3} i$	$-3.07299(1) \cdot 10^{-3} + 3.897481(1) \cdot 10^{-3} i$
Pred.	$+2.001982671 \cdot 10^{-4}$	$+7.482078292 \cdot 10^{-4} + 1.257882810 \cdot 10^{-3} i$	
$A_{10}$	$+3.636573767 \cdot 10^{-4}$	$+1.390161598 \cdot 10^{-3} + 2.284926686 \cdot 10^{-3} i$	$-4.77622(2) \cdot 10^{-3} + 7.274828(2) \cdot 10^{-3} i$
Pred.	$+3.636573768 \cdot 10^{-4}$	$+1.390161596 \cdot 10^{-3} + 2.284926686 \cdot 10^{-3} i$	
$A_{11}$	$+5.388240322 \cdot 10^{-6}$	$-1.272166624 \cdot 10^{-4} + 3.385531242 \cdot 10^{-5} i$	$+1.04254(1) \cdot 10^{-3} - 8.20955(1) \cdot 10^{-4} i$
Pred.	$+5.388240348 \cdot 10^{-6}$	$-1.272166651 \cdot 10^{-4} + 3.385531259 \cdot 10^{-5} i$	
$A_{12}$	$-5.388240322 \cdot 10^{-6}$	$+1.272166624 \cdot 10^{-4} - 3.385531242 \cdot 10^{-5} i$	$-1.04254(1) \cdot 10^{-3} + 8.20955(1) \cdot 10^{-4} i$
Pred.	$-5.388240348 \cdot 10^{-6}$	$+1.272166651 \cdot 10^{-4} - 3.385531259 \cdot 10^{-5} i$	
$A_{13}$	$-3.636573767 \cdot 10^{-4}$	$-1.390161598 \cdot 10^{-3} - 2.284926686 \cdot 10^{-3} i$	$+4.77622(2) \cdot 10^{-3} - 7.274828(2) \cdot 10^{-3} i$
Pred.	$-3.636573768 \cdot 10^{-4}$	$-1.390161596 \cdot 10^{-3} - 2.284926686 \cdot 10^{-3} i$	
$A_{14}$	$-2.001982671 \cdot 10^{-4}$	$-7.482078266 \cdot 10^{-4} - 1.257882810 \cdot 10^{-3} i$	$+3.07299(1) \cdot 10^{-3} - 3.897481(1) \cdot 10^{-3} i$
Pred.	$-2.001982671 \cdot 10^{-4}$	$-7.482078292 \cdot 10^{-4} - 1.257882810 \cdot 10^{-3} i$	
$A_{15}$	$+3.055600405 \cdot 10^{-4}$	$+1.158849558 \cdot 10^{-3} + 1.919890357 \cdot 10^{-3} i$	$-4.35036(1) \cdot 10^{-3} + 6.0546699(5) \cdot 10^{-3} i$
Pred.	$+3.055600405 \cdot 10^{-4}$	$+1.158849559 \cdot 10^{-3} + 1.919890357 \cdot 10^{-3} i$	
$A_{16}$	$+1.898361362 \cdot 10^{-4}$	$+6.165488820 \cdot 10^{-4} + 1.192775622 \cdot 10^{-3} i$	$-2.233448(2) \cdot 10^{-3} + 3.11183978(6) \cdot 10^{-3} i$
Pred.	$+1.898361362 \cdot 10^{-4}$	$+6.165488809 \cdot 10^{-4} + 1.192775622 \cdot 10^{-3} i$	
$A_{17}$	$-4.235989659 \cdot 10^{-8}$	$-1.659620988 \cdot 10^{-7} - 2.661550798 \cdot 10^{-7} i$	$+8.1249(2) \cdot 10^{-7} - 8.72727(4) \cdot 10^{-7} i$
Pred.	$-4.235989677 \cdot 10^{-8}$	$-1.659621000 \cdot 10^{-7} - 2.661550810 \cdot 10^{-7} i$	
$A_{18}$	$-9.857950093 \cdot 10^{-8}$	$-9.594603102 \cdot 10^{-7} - 6.193932718 \cdot 10^{-7} i$	$+4.4198(6) \cdot 10^{-7} - 5.632743(5) \cdot 10^{-6} i$
Pred.	$-9.857950139 \cdot 10^{-8}$	$-9.594603103 \cdot 10^{-7} - 6.193932747 \cdot 10^{-7} i$	
$A_{19}$	$+8.932087549 \cdot 10^{-7}$	$+3.205282901 \cdot 10^{-6} + 5.612196125 \cdot 10^{-6} i$	$-7.43447(5) \cdot 10^{-6} + 1.6553816(4) \cdot 10^{-5} i$
Pred.	$+8.932087551 \cdot 10^{-7}$	$+3.205282889 \cdot 10^{-6} + 5.612196126 \cdot 10^{-6} i$	
$A_{20}$	$-4.235989659 \cdot 10^{-8}$	$-1.659620988 \cdot 10^{-7} - 2.661550798 \cdot 10^{-7} i$	$+8.1249(2) \cdot 10^{-7} - 8.72727(4) \cdot 10^{-7} i$
Pred.	$-4.235989677 \cdot 10^{-8}$	$-1.659621000 \cdot 10^{-7} - 2.661550810 \cdot 10^{-7} i$	

Table F.1: Numerical poles for the Euclidean phase-space point  $s/m_t^2 = -191$ ,  $t/m_t^2 = -337$ ,  $m_Z^2/m_t^2 = -853$ ,  $m_t = 1$  compared against the predicted values. Also shown are the  $\epsilon^0$  terms before IR subtraction with the digits in parentheses denoting the uncertainty in the last digit.

FF	$1/\epsilon^2$	$1/\epsilon$	$\epsilon^0$
$A_1$	$-5.726898 \cdot 10^{-1} - 4.634791 \cdot 10^{-1}i$	$-6.75706 \cdot 10^{-1} - 4.05460 i$	$6.87787(1) - 7.90340(1) i$
Pred.	$-5.726897 \cdot 10^{-1} - 4.634791 \cdot 10^{-1}i$	$-6.75704 \cdot 10^{-1} - 4.05460 i$	
$A_2$	$+4.153857 \cdot 10^{-1} + 1.097935 \cdot 10^{-1}i$	$+1.40864 + 2.02204 i$	$-2.53566(2) + 7.06651(3) i$
Pred.	$+4.153857 \cdot 10^{-1} + 1.097934 \cdot 10^{-1}i$	$+1.40865 + 2.02204 i$	
$A_3$	$+2.003102 \cdot 10^{-1} + 3.116062 \cdot 10^{-1}i$	$-5.02052 \cdot 10^{-1} + 1.86425 i$	$-3.99592(2) + 2.59711(2) i$
Pred.	$+2.003101 \cdot 10^{-1} + 3.116062 \cdot 10^{-1}i$	$-5.02053 \cdot 10^{-1} + 1.86425 i$	
$A_4$	$+3.147592 \cdot 10^{-2} + 9.237206 \cdot 10^{-4}i$	$+1.39272 \cdot 10^{-1} + 1.16086 \cdot 10^{-1}i$	$-4.1039(4) \cdot 10^{-2} + 5.40365(5) \cdot 10^{-1} i$
Pred.	$+3.147591 \cdot 10^{-2} + 9.237121 \cdot 10^{-4}i$	$+1.39272 \cdot 10^{-1} + 1.16086 \cdot 10^{-1}i$	
$A_5$	$+1.041667 \cdot 10^{-1} + 5.382124 \cdot 10^{-2}i$	$+2.44023 \cdot 10^{-1} + 5.97453 \cdot 10^{-1}i$	$-8.96421(5) \cdot 10^{-1} + 1.736695(6) i$
Pred.	$+1.041667 \cdot 10^{-1} + 5.382123 \cdot 10^{-2}i$	$+2.44022 \cdot 10^{-1} + 5.97453 \cdot 10^{-1}i$	
$A_6$	$+1.242527 \cdot 10^{-1} + 6.941130 \cdot 10^{-2}i$	$+2.52191 \cdot 10^{-1} + 7.24307 \cdot 10^{-1}i$	$-1.20930(2) + 1.93865(2) i$
Pred.	$+1.242527 \cdot 10^{-1} + 6.941131 \cdot 10^{-2}i$	$+2.52189 \cdot 10^{-1} + 7.24307 \cdot 10^{-1}i$	
$A_7$	$+3.147592 \cdot 10^{-2} + 9.237206 \cdot 10^{-4}i$	$+1.39272 \cdot 10^{-1} + 1.16086 \cdot 10^{-1}i$	$-4.1039(4) \cdot 10^{-2} + 5.40365(4) \cdot 10^{-1} i$
Pred.	$+3.147591 \cdot 10^{-2} + 9.237121 \cdot 10^{-4}i$	$+1.39272 \cdot 10^{-1} + 1.16086 \cdot 10^{-1}i$	
$A_8$	$-1.017708 \cdot 10^{-2} + 8.808524 \cdot 10^{-2}i$	$-4.41618 \cdot 10^{-1} + 2.61228 \cdot 10^{-1}i$	$-1.00384(5) - 4.4284(4) \cdot 10^{-1} i$
Pred.	$-1.017707 \cdot 10^{-2} + 8.808519 \cdot 10^{-2}i$	$-4.41613 \cdot 10^{-1} + 2.61225 \cdot 10^{-1}i$	
$A_9$	$+7.168287 \cdot 10^{-2} - 5.063902 \cdot 10^{-2}i$	$+5.37076 \cdot 10^{-1} + 9.24698 \cdot 10^{-2}i$	$3.07426(8) \cdot 10^{-1} 1.266108(9) i$
Pred.	$+7.168286 \cdot 10^{-2} - 5.063902 \cdot 10^{-2}i$	$+5.37075 \cdot 10^{-1} + 9.24707 \cdot 10^{-2}i$	
$A_{10}$	$+1.873343 \cdot 10^{-2} - 8.497011 \cdot 10^{-2}i$	$+4.70733 \cdot 10^{-1} - 2.17284 \cdot 10^{-1}i$	$+9.3643(1) \cdot 10^{-1} + 6.3029(1) \cdot 10^{-1} i$
Pred.	$+1.873344 \cdot 10^{-2} - 8.497010 \cdot 10^{-2}i$	$+4.70734 \cdot 10^{-1} - 2.17286 \cdot 10^{-1}i$	
$A_{11}$	$-7.675742 \cdot 10^{-2} + 5.097567 \cdot 10^{-2}i$	$-5.57824 \cdot 10^{-1} - 1.06514 \cdot 10^{-1}i$	$-3.1397(3) \cdot 10^{-1} - 1.35727(4) i$
Pred.	$-7.675741 \cdot 10^{-2} + 5.097571 \cdot 10^{-2}i$	$-5.57827 \cdot 10^{-1} - 1.06513 \cdot 10^{-1}i$	
$A_{12}$	$+7.675742 \cdot 10^{-2} - 5.097567 \cdot 10^{-2}i$	$+5.57824 \cdot 10^{-1} + 1.06514 \cdot 10^{-1}i$	$+3.1397(3) \cdot 10^{-1} + 1.35727(4) i$
Pred.	$+7.675741 \cdot 10^{-2} - 5.097571 \cdot 10^{-2}i$	$+5.57827 \cdot 10^{-1} + 1.06513 \cdot 10^{-1}i$	
$A_{13}$	$-1.873343 \cdot 10^{-2} + 8.497011 \cdot 10^{-2}i$	$-4.70733 \cdot 10^{-1} + 2.17284 \cdot 10^{-1}i$	$-9.3644(1) \cdot 10^{-1} - 6.3029(1) \cdot 10^{-1} i$
Pred.	$-1.873344 \cdot 10^{-2} + 8.497010 \cdot 10^{-2}i$	$-4.70734 \cdot 10^{-1} + 2.17286 \cdot 10^{-1}i$	
$A_{14}$	$-7.168287 \cdot 10^{-2} + 5.063902 \cdot 10^{-2}i$	$-5.37076 \cdot 10^{-1} - 9.24698 \cdot 10^{-2}i$	$-3.07426(8) \cdot 10^{-1} - 1.266108(9) i$
Pred.	$-7.168286 \cdot 10^{-2} + 5.063902 \cdot 10^{-2}i$	$-5.37075 \cdot 10^{-1} - 9.24707 \cdot 10^{-2}i$	
$A_{15}$	$+1.017708 \cdot 10^{-2} - 8.808524 \cdot 10^{-2}i$	$+4.41618 \cdot 10^{-1} - 2.61228 \cdot 10^{-1}i$	$1.00384(4) + 4.4283(4) \cdot 10^{-1} i$
Pred.	$+1.017707 \cdot 10^{-2} - 8.808519 \cdot 10^{-2}i$	$+4.41613 \cdot 10^{-1} - 2.61225 \cdot 10^{-1}i$	
$A_{16}$	$-6.195421 \cdot 10^{-2} - 9.197693 \cdot 10^{-2}i$	$+1.25592 \cdot 10^{-1} - 6.06299 \cdot 10^{-1}i$	$1.76383(3) - 9.4291(3) \cdot 10^{-1} i$
Pred.	$-6.195417 \cdot 10^{-2} - 9.197695 \cdot 10^{-2}i$	$+1.25596 \cdot 10^{-1} - 6.06299 \cdot 10^{-1}i$	
$A_{17}$	$+9.152404 \cdot 10^{-4} + 4.922399 \cdot 10^{-3}i$	$-1.47185 \cdot 10^{-2} + 2.71477 \cdot 10^{-2}i$	$-8.6390(6) \cdot 10^{-2} + 2.7504(7) \cdot 10^{-2} i$
Pred.	$+9.152368 \cdot 10^{-4} + 4.922402 \cdot 10^{-3}i$	$-1.47187 \cdot 10^{-2} + 2.71472 \cdot 10^{-2}i$	
$A_{18}$	$+6.800443 \cdot 10^{-3} + 5.687424 \cdot 10^{-3}i$	$+7.80438 \cdot 10^{-3} + 4.98318 \cdot 10^{-2}i$	$-1.02182(8) \cdot 10^{-1} + 1.37512(8) \cdot 10^{-1} i$
Pred.	$+6.800439 \cdot 10^{-3} + 5.687435 \cdot 10^{-3}i$	$+7.80405 \cdot 10^{-3} + 4.98315 \cdot 10^{-2}i$	
$A_{19}$	$+4.208648 \cdot 10^{-3} + 4.547692 \cdot 10^{-3}i$	$-3.01730 \cdot 10^{-4} + 3.55035 \cdot 10^{-2}i$	$-7.895(10) \cdot 10^{-2} + 7.980(11) \cdot 10^{-2} i$
Pred.	$+4.208616 \cdot 10^{-3} + 4.547808 \cdot 10^{-3}i$	$-3.13880 \cdot 10^{-4} + 3.55067 \cdot 10^{-2}i$	
$A_{20}$	$+9.152403 \cdot 10^{-4} + 4.922399 \cdot 10^{-3}i$	$-1.47185 \cdot 10^{-2} + 2.71477 \cdot 10^{-2}i$	$-8.6391(6) \cdot 10^{-2} + 2.7504(7) \cdot 10^{-2} i$
Pred.	$+9.152368 \cdot 10^{-4} + 4.922402 \cdot 10^{-3}i$	$-1.47187 \cdot 10^{-2} + 2.71472 \cdot 10^{-2}i$	

Table F.2: Numerical poles for the physical phase-space point  $s/m_t^2 = 142/17$ ,  $t/m_t^2 = -125/22$ ,  $m_Z^2/m_t^2 = 5/18$ ,  $m_t = 1$  compared against the predicted values. Also shown are the  $\epsilon^0$  terms before IR subtraction with the digits in parentheses denoting the uncertainty in the last digit.

# BIBLIOGRAPHY

# BIBLIOGRAPHY

- [1] **Particle Data Group** Collaboration, P. Zyla et al., *Review of Particle Physics*, *PTEP* **2020** (2020), no. 8 083C01.
- [2] **LHC Higgs Cross Section Working Group** Collaboration, J. R. Andersen et al., *Handbook of LHC Higgs Cross Sections: 3. Higgs Properties*, [arXiv:1307.1347](#).
- [3] S. Borowka, G. Heinrich, S. Jahn, S. P. Jones, M. Kerner, J. Schlenk, and T. Zirke, *pySecDec: a toolbox for the numerical evaluation of multi-scale integrals*, *Comput. Phys. Commun.* **222** (2018) 313–326, [[arXiv:1703.09692](#)].
- [4] Z. Li, J. Wang, Q.-S. Yan, and X. Zhao, *Efficient numerical evaluation of Feynman integrals*, *Chin. Phys. C* **40** (2016), no. 3 033103, [[arXiv:1508.02512](#)].
- [5] S. Borowka, G. Heinrich, S. Jahn, S. Jones, M. Kerner, and J. Schlenk, *A GPU compatible quasi-Monte Carlo integrator interfaced to pySecDec*, *Comput. Phys. Commun.* **240** (2019) 120–137, [[arXiv:1811.11720](#)].
- [6] B. Agarwal, S. P. Jones, and A. von Manteuffel, *Two-loop helicity amplitudes for  $gg \rightarrow ZZ$  with full top-quark mass effects*, [arXiv:2011.15113](#).
- [7] **LHC Higgs Cross Section Working Group** Collaboration, S. Dittmaier et al., *Handbook of LHC Higgs Cross Sections: 1. Inclusive Observables*, [arXiv:1101.0593](#).
- [8] **ATLAS** Collaboration, G. Aad et al., *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett. B* **716** (2012) 1–29, [[arXiv:1207.7214](#)].
- [9] **CMS** Collaboration, S. Chatrchyan et al., *Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC*, *Phys. Lett. B* **716** (2012) 30–61, [[arXiv:1207.7235](#)].
- [10] **ATLAS** Collaboration, *A combination of measurements of Higgs boson production and decay using up to 139 fb<sup>-1</sup> of proton–proton collision data at  $\sqrt{s} = 13$  TeV collected with the ATLAS experiment*, .
- [11] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog, and B. Mistlberger, *Higgs Boson Gluon-Fusion Production in QCD at Three Loops*, *Phys. Rev. Lett.* **114** (2015) 212001, [[arXiv:1503.06056](#)].

- [12] J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, *gg* → *ZZ*: analytic two-loop results for the low- and high-energy regions, *JHEP* **04** (2020) 024, [arXiv:2002.05558].
- [13] S. Catani, *The Singular behavior of QCD amplitudes at two loop order*, *Phys. Lett. B* **427** (1998) 161–171, [hep-ph/9802439].
- [14] R. H. Parker, C. Yu, W. Zhong, B. Estey, and H. Müller, *Measurement of the fine-structure constant as a test of the standard model*, *Science* **360** (2018), no. 6385 191–195, [https://science.sciencemag.org/content/360/6385/191.full.pdf].
- [15] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, *Tenth-Order Electron Anomalous Magnetic Moment — Contribution of Diagrams without Closed Lepton Loops*, *Phys. Rev. D* **91** (2015), no. 3 033006, [arXiv:1412.8284]. [Erratum: *Phys.Rev.D* 96, 019901 (2017)].
- [16] D. Hanneke, S. F. Hoogerheide, and G. Gabrielse, *Cavity Control of a Single-Electron Quantum Cyclotron: Measuring the Electron Magnetic Moment*, *Phys. Rev. A* **83** (2011) 052122, [arXiv:1009.4831].
- [17] P. W. Higgs, *Broken symmetries, massless particles and gauge fields*, *Phys. Lett.* **12** (1964) 132–133.
- [18] P. W. Higgs, *Broken Symmetries and the Masses of Gauge Bosons*, *Phys. Rev. Lett.* **13** (1964) 508–509.
- [19] F. Englert and R. Brout, *Broken Symmetry and the Mass of Gauge Vector Mesons*, *Phys. Rev. Lett.* **13** (1964) 321–323.
- [20] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, *Global Conservation Laws and Massless Particles*, *Phys. Rev. Lett.* **13** (1964) 585–587.
- [21] J. Goodman, M. Ibe, A. Rajaraman, W. Shepherd, T. M. P. Tait, and H.-B. Yu, *Constraints on Dark Matter from Colliders*, *Phys. Rev. D* **82** (2010) 116010, [arXiv:1008.1783].
- [22] P. Cushman et al., *Working Group Report: WIMP Dark Matter Direct Detection*, in *Community Summer Study 2013: Snowmass on the Mississippi*, 10, 2013. arXiv:1310.8327.
- [23] M. Battaglieri et al., *US Cosmic Visions: New Ideas in Dark Matter 2017: Community Report*, in *U.S. Cosmic Visions: New Ideas in Dark Matter*, 7, 2017. arXiv:1707.04591.
- [24] G. Arcadi, M. Dutra, P. Ghosh, M. Lindner, Y. Mambrini, M. Pierre, S. Profumo, and F. S. Queiroz, *The waning of the WIMP? A review of models, searches, and constraints*, *Eur. Phys. J. C* **78** (2018), no. 3 203, [arXiv:1703.07364].

- [25] E. J. Copeland, M. Sami, and S. Tsujikawa, *Dynamics of dark energy*, *Int. J. Mod. Phys. D* **15** (2006) 1753–1936, [[hep-th/0603057](#)].
- [26] A. Strumia and F. Vissani, *Neutrino masses and mixings and...*, [hep-ph/0606054](#).
- [27] S. Weinberg, *Gauge Hierarchies*, *Phys. Lett. B* **82** (1979) 387–391.
- [28] A. de Gouvea, D. Hernandez, and T. M. P. Tait, *Criteria for Natural Hierarchies*, *Phys. Rev. D* **89** (2014), no. 11 115005, [[arXiv:1402.2658](#)].
- [29] M. Dine, *TASI lectures on the strong CP problem*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2000): Flavor Physics for the Millennium*, 6, 2000. [hep-ph/0011376](#).
- [30] L. Canetti, M. Drewes, and M. Shaposhnikov, *Matter and Antimatter in the Universe*, *New J. Phys.* **14** (2012) 095012, [[arXiv:1204.4186](#)].
- [31] C.-N. Yang and R. L. Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, *Phys. Rev.* **96** (1954) 191–195.
- [32] Y. Nambu, *Quasi-particles and gauge invariance in the theory of superconductivity*, *Phys. Rev.* **117** (Feb, 1960) 648–663.
- [33] J. Goldstone, *Field Theories with Superconductor Solutions*, *Nuovo Cim.* **19** (1961) 154–164.
- [34] P. W. Higgs, *Spontaneous Symmetry Breakdown without Massless Bosons*, *Phys. Rev.* **145** (1966) 1156–1163.
- [35] S. Weinberg, *A model of leptons*, *Phys. Rev. Lett.* **19** (Nov, 1967) 1264–1266.
- [36] B. W. Lee, C. Quigg, and H. Thacker, *Weak Interactions at Very High-Energies: The Role of the Higgs Boson Mass*, *Phys. Rev. D* **16** (1977) 1519.
- [37] M. S. Chanowitz and M. K. Gaillard, *The TeV Physics of Strongly Interacting W's and Z's*, *Nucl. Phys. B* **261** (1985) 379–431.
- [38] **ATLAS** Collaboration, M. Aaboud et al., *Observation of  $H \rightarrow b\bar{b}$  decays and VH production with the ATLAS detector*, *Phys. Lett. B* **786** (2018) 59–86, [[arXiv:1808.08238](#)].
- [39] **CMS** Collaboration, A. M. Sirunyan et al., *Observation of Higgs boson decay to bottom quarks*, *Phys. Rev. Lett.* **121** (2018), no. 12 121801, [[arXiv:1808.08242](#)].
- [40] **ATLAS** Collaboration, G. Aad et al., *A search for the dimuon decay of the Standard Model Higgs boson with the ATLAS detector*, *Phys. Lett. B* **812** (2021) 135980, [[arXiv:2007.07830](#)].
- [41] **CMS** Collaboration, A. M. Sirunyan et al., *Evidence for Higgs boson decay to a pair of muons*, *JHEP* **01** (2021) 148, [[arXiv:2009.04363](#)].

- [42] S. Dittmaier et al., *Handbook of LHC Higgs Cross Sections: 2. Differential Distributions*, arXiv:1201.3084.
- [43] **LHC Higgs Cross Section Working Group** Collaboration, D. de Florian et al., *Handbook of LHC Higgs Cross Sections: 4. Deciphering the Nature of the Higgs Sector*, arXiv:1610.07922.
- [44] D. De Florian and J. Mazzitelli, *Soft gluon resummation for Higgs boson pair production including finite  $M_t$  effects*, *JHEP* **08** (2018) 156, [arXiv:1807.03704].
- [45] J. Chang, K. Cheung, J. S. Lee, and J. Park, *Probing the trilinear Higgs boson self-coupling at the high-luminosity LHC via multivariate analysis*, *Phys. Rev. D* **101** (2020), no. 1 016004, [arXiv:1908.00753].
- [46] M. L. Mangano, G. Ortona, and M. Selvaggi, *Measuring the Higgs self-coupling via Higgs-pair production at a 100 TeV p-p collider*, *Eur. Phys. J. C* **80** (2020), no. 11 1030, [arXiv:2004.03505].
- [47] M. Chiesa, F. Maltoni, L. Mantani, B. Mele, F. Piccinini, and X. Zhao, *Measuring the quartic Higgs self-coupling at a multi-TeV muon collider*, *JHEP* **09** (2020) 098, [arXiv:2003.13628].
- [48] **CMS** Collaboration, V. Khachatryan et al., *Precise determination of the mass of the Higgs boson and tests of compatibility of its couplings with the standard model predictions using proton collisions at 7 and 8 TeV*, *Eur. Phys. J. C* **75** (2015), no. 5 212, [arXiv:1412.8662].
- [49] **ATLAS** Collaboration, G. Aad et al., *Measurement of the Higgs boson mass from the  $H \rightarrow \gamma\gamma$  and  $H \rightarrow ZZ^* \rightarrow 4\ell$  channels with the ATLAS detector using 25 fb<sup>-1</sup> of pp collision data*, *Phys. Rev. D* **90** (2014), no. 5 052004, [arXiv:1406.3827].
- [50] **CMS** Collaboration, A. M. Sirunyan et al., *Measurements of properties of the Higgs boson decaying into the four-lepton final state in pp collisions at  $\sqrt{s} = 13$  TeV*, *JHEP* **11** (2017) 047, [arXiv:1706.09936].
- [51] N. Kauer and G. Passarino, *Inadequacy of zero-width approximation for a light Higgs boson signal*, *JHEP* **08** (2012) 116, [arXiv:1206.4803].
- [52] F. Caola and K. Melnikov, *Constraining the Higgs boson width with ZZ production at the LHC*, *Phys. Rev. D* **88** (2013) 054024, [arXiv:1307.4935].
- [53] J. M. Campbell, R. K. Ellis, and C. Williams, *Bounding the Higgs Width at the LHC Using Full Analytic Results for  $gg \rightarrow e^-e^+\mu^-\mu^+$* , *JHEP* **04** (2014) 060, [arXiv:1311.3589].
- [54] **CMS** Collaboration, A. M. Sirunyan et al., *Measurements of the Higgs boson width and anomalous HVV couplings from on-shell and off-shell production in the four-lepton final state*, *Phys. Rev. D* **99** (2019), no. 11 112003, [arXiv:1901.00174].

- [55] M. Gell-Mann, *Isotopic spin and new unstable particles*, *Phys. Rev.* **92** (Nov, 1953) 833–834.
- [56] T. Nakano and K. Nishijima, *Charge Independence for V-particles\**, *Progress of Theoretical Physics* **10** (11, 1953) 581–582, [<https://academic.oup.com/ptp/article-pdf/10/5/581/5364926/10-5-581.pdf>].
- [57] M. Gell-Mann, *The eightfold way: A theory of strong interaction symmetry*, .
- [58] Y. Ne’eman, *Derivation of strong interactions from a gauge invariance*, *Nuclear Physics* **26** (1961), no. 2 222–229.
- [59] M. Gell-Mann, *A schematic model of baryons and mesons*, *Physics Letters* **8** (1964), no. 3 214–215.
- [60] G. Zweig, *An SU(3) model for strong interaction symmetry and its breaking. Version 2*, 2, 1964.
- [61] O. W. Greenberg, *Spin and Unitary Spin Independence in a Paraquark Model of Baryons and Mesons*, *Phys. Rev. Lett.* **13** (1964) 598–602.
- [62] M. Y. Han and Y. Nambu, *Three Triplet Model with Double SU(3) Symmetry*, *Phys. Rev.* **139** (1965) B1006–B1010.
- [63] H. Fritzsch, M. Gell-Mann, and H. Leutwyler, *Advantages of the Color Octet Gluon Picture*, *Phys. Lett. B* **47** (1973) 365–368.
- [64] L. D. Faddeev and V. N. Popov, *Feynman Diagrams for the Yang-Mills Field*, *Phys. Lett. B* **25** (1967) 29–30.
- [65] D. F. Litim and J. M. Pawłowski, *Flow equations for Yang-Mills theories in general axial gauges*, *Phys. Lett. B* **435** (1998) 181–188, [[hep-th/9802064](https://arxiv.org/abs/hep-th/9802064)].
- [66] C. G. Bollini and J. J. Giambiagi, *Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter*, *Nuovo Cim. B* **12** (1972) 20–26.
- [67] G. M. Cicuta and E. Montaldi, *Analytic renormalization via continuous space dimension*, *Lett. Nuovo Cim.* **4** (1972) 329–332.
- [68] G. ’t Hooft and M. J. G. Veltman, *Regularization and Renormalization of Gauge Fields*, *Nucl. Phys. B* **44** (1972) 189–213.
- [69] J. F. Ashmore, *A Method of Gauge Invariant Regularization*, *Lett. Nuovo Cim.* **4** (1972) 289–290.
- [70] W. J. Marciano, *Dimensional Regularization and Mass Singularities*, *Phys. Rev. D* **12** (1975) 3861.
- [71] T. Peraro and L. Tancredi, *Physical projectors for multi-leg helicity amplitudes*, *JHEP* **07** (2019) 114, [[arXiv:1906.03298](https://arxiv.org/abs/1906.03298)].

- [72] T. Peraro and L. Tancredi, *Tensor decomposition for bosonic and fermionic scattering amplitudes*, *Phys. Rev. D* **103** (2021), no. 5 054042, [[arXiv:2012.00820](https://arxiv.org/abs/2012.00820)].
- [73] G. 't Hooft, *Dimensional regularization and the renormalization group*, *Nucl. Phys. B* **61** (1973) 455–468.
- [74] S. Weinberg, *New approach to the renormalization group*, *Phys. Rev. D* **8** (1973) 3497–3509.
- [75] N. N. Bogoliubov and O. S. Parasiuk, *On the Multiplication of the causal function in the quantum theory of fields*, *Acta Math.* **97** (1957) 227–266.
- [76] K. Hepp, *Proof of the Bogolyubov-Parasiuk theorem on renormalization*, *Commun. Math. Phys.* **2** (1966) 301–326.
- [77] W. Zimmermann, *Convergence of Bogolyubov's method of renormalization in momentum space*, *Commun. Math. Phys.* **15** (1969) 208–234.
- [78] G. 't Hooft, *Renormalizable Lagrangians for Massive Yang-Mills Fields*, *Nucl. Phys. B* **35** (1971) 167–188.
- [79] F. Bloch and A. Nordsieck, *Note on the Radiation Field of the electron*, *Phys. Rev.* **52** (1937) 54–59.
- [80] D. R. Yennie, S. C. Frautschi, and H. Suura, *The infrared divergence phenomena and high-energy processes*, *Annals Phys.* **13** (1961) 379–452.
- [81] T. Kinoshita, *Mass singularities of Feynman amplitudes*, *J. Math. Phys.* **3** (1962) 650–677.
- [82] T. D. Lee and M. Nauenberg, *Degenerate Systems and Mass Singularities*, *Phys. Rev.* **133** (1964) B1549–B1562.
- [83] N. Nakanishi, *General Theory of Infrared Divergence*, *Progress of Theoretical Physics* **19** (02, 1958) 159–168, [<https://academic.oup.com/ptp/article-pdf/19/2/159/5387237/19-2-159.pdf>].
- [84] S. Catani, S. Dittmaier, M. H. Seymour, and Z. Trocsanyi, *The Dipole formalism for next-to-leading order QCD calculations with massive partons*, *Nucl. Phys. B* **627** (2002) 189–265, [[hep-ph/0201036](https://arxiv.org/abs/hep-ph/0201036)].
- [85] S. Frixione, Z. Kunszt, and A. Signer, *Three jet cross-sections to next-to-leading order*, *Nucl. Phys. B* **467** (1996) 399–442, [[hep-ph/9512328](https://arxiv.org/abs/hep-ph/9512328)].
- [86] S. Frixione, *A General approach to jet cross-sections in QCD*, *Nucl. Phys. B* **507** (1997) 295–314, [[hep-ph/9706545](https://arxiv.org/abs/hep-ph/9706545)].
- [87] Z. Nagy and D. E. Soper, *General subtraction method for numerical calculation of one loop QCD matrix elements*, *JHEP* **09** (2003) 055, [[hep-ph/0308127](https://arxiv.org/abs/hep-ph/0308127)].

- [88] T. Gleisberg and F. Krauss, *Automating dipole subtraction for QCD NLO calculations*, *Eur. Phys. J. C* **53** (2008) 501–523, [arXiv:0709.2881].
- [89] R. Frederix, S. Frixione, F. Maltoni, and T. Stelzer, *Automation of next-to-leading order computations in QCD: The FKS subtraction*, *JHEP* **10** (2009) 003, [arXiv:0908.4272].
- [90] W. J. Torres Bobadilla et al., *May the four be with you: Novel IR-subtraction methods to tackle NNLO calculations*, *Eur. Phys. J. C* **81** (2021), no. 3 250, [arXiv:2012.02567].
- [91] D. J. Gross and F. Wilczek, *Ultraviolet Behavior of Nonabelian Gauge Theories*, *Phys. Rev. Lett.* **30** (1973) 1343–1346.
- [92] H. D. Politzer, *Reliable Perturbative Results for Strong Interactions?*, *Phys. Rev. Lett.* **30** (1973) 1346–1349.
- [93] *78 - the removal of infinities in quantum electrodynamics*, in *Collected Papers of L.D. Landau* (D. TER HAAR, ed.), pp. 607–610. Pergamon, 1965.
- [94] R. P. Feynman, *Very high-energy collisions of hadrons*, *Phys. Rev. Lett.* **23** (1969) 1415–1417.
- [95] NNPDF Collaboration, R. D. Ball et al., *Parton distributions from high-precision collider data*, *Eur. Phys. J. C* **77** (2017), no. 10 663, [arXiv:1706.00428].
- [96] T.-J. Hou et al., *New CTEQ global analysis of quantum chromodynamics with high-precision data from the LHC*, *Phys. Rev. D* **103** (2021), no. 1 014013, [arXiv:1912.10053].
- [97] S. Bailey, T. Cridge, L. A. Harland-Lang, A. D. Martin, and R. S. Thorne, *Parton distributions from LHC, HERA, Tevatron and fixed target data: MSHT20 PDFs*, *Eur. Phys. J. C* **81** (2021), no. 4 341, [arXiv:2012.04684].
- [98] Y. L. Dokshitzer, *Calculation of the Structure Functions for Deep Inelastic Scattering and  $e^+ e^-$  Annihilation by Perturbation Theory in Quantum Chromodynamics.*, *Sov. Phys. JETP* **46** (1977) 641–653.
- [99] V. N. Gribov and L. N. Lipatov, *Deep inelastic  $e p$  scattering in perturbation theory*, *Sov. J. Nucl. Phys.* **15** (1972) 438–450.
- [100] G. Altarelli and G. Parisi, *Asymptotic Freedom in Parton Language*, *Nucl. Phys. B* **126** (1977) 298–318.
- [101] G. T. Bodwin, *Factorization of the Drell-Yan Cross-Section in Perturbation Theory*, *Phys. Rev. D* **31** (1985) 2616. [Erratum: *Phys.Rev.D* 34, 3932 (1986)].
- [102] J. C. Collins, D. E. Soper, and G. F. Sterman, *Factorization for Short Distance Hadron - Hadron Scattering*, *Nucl. Phys. B* **261** (1985) 104–142.

- [103] J. C. Collins, D. E. Soper, and G. F. Sterman, *Soft Gluons and Factorization*, *Nucl. Phys. B* **308** (1988) 833–856.
- [104] J. C. Collins, D. E. Soper, and G. F. Sterman, *Factorization of Hard Processes in QCD*, *Adv. Ser. Direct. High Energy Phys.* **5** (1989) 1–91, [[hep-ph/0409313](#)].
- [105] S. D. Ellis, J. Huston, K. Hatakeyama, P. Loch, and M. Tonnesmann, *Jets in hadron-hadron collisions*, *Prog. Part. Nucl. Phys.* **60** (2008) 484–551, [[arXiv:0712.2447](#)].
- [106] G. P. Salam, *Towards Jetography*, *Eur. Phys. J. C* **67** (2010) 637–686, [[arXiv:0906.1833](#)].
- [107] M. Cacciari, *Phenomenological and theoretical developments in jet physics at the LHC*, *Int. J. Mod. Phys. A* **30** (2015), no. 31 1546001, [[arXiv:1509.02272](#)].
- [108] S. L. Glashow, *The renormalizability of vector meson interactions*, *Nucl. Phys.* **10** (1959) 107–117.
- [109] A. Salam and J. C. Ward, *Weak and electromagnetic interactions*, *Nuovo Cim.* **11** (1959) 568–577.
- [110] A. Salam and J. C. Ward, *Electromagnetic and weak interactions*, *Phys. Lett.* **13** (1964) 168–171.
- [111] S. Weinberg, *A Model of Leptons*, *Phys. Rev. Lett.* **19** (1967) 1264–1266.
- [112] P. Sikivie, L. Susskind, M. B. Voloshin, and V. I. Zakharov, *Isospin Breaking in Technicolor Models*, *Nucl. Phys. B* **173** (1980) 189–207.
- [113] H. Georgi, *Effective field theory*, *Ann. Rev. Nucl. Part. Sci.* **43** (1993) 209–252.
- [114] S. Bilenky, *Neutrino oscillations: From a historical perspective to the present status*, *Nucl. Phys. B* **908** (2016) 2–13, [[arXiv:1602.00170](#)].
- [115] F. Capozzi, E. Lisi, A. Marrone, D. Montanino, and A. Palazzo, *Neutrino masses and mixings: Status of known and unknown  $3\nu$  parameters*, *Nucl. Phys. B* **908** (2016) 218–234, [[arXiv:1601.07777](#)].
- [116] **ATLAS** Collaboration, M. Aaboud et al.,  *$ZZ \rightarrow \ell^+\ell^-\ell'^+\ell'^-$  cross-section measurements and search for anomalous triple gauge couplings in 13 TeV pp collisions with the ATLAS detector*, *Phys. Rev. D* **97** (2018), no. 3 032005, [[arXiv:1709.07703](#)].
- [117] **ATLAS** Collaboration, M. Aaboud et al., *Measurement of the four-lepton invariant mass spectrum in 13 TeV proton-proton collisions with the ATLAS detector*, *JHEP* **04** (2019) 048, [[arXiv:1902.05892](#)].

- [118] **ATLAS** Collaboration, M. Aaboud et al., *Measurement of  $ZZ$  production in the  $\ell\nu\nu$  final state with the ATLAS detector in  $pp$  collisions at  $\sqrt{s} = 13$  TeV*, *JHEP* **10** (2019) 127, [[arXiv:1905.07163](#)].
- [119] **CMS** Collaboration, A. M. Sirunyan et al., *Measurements of the  $pp \rightarrow ZZ$  production cross section and the  $Z \rightarrow 4\ell$  branching fraction, and constraints on anomalous triple gauge couplings at  $\sqrt{s} = 13$  TeV*, *Eur. Phys. J. C* **78** (2018) 165, [[arXiv:1709.08601](#)]. [Erratum: *Eur.Phys.J.C* 78, 515 (2018)].
- [120] **CMS** Collaboration, A. M. Sirunyan et al., *Measurements of  $pp \rightarrow ZZ$  production cross sections and constraints on anomalous triple gauge couplings at  $\sqrt{s} = 13$  TeV*, [arXiv:2009.01186](#).
- [121] **ATLAS** Collaboration, M. Aaboud et al., *Constraints on off-shell Higgs boson production and the Higgs boson total width in  $ZZ \rightarrow 4\ell$  and  $ZZ \rightarrow 2\ell 2\nu$  final states with the ATLAS detector*, *Phys. Lett. B* **786** (2018) 223–244, [[arXiv:1808.01191](#)].
- [122] **CMS** Collaboration, A. M. Sirunyan et al., *Measurement and interpretation of differential cross sections for Higgs boson production at  $\sqrt{s} = 13$  TeV*, *Phys. Lett. B* **792** (2019) 369–396, [[arXiv:1812.06504](#)].
- [123] **ATLAS** Collaboration, G. Aad et al., *Measurements of the Higgs boson inclusive and differential fiducial cross sections in the  $4\ell$  decay channel at  $\sqrt{s} = 13$  TeV*, *Eur. Phys. J. C* **80** (2020), no. 10 942, [[arXiv:2004.03969](#)].
- [124] N. Kauer, *Interference effects for  $H \rightarrow WW/ZZ \rightarrow \ell\bar{\nu}_\ell\bar{\ell}\nu_\ell$  searches in gluon fusion at the LHC*, *JHEP* **12** (2013) 082, [[arXiv:1310.7011](#)].
- [125] F. Cascioli, T. Gehrmann, M. Grazzini, S. Kallweit, P. Maierhöfer, A. von Manteuffel, S. Pozzorini, D. Rathlev, L. Tancredi, and E. Weihs,  *$ZZ$  production at hadron colliders in NNLO QCD*, *Phys. Lett.* **B735** (2014) 311–313, [[arXiv:1405.2219](#)].
- [126] F. Caola, K. Melnikov, R. Röntsch, and L. Tancredi, *QCD corrections to  $ZZ$  production in gluon fusion at the LHC*, *Phys. Rev.* **D92** (2015), no. 9 094028, [[arXiv:1509.06734](#)].
- [127] M. Grazzini, S. Kallweit, M. Wiesemann, and J. Y. Yook,  *$ZZ$  production at the LHC: NLO QCD corrections to the loop-induced gluon fusion channel*, *JHEP* **03** (2019) 070, [[arXiv:1811.09593](#)].
- [128] J. Ohnemus, *Order  $\alpha^{-s}$  calculations of hadronic  $W^\pm\gamma$  and  $Z\gamma$  production*, *Phys. Rev. D* **47** (1993) 940–955.
- [129] U. Baur, T. Han, and J. Ohnemus, *QCD corrections to hadronic  $W\gamma$  production with nonstandard  $WW\gamma$  couplings*, *Phys. Rev. D* **48** (1993) 5140–5161, [[hep-ph/9305314](#)].
- [130] U. Baur, T. Han, and J. Ohnemus, *QCD corrections and anomalous couplings in  $Z\gamma$  production at hadron colliders*, *Phys. Rev. D* **57** (1998) 2823–2836, [[hep-ph/9710416](#)].

- [131] L. J. Dixon, Z. Kunszt, and A. Signer, *Helicity amplitudes for  $O(\alpha_s)$  production of  $W^+W^-$ ,  $W^\pm Z$ ,  $ZZ$ ,  $W^\pm\gamma$ , or  $Z\gamma$  pairs at hadron colliders*, *Nucl. Phys. B* **531** (1998) 3–23, [[hep-ph/9803250](#)].
- [132] J. M. Campbell and R. K. Ellis, *An Update on vector boson pair production at hadron colliders*, *Phys. Rev. D* **60** (1999) 113006, [[hep-ph/9905386](#)].
- [133] L. J. Dixon, Z. Kunszt, and A. Signer, *Vector boson pair production in hadronic collisions at order  $\alpha_s$  : Lepton correlations and anomalous couplings*, *Phys. Rev. D* **60** (1999) 114037, [[hep-ph/9907305](#)].
- [134] E. Accomando, A. Denner, and A. Kaiser, *Logarithmic electroweak corrections to gauge-boson pair production at the LHC*, *Nucl. Phys. B* **706** (2005) 325–371, [[hep-ph/0409247](#)].
- [135] E. Accomando, A. Denner, and C. Meier, *Electroweak corrections to  $W\gamma$  and  $Z\gamma$  production at the LHC*, *Eur. Phys. J. C* **47** (2006) 125–146, [[hep-ph/0509234](#)].
- [136] E. Accomando and A. Kaiser, *Electroweak corrections and anomalous triple gauge-boson couplings in  $W^+W^-$  and  $W^\pm Z$  production at the LHC*, *Phys. Rev. D* **73** (2006) 093006, [[hep-ph/0511088](#)].
- [137] A. Bierweiler, T. Kasprzik, and J. H. Kühn, *Vector-boson pair production at the LHC to  $O(\alpha^3)$  accuracy*, *JHEP* **12** (2013) 071, [[arXiv:1305.5402](#)].
- [138] J. Baglio, L. D. Ninh, and M. M. Weber, *Massive gauge boson pair production at the LHC: a next-to-leading order story*, *Phys. Rev. D* **88** (2013) 113005, [[arXiv:1307.4331](#)]. [Erratum: *Phys.Rev.D* 94, 099902 (2016)].
- [139] M. Billoni, S. Dittmaier, B. Jäger, and C. Speckner, *Next-to-leading order electroweak corrections to  $pp \rightarrow W+W^- \rightarrow 4$  leptons at the LHC in double-pole approximation*, *JHEP* **12** (2013) 043, [[arXiv:1310.1564](#)].
- [140] S. Gieseke, T. Kasprzik, and J. H. Kühn, *Vector-boson pair production and electroweak corrections in HERWIG++*, *Eur. Phys. J. C* **74** (2014), no. 8 2988, [[arXiv:1401.3964](#)].
- [141] A. Denner, S. Dittmaier, M. Hecht, and C. Pasold, *NLO QCD and electroweak corrections to  $W+\gamma$  production with leptonic  $W$ -boson decays*, *JHEP* **04** (2015) 018, [[arXiv:1412.7421](#)].
- [142] R. Britto, F. Cachazo, and B. Feng, *Generalized unitarity and one-loop amplitudes in  $N=4$  super-Yang-Mills*, *Nucl. Phys. B* **725** (2005) 275–305, [[hep-th/0412103](#)].
- [143] G. Ossola, C. G. Papadopoulos, and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys. B* **763** (2007) 147–169, [[hep-ph/0609007](#)].

- [144] R. K. Ellis, W. T. Giele, Z. Kunszt, and K. Melnikov, *Masses, fermions and generalized D-dimensional unitarity*, *Nucl. Phys. B* **822** (2009) 270–282, [[arXiv:0806.3467](#)].
- [145] C. F. Berger and D. Forde, *Multi-Parton Scattering Amplitudes via On-Shell Methods*, *Ann. Rev. Nucl. Part. Sci.* **60** (2010) 181–205, [[arXiv:0912.3534](#)].
- [146] F. Cascioli, P. Maierhofer, and S. Pozzorini, *Scattering Amplitudes with Open Loops*, *Phys. Rev. Lett.* **108** (2012) 111601, [[arXiv:1111.5206](#)].
- [147] Z. Bern, L. J. Dixon, and D. A. Kosower, *On-Shell Methods in Perturbative QCD*, *Annals Phys.* **322** (2007) 1587–1634, [[arXiv:0704.2798](#)].
- [148] R. K. Ellis, Z. Kunszt, K. Melnikov, and G. Zanderighi, *One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts*, *Phys. Rept.* **518** (2012) 141–250, [[arXiv:1105.4319](#)].
- [149] V. Costantini, B. De Tollis, and G. Pistoni, *Nonlinear effects in quantum electrodynamics*, *Nuovo Cim. A* **2** (1971), no. 3 733–787.
- [150] D. A. Dicus, C. Kao, and W. Repko, *Gluon Production of Gauge Bosons*, *Phys. Rev. D* **36** (1987) 1570.
- [151] J. J. van der Bij and E. W. N. Glover, *PHOTON Z BOSON PAIR PRODUCTION VIA GLUON FUSION*, *Phys. Lett. B* **206** (1988) 701–704.
- [152] E. W. N. Glover and J. J. van der Bij, *VECTOR BOSON PAIR PRODUCTION VIA GLUON FUSION*, *Phys. Lett. B* **219** (1989) 488–492.
- [153] E. W. N. Glover and J. J. van der Bij, *Z BOSON PAIR PRODUCTION VIA GLUON FUSION*, *Nucl. Phys.* **B321** (1989) 561–590.
- [154] C. Zecher, T. Matsuura, and J. van der Bij, *Leptonic signals from off-shell Z boson pairs at hadron colliders*, *Z. Phys. C* **64** (1994) 219–226, [[hep-ph/9404295](#)].
- [155] K. L. Adamson, D. de Florian, and A. Signer, *Gluon induced contributions to WZ and W gamma production at NNLO*, *Phys. Rev. D* **65** (2002) 094041, [[hep-ph/0202132](#)].
- [156] K. L. Adamson, D. de Florian, and A. Signer, *Gluon induced contributions to Z gamma production at hadron colliders*, *Phys. Rev. D* **67** (2003) 034016, [[hep-ph/0211295](#)].
- [157] T. Binoth, M. Ciccolini, N. Kauer, and M. Kramer, *Gluon-induced WW background to Higgs boson searches at the LHC*, *JHEP* **03** (2005) 065, [[hep-ph/0503094](#)].
- [158] T. Binoth, M. Ciccolini, N. Kauer, and M. Kramer, *Gluon-induced W-boson pair production at the LHC*, *JHEP* **12** (2006) 046, [[hep-ph/0611170](#)].
- [159] A. von Manteuffel and L. Tancredi, *The two-loop helicity amplitudes for  $gg \rightarrow V_1 V_2 \rightarrow 4$  leptons*, *JHEP* **06** (2015) 197, [[arXiv:1503.08835](#)].

- [160] F. Caola, J. M. Henn, K. Melnikov, A. V. Smirnov, and V. A. Smirnov, *Two-loop helicity amplitudes for the production of two off-shell electroweak bosons in gluon fusion*, *JHEP* **06** (2015) 129, [[arXiv:1503.08759](#)].
- [161] A. Azatov, C. Grojean, A. Paul, and E. Salvioni, *Resolving gluon fusion loops at current and future hadron colliders*, *JHEP* **09** (2016) 123, [[arXiv:1608.00977](#)].
- [162] Q.-H. Cao, B. Yan, C. Yuan, and Y. Zhang, *Probing  $Zt\bar{t}$  couplings using  $Z$  boson polarization in  $ZZ$  production at hadron colliders*, *Phys. Rev. D* **102** (2020), no. 5 055010, [[arXiv:2004.02031](#)].
- [163] K. Melnikov and M. Dowling, *Production of two  $Z$ -bosons in gluon fusion in the heavy top quark approximation*, *Phys. Lett.* **B744** (2015) 43–47, [[arXiv:1503.01274](#)].
- [164] F. Caola, M. Dowling, K. Melnikov, R. Röntsch, and L. Tancredi,  *$QCD$  corrections to vector boson pair production in gluon fusion including interference effects with off-shell Higgs at the LHC*, *JHEP* **07** (2016) 087, [[arXiv:1605.04610](#)].
- [165] J. M. Campbell, R. K. Ellis, M. Czakon, and S. Kirchner, *Two loop correction to interference in  $gg \rightarrow ZZ$* , *JHEP* **08** (2016) 011, [[arXiv:1605.01380](#)].
- [166] R. Gröber, A. Maier, and T. Rauh, *Top quark mass effects in  $gg \rightarrow ZZ$  at two loops and off-shell Higgs interference*, *Phys. Rev.* **D100** (2019), no. 11 114013, [[arXiv:1908.04061](#)].
- [167] M. Spira, A. Djouadi, D. Graudenz, and P. Zerwas, *Higgs boson production at the LHC*, *Nucl. Phys. B* **453** (1995) 17–82, [[hep-ph/9504378](#)].
- [168] R. Harlander and P. Kant, *Higgs production and decay: Analytic results at next-to-leading order  $QCD$* , *JHEP* **12** (2005) 015, [[hep-ph/0509189](#)].
- [169] C. Anastasiou, S. Beerli, S. Bucherer, A. Daleo, and Z. Kunszt, *Two-loop amplitudes and master integrals for the production of a Higgs boson via a massive quark and a scalar-quark loop*, *JHEP* **01** (2007) 082, [[hep-ph/0611236](#)].
- [170] U. Aglietti, R. Bonciani, G. Degrossi, and A. Vicini, *Analytic Results for Virtual  $QCD$  Corrections to Higgs Production and Decay*, *JHEP* **01** (2007) 021, [[hep-ph/0611266](#)].
- [171] C. Brønnum-Hansen and C.-Y. Wang, *Contribution of third generation quarks to two-loop helicity amplitudes for  $W$  boson pair production in gluon fusion*, [arXiv:2009.03742](#).
- [172] C. Brønnum-Hansen and C.-Y. Wang, *Top quark contribution to two-loop helicity amplitudes for  $Z$  boson pair production in gluon fusion*, [arXiv:2101.12095](#).
- [173] **ATLAS** Collaboration, M. Aaboud et al., *Search for resonances in diphoton events at  $\sqrt{s}=13$  TeV with the ATLAS detector*, *JHEP* **09** (2016) 001, [[arXiv:1606.03833](#)].

- [174] CMS Collaboration, A. M. Sirunyan et al., *Search for physics beyond the standard model in high-mass diphoton events from proton-proton collisions at  $\sqrt{s} = 13$  TeV*, *Phys. Rev. D* **98** (2018), no. 9 092001, [arXiv:1809.00327].
- [175] S. Amoroso et al., *Les Houches 2019: Physics at TeV Colliders: Standard Model Working Group Report*, in *11th Les Houches Workshop on Physics at TeV Colliders: PhysTeV Les Houches*, 3, 2020. arXiv:2003.01700.
- [176] E. W. N. Glover and M. E. Tejeda-Yeomans, *Two loop QCD helicity amplitudes for massless quark massless gauge boson scattering*, *JHEP* **06** (2003) 033, [hep-ph/0304169].
- [177] Z. Bern, A. De Freitas, and L. J. Dixon, *Two loop amplitudes for gluon fusion into two photons*, *JHEP* **09** (2001) 037, [hep-ph/0109078].
- [178] S. Catani, L. Cieri, D. de Florian, G. Ferrera, and M. Grazzini, *Diphoton production at hadron colliders: a fully-differential QCD calculation at NNLO*, *Phys. Rev. Lett.* **108** (2012) 072001, [arXiv:1110.2375]. [Erratum: Phys.Rev.Lett. 117, 089901 (2016)].
- [179] T. Gehrmann, N. Glover, A. Huss, and J. Whitehead, *Scale and isolation sensitivity of diphoton distributions at the LHC*, *JHEP* **01** (2021) 108, [arXiv:2009.11310].
- [180] T. Ahmed, J. Henn, and B. Mistlberger, *Four-particle scattering amplitudes in QCD at NNLO to higher orders in the dimensional regulator*, *JHEP* **12** (2019) 177, [arXiv:1910.06684].
- [181] J. Henn, B. Mistlberger, V. A. Smirnov, and P. Wasser, *Constructing  $d$ -log integrands and computing master integrals for three-loop four-particle scattering*, *JHEP* **04** (2020) 167, [arXiv:2002.09492].
- [182] F. Caola, A. Von Manteuffel, and L. Tancredi, *Diphoton Amplitudes in Three-Loop Quantum Chromodynamics*, *Phys. Rev. Lett.* **126** (2021), no. 11 112004, [arXiv:2011.13946].
- [183] V. Del Duca, F. Maltoni, Z. Nagy, and Z. Trocsanyi, *QCD radiative corrections to prompt diphoton production in association with a jet at hadron colliders*, *JHEP* **04** (2003) 059, [hep-ph/0303012].
- [184] T. Gehrmann, N. Greiner, and G. Heinrich, *Photon isolation effects at NLO in  $\gamma\gamma +$  jet final states in hadronic collisions*, *JHEP* **06** (2013) 058, [arXiv:1303.0824]. [Erratum: JHEP 06, 076 (2014)].
- [185] T. Gehrmann, J. M. Henn, and N. A. Lo Presti, *Analytic form of the two-loop planar five-gluon all-plus-helicity amplitude in QCD*, *Phys. Rev. Lett.* **116** (2016), no. 6 062001, [arXiv:1511.05409]. [Erratum: Phys.Rev.Lett. 116, 189903 (2016)].

- [186] S. Badger, C. Brønnum-Hansen, H. B. Hartanto, and T. Peraro, *First look at two-loop five-gluon scattering in QCD*, *Phys. Rev. Lett.* **120** (2018), no. 9 092001, [[arXiv:1712.02229](#)].
- [187] S. Abreu, F. Febres Cordero, H. Ita, B. Page, and M. Zeng, *Planar Two-Loop Five-Gluon Amplitudes from Numerical Unitarity*, *Phys. Rev.* **D97** (2018), no. 11 116014, [[arXiv:1712.03946](#)].
- [188] S. Abreu, F. Febres Cordero, H. Ita, B. Page, and V. Sotnikov, *Planar Two-Loop Five-Parton Amplitudes from Numerical Unitarity*, *JHEP* **11** (2018) 116, [[arXiv:1809.09067](#)].
- [189] S. Abreu, B. Page, E. Pascual, and V. Sotnikov, *Leading-Color Two-Loop QCD Corrections for Three-Photon Production at Hadron Colliders*, *JHEP* **01** (2021) 078, [[arXiv:2010.15834](#)].
- [190] S. Badger, C. Brønnum-Hansen, H. B. Hartanto, and T. Peraro, *Analytic helicity amplitudes for two-loop five-gluon scattering: the single-minus case*, *JHEP* **01** (2019) 186, [[arXiv:1811.11699](#)].
- [191] H. A. Chawdhry, M. Czakon, A. Mitov, and R. Poncelet, *Two-loop leading-color helicity amplitudes for three-photon production at the LHC*, [arXiv:2012.13553](#).
- [192] G. De Laurentis and D. Maître, *Two-Loop Five-Parton Leading-Colour Finite Remainders in the Spinor-Helicity Formalism*, *JHEP* **02** (2021) 016, [[arXiv:2010.14525](#)].
- [193] B. Agarwal, F. Buccioni, A. von Manteuffel, and L. Tancredi, *Two-loop leading colour QCD corrections to  $q\bar{q} \rightarrow \gamma\gamma g$  and  $qg \rightarrow \gamma\gamma q$* , *JHEP* **04** (2021) 201, [[arXiv:2102.01820](#)].
- [194] S. Badger, H. B. Hartanto, and S. Zoia, *Two-loop QCD corrections to  $Wb\bar{b}$  production at hadron colliders*, [arXiv:2102.02516](#).
- [195] S. Abreu, F. Febres Cordero, H. Ita, B. Page, and V. Sotnikov, *Leading-Color Two-Loop QCD Corrections for Three-Jet Production at Hadron Colliders*, [arXiv:2102.13609](#).
- [196] S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J. M. Henn, T. Peraro, P. Wasser, Y. Zhang, and S. Zoia, *Analytic form of the full two-loop five-gluon all-plus helicity amplitude*, *Phys. Rev. Lett.* **123** (2019), no. 7 071601, [[arXiv:1905.03733](#)].
- [197] B. Agarwal, F. Buccioni, A. von Manteuffel, and L. Tancredi, *Two-loop helicity amplitudes for diphoton plus jet production in full color*, [arXiv:2105.04585](#).
- [198] S. Badger, C. Brønnum-Hansen, D. Chicherin, T. Gehrmann, H. B. Hartanto, J. Henn, M. Marcoli, R. Moodie, T. Peraro, and S. Zoia, *Virtual QCD corrections to gluon-initiated diphoton plus jet production at hadron colliders*, [arXiv:2106.08664](#).

- [199] F. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, and T. T. Wu, *Single bremsstrahlung processes in gauge theories*, *Physics Letters B* **103** (1981), no. 2 124–128.
- [200] Z. Xu, D.-H. Zhang, and L. Chang, *Helicity Amplitudes for Multiple Bremsstrahlung in Massless Nonabelian Gauge Theories*, *Nucl. Phys. B* **291** (1987) 392–428.
- [201] L. J. Dixon, *Calculating scattering amplitudes efficiently*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 95): QCD and Beyond*, 1, 1996. [hep-ph/9601359](#).
- [202] M. L. Mangano and S. J. Parke, *Multiparton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301–367, [[hep-th/0509223](#)].
- [203] M. E. Peskin, *Simplifying Multi-Jet QCD Computation*, in *13th Mexican School of Particles and Fields*, 1, 2011. [arXiv:1101.2414](#).
- [204] S. J. Parke and T. R. Taylor, *Amplitude for  $n$ -gluon scattering*, *Phys. Rev. Lett.* **56** (Jun, 1986) 2459–2460.
- [205] F. Berends and W. Giele, *Recursive calculations for processes with  $n$  gluons*, *Nuclear Physics B* **306** (1988), no. 4 759–808.
- [206] P. Nogueira, *Automatic feynman graph generation*, *Journal of Computational Physics* **105** (1993), no. 2 279 – 289.
- [207] B. Ruijl, T. Ueda, and J. Vermaseren, *FORM version 4.2*, [arXiv:1707.06453](#).
- [208] J. Kuipers, T. Ueda, J. A. M. Vermaseren, and J. Vollinga, *FORM version 4.0*, *Comput. Phys. Commun.* **184** (2013) 1453–1467, [[arXiv:1203.6543](#)].
- [209] J. A. M. Vermaseren, *New features of FORM*, [math-ph/0010025](#).
- [210] D. Kreimer, *The  $\gamma(5)$  Problem and Anomalies: A Clifford Algebra Approach*, *Phys. Lett. B* **237** (1990) 59–62.
- [211] J. Korner, D. Kreimer, and K. Schilcher, *A Practicable  $\gamma(5)$  scheme in dimensional regularization*, *Z. Phys. C* **54** (1992) 503–512.
- [212] D. Kreimer, *The Role of  $\gamma(5)$  in dimensional regularization*, [hep-ph/9401354](#).
- [213] P. Breitenlohner and D. Maison, *Dimensionally Renormalized Green’s Functions for Theories with Massless Particles. 1.*, *Commun. Math. Phys.* **52** (1977) 39.
- [214] P. Breitenlohner and D. Maison, *Dimensionally Renormalized Green’s Functions for Theories with Massless Particles. 2.*, *Commun. Math. Phys.* **52** (1977) 55.
- [215] P. Breitenlohner and D. Maison, *Dimensional Renormalization and the Action Principle*, *Commun. Math. Phys.* **52** (1977) 11–38.

- [216] D. B. Melrose, *Reduction of Feynman diagrams*, *Nuovo Cim.* **40** (1965) 181–213.
- [217] W. L. van Neerven and J. A. M. Vermaseren, *LARGE LOOP INTEGRALS*, *Phys. Lett. B* **137** (1984) 241–244.
- [218] Z. Bern, L. J. Dixon, and D. A. Kosower, *Dimensionally regulated one loop integrals*, *Phys. Lett. B* **302** (1993) 299–308, [[hep-ph/9212308](#)]. [Erratum: *Phys.Lett.B* 318, 649 (1993)].
- [219] G. Passarino and M. J. G. Veltman, *One Loop Corrections for  $e^+ e^-$  Annihilation Into  $\mu^+ \mu^-$  in the Weinberg Model*, *Nucl. Phys. B* **160** (1979) 151–207.
- [220] T. Hahn and M. Perez-Victoria, *Automatized one loop calculations in four-dimensions and D-dimensions*, *Comput. Phys. Commun.* **118** (1999) 153–165, [[hep-ph/9807565](#)].
- [221] R. K. Ellis and G. Zanderighi, *Scalar one-loop integrals for QCD*, *JHEP* **02** (2008) 002, [[arXiv:0712.1851](#)].
- [222] A. van Hameren, *OneLoop: For the evaluation of one-loop scalar functions*, *Comput. Phys. Commun.* **182** (2011) 2427–2438, [[arXiv:1007.4716](#)].
- [223] G. Cullen, J. P. Guillet, G. Heinrich, T. Kleinschmidt, E. Pilon, T. Reiter, and M. Rodgers, *Golem95C: A library for one-loop integrals with complex masses*, *Comput. Phys. Commun.* **182** (2011) 2276–2284, [[arXiv:1101.5595](#)].
- [224] A. Denner, S. Dittmaier, and L. Hofer, *COLLIER - A fortran-library for one-loop integrals*, *PoS* **LL2014** (2014) 071, [[arXiv:1407.0087](#)].
- [225] H. H. Patel, *Package-X: A Mathematica package for the analytic calculation of one-loop integrals*, *Comput. Phys. Commun.* **197** (2015) 276–290, [[arXiv:1503.01469](#)].
- [226] L. J. Dixon, *A brief introduction to modern amplitude methods*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Particle Physics: The Higgs Boson and Beyond*, 10, 2013. [arXiv:1310.5353](#).
- [227] Z. Bern, L. J. Dixon, and D. A. Kosower, *One loop amplitudes for  $e^+ e^-$  to four partons*, *Nucl. Phys. B* **513** (1998) 3–86, [[hep-ph/9708239](#)].
- [228] R. K. Ellis, W. T. Giele, and Z. Kunszt, *A Numerical Unitarity Formalism for Evaluating One-Loop Amplitudes*, *JHEP* **03** (2008) 003, [[arXiv:0708.2398](#)].
- [229] W. T. Giele, Z. Kunszt, and K. Melnikov, *Full one-loop amplitudes from tree amplitudes*, *JHEP* **04** (2008) 049, [[arXiv:0801.2237](#)].
- [230] P. Mastrolia, G. Ossola, C. G. Papadopoulos, and R. Pittau, *Optimizing the Reduction of One-Loop Amplitudes*, *JHEP* **06** (2008) 030, [[arXiv:0803.3964](#)].

- [231] P. Mastrolia, E. Mirabella, and T. Peraro, *Integrand reduction of one-loop scattering amplitudes through Laurent series expansion*, *JHEP* **06** (2012) 095, [[arXiv:1203.0291](#)]. [Erratum: *JHEP* **11**, 128 (2012)].
- [232] G. Ossola, C. G. Papadopoulos, and R. Pittau, *CutTools: A Program implementing the OPP reduction method to compute one-loop amplitudes*, *JHEP* **03** (2008) 042, [[arXiv:0711.3596](#)].
- [233] P. Mastrolia, G. Ossola, T. Reiter, and F. Tramontano, *Scattering AMplitudes from Unitarity-based Reduction Algorithm at the Integrand-level*, *JHEP* **08** (2010) 080, [[arXiv:1006.0710](#)].
- [234] T. Peraro, *Ninja: Automated Integrand Reduction via Laurent Expansion for One-Loop Amplitudes*, *Comput. Phys. Commun.* **185** (2014) 2771–2797, [[arXiv:1403.1229](#)].
- [235] C. F. Berger, Z. Bern, L. J. Dixon, F. Febres Cordero, D. Forde, H. Ita, D. A. Kosower, and D. Maitre, *An Automated Implementation of On-Shell Methods for One-Loop Amplitudes*, *Phys. Rev. D* **78** (2008) 036003, [[arXiv:0803.4180](#)].
- [236] G. Bevilacqua, M. Czakon, M. V. Garzelli, A. van Hameren, A. Kardos, C. G. Papadopoulos, R. Pittau, and M. Worek, *HELAC-NLO*, *Comput. Phys. Commun.* **184** (2013) 986–997, [[arXiv:1110.1499](#)].
- [237] J. Alwall, R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, O. Mattelaer, H. S. Shao, T. Stelzer, P. Torrielli, and M. Zaro, *The automated computation of tree-level and next-to-leading order differential cross sections, and their matching to parton shower simulations*, *JHEP* **07** (2014) 079, [[arXiv:1405.0301](#)].
- [238] G. Cullen et al., *GOSAM-2.0: a tool for automated one-loop calculations within the Standard Model and beyond*, *Eur. Phys. J. C* **74** (2014), no. 8 3001, [[arXiv:1404.7096](#)].
- [239] F. Buccioni, J.-N. Lang, J. M. Lindert, P. Maierhöfer, S. Pozzorini, H. Zhang, and M. F. Zoller, *OpenLoops 2*, *Eur. Phys. J. C* **79** (2019), no. 10 866, [[arXiv:1907.13071](#)].
- [240] F. del Aguila and R. Pittau, *Recursive numerical calculus of one-loop tensor integrals*, *JHEP* **07** (2004) 017, [[hep-ph/0404120](#)].
- [241] K. G. Chetyrkin and F. V. Tkachov, *Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops*, *Nucl. Phys.* **B192** (1981) 159–204.
- [242] C. Studerus, *Reduze-Feynman Integral Reduction in C++*, *Comput. Phys. Commun.* **181** (2010) 1293–1300, [[arXiv:0912.2546](#)].
- [243] A. von Manteuffel and C. Studerus, *Reduze 2 - Distributed Feynman Integral Reduction*, [arXiv:1201.4330](#).

- [244] C. W. Bauer, A. Frink, and R. Kreckel, *Introduction to the GiNaC framework for symbolic computation within the C++ programming language*, *J. Symb. Comput.* **33** (2002) 1, [[cs/0004015](#)].
- [245] R. H. Lewis, “*Computer Algebra System Fermat.*” <http://home.bway.net/lewis/>.
- [246] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys.* **A15** (2000) 5087–5159, [[hep-ph/0102033](#)].
- [247] C. Anastasiou and A. Lazopoulos, *Automatic integral reduction for higher order perturbative calculations*, *JHEP* **07** (2004) 046, [[hep-ph/0404258](#)].
- [248] R. N. Lee, *Presenting LiteRed: a tool for the Loop InTEgrals REDuction*, [arXiv:1212.2685](#).
- [249] P. Maierhöfer, J. Usovitsch, and P. Uwer, *Kira—A Feynman integral reduction program*, *Comput. Phys. Commun.* **230** (2018) 99–112, [[arXiv:1705.05610](#)].
- [250] A. Smirnov and F. Chuharev, *FIRE6: Feynman Integral REDuction with Modular Arithmetic*, [arXiv:1901.07808](#).
- [251] J. Gluza, K. Kajda, and D. A. Kosower, *Towards a Basis for Planar Two-Loop Integrals*, *Phys. Rev.* **D83** (2011) 045012, [[arXiv:1009.0472](#)].
- [252] R. M. Schabinger, *A New Algorithm For The Generation Of Unitarity-Compatible Integration By Parts Relations*, *JHEP* **01** (2012) 077, [[arXiv:1111.4220](#)].
- [253] P. A. Baikov, *Explicit solutions of the three loop vacuum integral recurrence relations*, *Phys. Lett.* **B385** (1996) 404–410, [[hep-ph/9603267](#)].
- [254] R. N. Lee, *Calculating multiloop integrals using dimensional recurrence relation and D-analyticity*, *Nucl. Phys. B Proc. Suppl.* **205-206** (2010) 135–140, [[arXiv:1007.2256](#)].
- [255] H. Frellesvig and C. G. Papadopoulos, *Cuts of Feynman Integrals in Baikov representation*, *JHEP* **04** (2017) 083, [[arXiv:1701.07356](#)].
- [256] M. Harley, F. Moriello, and R. M. Schabinger, *Baikov-Lee Representations Of Cut Feynman Integrals*, *JHEP* **06** (2017) 049, [[arXiv:1705.03478](#)].
- [257] R. E. Cutkosky, *Singularities and discontinuities of Feynman amplitudes*, *J. Math. Phys.* **1** (1960) 429–433.
- [258] R. N. Lee and V. A. Smirnov, *The Dimensional Recurrence and Analyticity Method for Multicomponent Master Integrals: Using Unitarity Cuts to Construct Homogeneous Solutions*, *JHEP* **12** (2012) 104, [[arXiv:1209.0339](#)].
- [259] S. Abreu, R. Britto, C. Duhr, and E. Gardi, *Cuts from residues: the one-loop case*, *JHEP* **06** (2017) 114, [[arXiv:1702.03163](#)].

- [260] G. 't Hooft and M. J. G. Veltman, *DIAGRAMMAR*, *NATO Sci. Ser. B* **4** (1974) 177–322.
- [261] M. J. G. Veltman, *Diagrammatica: The Path to Feynman rules*, vol. 4. Cambridge University Press, 5, 2012.
- [262] R. G. Newton, *Optical theorem and beyond*, *American Journal of Physics* **44** (1976), no. 7 639–642, [<https://doi.org/10.1119/1.10324>].
- [263] A. Primo and L. Tancredi, *On the maximal cut of Feynman integrals and the solution of their differential equations*, *Nucl. Phys. B* **916** (2017) 94–116, [[arXiv:1610.08397](https://arxiv.org/abs/1610.08397)].
- [264] D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang, and S. Zoia, *All Master Integrals for Three-Jet Production at Next-to-Next-to-Leading Order*, *Phys. Rev. Lett.* **123** (2019), no. 4 041603, [[arXiv:1812.11160](https://arxiv.org/abs/1812.11160)].
- [265] C. Dlapa, X. Li, and Y. Zhang, *Leading singularities in Baikov representation and Feynman integrals with uniform transcendental weight*, [arXiv:2103.04638](https://arxiv.org/abs/2103.04638).
- [266] H. Ita, *Two-loop Integrand Decomposition into Master Integrals and Surface Terms*, *Phys. Rev.* **D94** (2016), no. 11 116015, [[arXiv:1510.05626](https://arxiv.org/abs/1510.05626)].
- [267] K. J. Larsen and Y. Zhang, *Integration-by-parts reductions from unitarity cuts and algebraic geometry*, *Phys. Rev.* **D93** (2016), no. 4 041701, [[arXiv:1511.01071](https://arxiv.org/abs/1511.01071)].
- [268] R. N. Lee, *Modern techniques of multiloop calculations*, in *49th Rencontres de Moriond on QCD and High Energy Interactions*, pp. 297–300, 2014. [arXiv:1405.5616](https://arxiv.org/abs/1405.5616).
- [269] Y. Zhang, *Lecture Notes on Multi-loop Integral Reduction and Applied Algebraic Geometry*, 12, 2016. [arXiv:1612.02249](https://arxiv.org/abs/1612.02249).
- [270] J. Böhm, A. Georgoudis, K. J. Larsen, M. Schulze, and Y. Zhang, *Complete sets of logarithmic vector fields for integration-by-parts identities of Feynman integrals*, *Phys. Rev.* **D98** (2018), no. 2 025023, [[arXiv:1712.09737](https://arxiv.org/abs/1712.09737)].
- [271] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, “SINGULAR 4-1-2 — A computer algebra system for polynomial computations.” <http://www.singular.uni-kl.de>, 2019.
- [272] A. von Manteuffel and R. M. Schabinger, *A novel approach to integration by parts reduction*, *Phys. Lett.* **B744** (2015) 101–104, [[arXiv:1406.4513](https://arxiv.org/abs/1406.4513)].
- [273] T. Peraro, *Scattering amplitudes over finite fields and multivariate functional reconstruction*, *JHEP* **12** (2016) 030, [[arXiv:1608.01902](https://arxiv.org/abs/1608.01902)].
- [274] D. Cabarcas and J. Ding, *Linear Algebra to Compute Syzygies and Gröbner Bases*, *ACM Press* (2011), no. 67-74.

- [275] S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, and B. Page, *Analytic Form of Planar Two-Loop Five-Gluon Scattering Amplitudes in QCD*, *Phys. Rev. Lett.* **122** (2019), no. 8 082002, [arXiv:1812.04586].
- [276] M. Heller and A. von Manteuffel, *Multivariate Apart: Generalized Partial Fractions*, arXiv:2101.08283.
- [277] A. V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett.* **B254** (1991) 158–164.
- [278] E. Remiddi, *Differential equations for Feynman graph amplitudes*, *Nuovo Cim.* **A110** (1997) 1435–1452, [hep-th/9711188].
- [279] T. Gehrmann and E. Remiddi, *Differential equations for two loop four point functions*, *Nucl. Phys.* **B580** (2000) 485–518, [hep-ph/9912329].
- [280] M. Argeri and P. Mastrolia, *Feynman Diagrams and Differential Equations*, *Int. J. Mod. Phys.* **A22** (2007) 4375–4436, [arXiv:0707.4037].
- [281] J. M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110** (2013) 251601, [arXiv:1304.1806].
- [282] M. Heller, A. von Manteuffel, and R. M. Schabinger, *Multiple polylogarithms with algebraic arguments and the two-loop EW-QCD Drell-Yan master integrals*, *Phys. Rev. D* **102** (2020), no. 1 016025, [arXiv:1907.00491].
- [283] M. Roth and A. Denner, *High-energy approximation of one loop Feynman integrals*, *Nucl. Phys. B* **479** (1996) 495–514, [hep-ph/9605420].
- [284] T. Binoth and G. Heinrich, *An automatized algorithm to compute infrared divergent multiloop integrals*, *Nucl. Phys. B* **585** (2000) 741–759, [hep-ph/0004013].
- [285] C. Bogner and S. Weinzierl, *Resolution of singularities for multi-loop integrals*, *Comput. Phys. Commun.* **178** (2008) 596–610, [arXiv:0709.4092].
- [286] A. V. Kotikov, *The Property of maximal transcendentality in the  $N=4$  Supersymmetric Yang-Mills*, 5, 2010. arXiv:1005.5029.
- [287] E. Kummer, *Ueber die transcendenten, welche aus wiederholten integrationen rationaler formeln entstehen.*, .
- [288] N. Nielsen, “Der Eulersche Dilogarithmus und seine Verallgemeinerungen. Eine Monographie..” *Nova Acta Leop.* 90, 121-211 (1909)., 1909.
- [289] A. B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes*, *Math. Res. Lett.* **5** (1998) 497–516, [arXiv:1105.2076].
- [290] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops*, *Phys. Rev. Lett.* **105** (2010) 151605, [arXiv:1006.5703].

- [291] C. Duhr, H. Gangl, and J. R. Rhodes, *From polygons and symbols to polylogarithmic functions*, *JHEP* **10** (2012) 075, [[arXiv:1110.0458](#)].
- [292] C. Duhr, *Hopf algebras, coproducts and symbols: an application to Higgs boson amplitudes*, *JHEP* **08** (2012) 043, [[arXiv:1203.0454](#)].
- [293] C. Duhr, *Mathematical aspects of scattering amplitudes*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Journeys Through the Precision Frontier: Amplitudes for Colliders*, 11, 2014. [arXiv:1411.7538](#).
- [294] T. Gehrmann and E. Remiddi, *Numerical evaluation of two-dimensional harmonic polylogarithms*, *Comput. Phys. Commun.* **144** (2002) 200–223, [[hep-ph/0111255](#)].
- [295] T. Gehrmann and E. Remiddi, *Numerical evaluation of harmonic polylogarithms*, *Comput. Phys. Commun.* **141** (2001) 296–312, [[hep-ph/0107173](#)].
- [296] J. Vollinga and S. Weinzierl, *Numerical evaluation of multiple polylogarithms*, *Comput. Phys. Commun.* **167** (2005) 177, [[hep-ph/0410259](#)].
- [297] S. Buehler and C. Duhr, *CHAPLIN - Complex Harmonic Polylogarithms in Fortran*, *Comput. Phys. Commun.* **185** (2014) 2703–2713, [[arXiv:1106.5739](#)].
- [298] C. Bogner, *MPL—A program for computations with iterated integrals on moduli spaces of curves of genus zero*, *Comput. Phys. Commun.* **203** (2016) 339–353, [[arXiv:1510.04562](#)].
- [299] H. Frellesvig, D. Tommasini, and C. Wever, *On the reduction of generalized polylogarithms to  $Li_n$  and  $Li_{2,2}$  and on the evaluation thereof*, *JHEP* **03** (2016) 189, [[arXiv:1601.02649](#)].
- [300] J. Ablinger, J. Blümlein, M. Round, and C. Schneider, *Numerical Implementation of Harmonic Polylogarithms to Weight  $w = 8$* , *Comput. Phys. Commun.* **240** (2019) 189–201, [[arXiv:1809.07084](#)].
- [301] C. Duhr and F. Dulat, *PolyLogTools — polylogs for the masses*, *JHEP* **08** (2019) 135, [[arXiv:1904.07279](#)].
- [302] T. Gehrmann, A. von Manteuffel, L. Tancredi, and E. Weihs, *The two-loop master integrals for  $q\bar{q} \rightarrow VV$* , *JHEP* **06** (2014) 032, [[arXiv:1404.4853](#)].
- [303] T. Gehrmann, A. von Manteuffel, and L. Tancredi, *The two-loop helicity amplitudes for  $q\bar{q}' \rightarrow V_1V_2 \rightarrow 4$  leptons*, *JHEP* **09** (2015) 128, [[arXiv:1503.04812](#)].
- [304] J. M. Henn, K. Melnikov, and V. A. Smirnov, *Two-loop planar master integrals for the production of off-shell vector bosons in hadron collisions*, *JHEP* **05** (2014) 090, [[arXiv:1402.7078](#)].
- [305] F. Caola, J. M. Henn, K. Melnikov, and V. A. Smirnov, *Non-planar master integrals for the production of two off-shell vector bosons in collisions of massless partons*, *JHEP* **09** (2014) 043, [[arXiv:1404.5590](#)].

- [306] C. G. Papadopoulos, D. Tommasini, and C. Wever, *The Pentabox Master Integrals with the Simplified Differential Equations approach*, *JHEP* **04** (2016) 078, [[arXiv:1511.09404](#)].
- [307] T. Gehrmann, J. Henn, and N. Lo Presti, *Pentagon functions for massless planar scattering amplitudes*, *JHEP* **10** (2018) 103, [[arXiv:1807.09812](#)].
- [308] D. Chicherin, T. Gehrmann, J. Henn, N. Lo Presti, V. Mitev, and P. Wasser, *Analytic result for the nonplanar hexa-box integrals*, *JHEP* **03** (2019) 042, [[arXiv:1809.06240](#)].
- [309] D. Chicherin and V. Sotnikov, *Pentagon Functions for Scattering of Five Massless Particles*, *JHEP* **20** (2020) 167, [[arXiv:2009.07803](#)].
- [310] R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, F. Moriello, and V. A. Smirnov, *Two-loop planar master integrals for Higgs  $\rightarrow$  3 partons with full heavy-quark mass dependence*, *JHEP* **12** (2016) 096, [[arXiv:1609.06685](#)].
- [311] A. von Manteuffel and L. Tancredi, *A non-planar two-loop three-point function beyond multiple polylogarithms*, *JHEP* **06** (2017) 127, [[arXiv:1701.05905](#)].
- [312] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, *Elliptic symbol calculus: from elliptic polylogarithms to iterated integrals of Eisenstein series*, *JHEP* **08** (2018) 014, [[arXiv:1803.10256](#)].
- [313] L. Adams, E. Chaubey, and S. Weinzierl, *Analytic results for the planar double box integral relevant to top-pair production with a closed top loop*, *JHEP* **10** (2018) 206, [[arXiv:1806.04981](#)].
- [314] R. N. Lee, *Symmetric  $\epsilon$ - and  $(\epsilon + 1/2)$ -forms and quadratic constraints in "elliptic" sectors*, *JHEP* **10** (2018) 176, [[arXiv:1806.04846](#)].
- [315] M. Walden and S. Weinzierl, *Numerical evaluation of iterated integrals related to elliptic Feynman integrals*, [arXiv:2010.05271](#).
- [316] F. Brown and C. Duhr, *A double integral of dlog forms which is not polylogarithmic*, 6, 2020. [arXiv:2006.09413](#).
- [317] U. Aglietti, R. Bonciani, L. Grassi, and E. Remiddi, *The Two loop crossed ladder vertex diagram with two massive exchanges*, *Nucl. Phys. B* **789** (2008) 45–83, [[arXiv:0705.2616](#)].
- [318] R. N. Lee, A. V. Smirnov, and V. A. Smirnov, *Solving differential equations for Feynman integrals by expansions near singular points*, *JHEP* **03** (2018) 008, [[arXiv:1709.07525](#)].
- [319] R. Bonciani, V. Del Duca, H. Frellesvig, J. Henn, M. Hidding, L. Maestri, F. Moriello, G. Salvatori, and V. Smirnov, *Evaluating a family of two-loop non-planar master integrals for Higgs + jet production with full heavy-quark mass dependence*, *JHEP* **01** (2020) 132, [[arXiv:1907.13156](#)].

- [320] H. Frellesvig, M. Hidding, L. Maestri, F. Moriello, and G. Salvatori, *The complete set of two-loop master integrals for Higgs + jet production in QCD*, *JHEP* **06** (2020) 093, [arXiv:1911.06308].
- [321] S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow, and M. Zeng, *Two-Loop Integrals for Planar Five-Point One-Mass Processes*, *JHEP* **11** (2020) 117, [arXiv:2005.04195].
- [322] L. Adams and S. Weinzierl, *The  $\varepsilon$ -form of the differential equations for Feynman integrals in the elliptic case*, *Phys. Lett. B* **781** (2018) 270–278, [arXiv:1802.05020].
- [323] C. Bogner, S. Müller-Stach, and S. Weinzierl, *The unequal mass sunrise integral expressed through iterated integrals on  $\overline{\mathcal{M}}_{1,3}$* , *Nucl. Phys. B* **954** (2020) 114991, [arXiv:1907.01251].
- [324] V. A. Smirnov, *Evaluating Feynman integrals*, *Springer Tracts Mod. Phys.* **211** (2004).
- [325] G. Heinrich, *Collider Physics at the Precision Frontier*, arXiv:2009.00516.
- [326] V. A. Smirnov, *Analytic tools for Feynman integrals*, *Springer Tracts Mod. Phys.* **250** (2012).
- [327] F. C. S. Brown, *On the periods of some feynman integrals*, 2010.
- [328] E. Panzer, *Feynman integrals and hyperlogarithms*. PhD thesis, Humboldt U., 2015. arXiv:1506.07243.
- [329] E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, *Comput. Phys. Commun.* **188** (2015) 148–166, [arXiv:1403.3385].
- [330] H. Cheng and T. T. Wu, *EXPANDING PROTONS: SCATTERING AT HIGH-ENERGIES*. 1987.
- [331] E. Panzer, *On the analytic computation of massless propagators in dimensional regularization*, *Nucl. Phys. B* **874** (2013) 567–593, [arXiv:1305.2161].
- [332] A. von Manteuffel, E. Panzer, and R. M. Schabinger, *A quasi-finite basis for multi-loop Feynman integrals*, *JHEP* **02** (2015) 120, [arXiv:1411.7392].
- [333] G. Heinrich, *Sector Decomposition*, *Int. J. Mod. Phys. A* **23** (2008) 1457–1486, [arXiv:0803.4177].
- [334] A. V. Smirnov, *FIESTA4: Optimized Feynman integral calculations with GPU support*, *Comput. Phys. Commun.* **204** (2016) 189–199, [arXiv:1511.03614].
- [335] S. Borowka, N. Greiner, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, and T. Zirke, *Full top quark mass dependence in Higgs boson pair production at NLO*, *JHEP* **10** (2016) 107, [arXiv:1608.04798].

- [336] A. von Manteuffel and R. M. Schabinger, *Numerical Multi-Loop Calculations via Finite Integrals and One-Mass EW-QCD Drell-Yan Master Integrals*, *JHEP* **04** (2017) 129, [[arXiv:1701.06583](#)].
- [337] Z. Bern, L. J. Dixon, and D. A. Kosower, *The Five gluon amplitude and one loop integrals*, in *The Fermilab Meeting DPF 92. Proceedings, 7th Meeting of the American Physical Society, Division of Particles and Fields, Batavia, USA, November 10-14, 1992. Vol. 1, 2*, pp. 901–905, 1992. [hep-ph/9212237](#).
- [338] E. Panzer, *On hyperlogarithms and Feynman integrals with divergences and many scales*, *JHEP* **03** (2014) 071, [[arXiv:1401.4361](#)].
- [339] B. Agarwal and A. Von Manteuffel, *On the two-loop amplitude for  $gg \rightarrow ZZ$  production with full top-mass dependence*, *PoS RADCOR2019* (2019) 008, [[arXiv:1912.08794](#)].
- [340] S. Borowka, G. Heinrich, S. Jones, M. Kerner, J. Schlenk, and T. Zirke, *SecDec-3.0: numerical evaluation of multi-scale integrals beyond one loop*, *Comput. Phys. Commun.* **196** (2015) 470–491, [[arXiv:1502.06595](#)].
- [341] A. Smirnov and V. Smirnov, *How to choose master integrals*, [arXiv:2002.08042](#).
- [342] J. Usovitsch, *Factorization of denominators in integration-by-parts reductions*, [arXiv:2002.08173](#).
- [343] R. M. Schabinger, *Constructing multiloop scattering amplitudes with manifest singularity structure*, *Phys. Rev. D* **99** (2019), no. 10 105010, [[arXiv:1806.05682](#)].
- [344] A. Pak, *The Toolbox of modern multi-loop calculations: novel analytic and semi-analytic techniques*, *J. Phys. Conf. Ser.* **368** (2012) 012049, [[arXiv:1111.0868](#)].
- [345] J. Boehm, M. Wittmann, Z. Wu, Y. Xu, and Y. Zhang, *IBP reduction coefficients made simple*, [arXiv:2008.13194](#).
- [346] W. Beenakker, S. Dittmaier, M. Kramer, B. Plumper, M. Spira, and P. Zerwas, *NLO QCD corrections to  $t$  anti- $t$   $H$  production in hadron collisions*, *Nucl. Phys. B* **653** (2003) 151–203, [[hep-ph/0211352](#)].
- [347] S. Catani, L. Cieri, D. de Florian, G. Ferrera, and M. Grazzini, *Universality of transverse-momentum resummation and hard factors at the NNLO*, *Nucl. Phys. B* **881** (2014) 414–443, [[arXiv:1311.1654](#)].
- [348] S. Larin and J. Vermaseren, *The  $\alpha_s^{**3}$  corrections to the Bjorken sum rule for polarized electroproduction and to the Gross-Llewellyn Smith sum rule*, *Phys. Lett. B* **259** (1991) 345–352.
- [349] S. Larin, *The Renormalization of the axial anomaly in dimensional regularization*, *Phys. Lett. B* **303** (1993) 113–118, [[hep-ph/9302240](#)].

- [350] J. Dick, F. Y. Kuo, and I. H. Sloan, *High-dimensional integration: The quasi-monte carlo way*, *Acta Numerica* **22** (2013) 133–288.