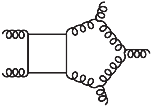
Two-loop Matrix Elements for Three-Jet Production at the LHC

at Leading-Color NNLO QCD



$$\sim \cdots + F_{m,n}(s_{ij}) \times \left(\frac{(11(-3s_{23}s_{45}^5 + s_{45}^5(3s_{15}+2s_{45}) + \cdots)}{(12s_{12}^3(s_{12}-s_{34}-s_{45})^3)} \right) + \cdots$$

Fernando Febres Cordero
Department of Physics, Florida State University

MSU Theory Seminar, East Lansing (Zoom), September 2021

With Samuel Abreu, Harald Ita, Ben Page, and Vasily Sotnikov
[arXiv:2102.13609]

Outline

PRECISE JET STUDIES @ the LHC
Data sets @ (HL-)LHC, NNLO QCD, Multi scale processes

NUMERICAL UNITARITY @ 2 LOOPS
Gravity '*toy*' example, Integrand ansatz, Analytics from numerics

3-JET 2-LOOP AMPLITUDES
Caravel for numerics, Analytic structure, LHC Phenomenology

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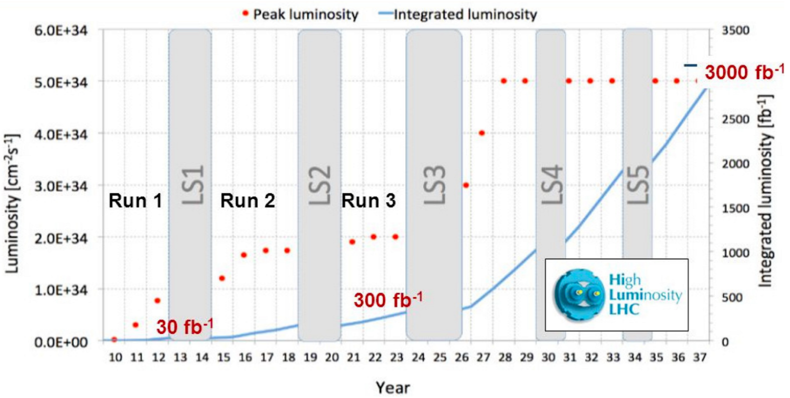
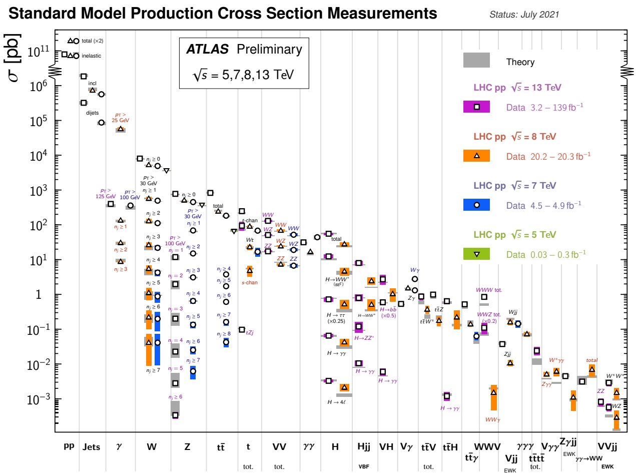


20-fold increase in data sets at the LHC experiments in the next decades

Reaching few-percent uncertainties in cross sections for processes with 3 (or more) objects in the final state

The *attobarn* Era



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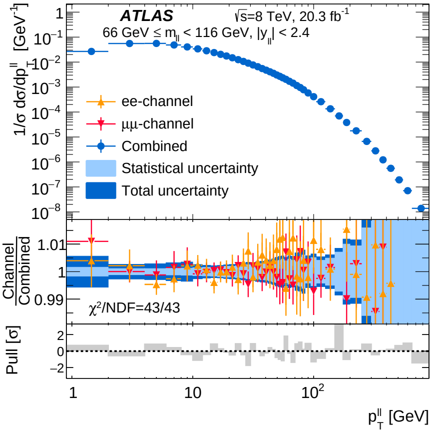
Few % Frontier at the LHC

[arXiv:1512.02192 [hep-ex]]

- p_T^{ll} in Drell-Yan, an impressive example of precise differential measurements by ATLAS (8 TeV)
- By normalizing to inclusive Z cross section, improvement in uncertainties
- Total uncertainties below 1% for $p_T^{ll} < 200$ GeV



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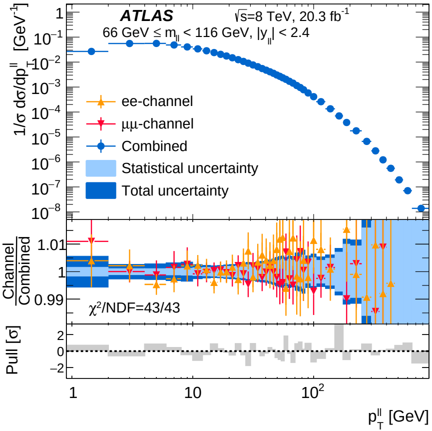

A quick jump: NNLO QCD a basic requirement for a variety of multi-particle/multi-jet processes in years to come!

NNLO QCD for Multi-Scale Processes

- Great advances over the last several years on NNLO QCD studies for $2 \to 2$ processes, with up to four scales
[Anastasiou, Angeles-Martinez, Asteriadis, Behring, Berger, Billis, Binoth, Bonciani, Boughezal, Brucherseifer, Buonocore, Cacciari, Campbell, Caola, Cascioli, Catani, Chen, Cieri, Cruz-Martinez, Currie, Czakon, de Florian, Del Duca, Delto, Devoto, Dreyer, Duhr, Ebert, Ellis, Ferrera, Fiedler, Focke, Frellesvig, Gao, Gauld, Gaunt, Gehrmann, Gehrmann-De Ridder, Giele, Glover, Grazzini, Hanga, Heinrich, Heymes, Huss, Höfer, Jaquier, Jones, Kallweit, Kardos, Karlberg, Kerner, Li, Lindert, Liu, Magnea, Maierhöfer, Maina, Majer, Mazzitelli, Melnikov, Michel, Mitov, Morgan, Neumann, Niehues, Pelliccioli, Petriello, Pires, Poncelet, Pozzorini, Rathlev, Rietkerk, Röntsch, Salam, Sapeta, Sargsyan, Schulze, Signorile-Signorile, Somogyi, Stahlhofen, Szőr, Tackmann, Tancredi, Torre, Torrielli, Tramontano, Trócsányi, Tulipánt, Uccirati, van Hameren, von Manteuffel, Walker, Walsh, Wang, Weihs, Wells, Wever, Wiesemann, Williams, Yuan, Zanderighi, Zhang, Zhu, $\cdots$]

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- Now $2 \to 3$ NNLO QCD starting to appear!

  [Chawdhry, Czakon, Mitov, Poncelet, arXiv:1911.00479] $\gamma\gamma\gamma$ production

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Major milestone for pQCD!

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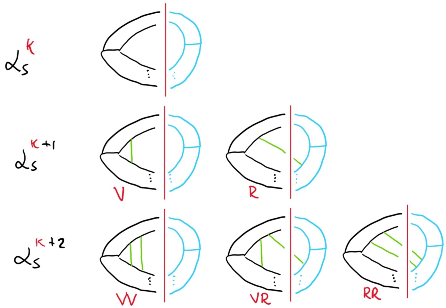
  Major milestone for pQCD!
- About 15 years ago, $2 \to 3$ was the frontier for NLO QCD calculations, and moving beyond relied mainly on efficient numerical algorithms (now available through many powerful tools, e.g. *BlackHat, GoSam, HELAC-1Loop/CutTools, Madgraph, NJet, NLOX, OpenLoops, Recola,* $\cdots$)

Key Building Blocks for NNLO QCD Corrections

- Strategy to handle and cancel IR divergences
- Two-loop matrix elements



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- Many recent advances and complete calculations (e.g. $t\bar{t}$, jj, VV', Vj, HH, 3γ, jjj, etc)
- Several well-developed approaches
  - Antenna subtraction
  - ColorfulNNLO
  - Nested soft-collinear subtractions
  - N-Jettiness slicing
  - Projection to born
  - q_T slicing
  - Sector improved residue subtraction scheme formalism
  - $\ldots$
- Different degrees of automation, important for progress in handling $2 \to 3$ processes

Key Building Blocks for NNLO QCD Corrections

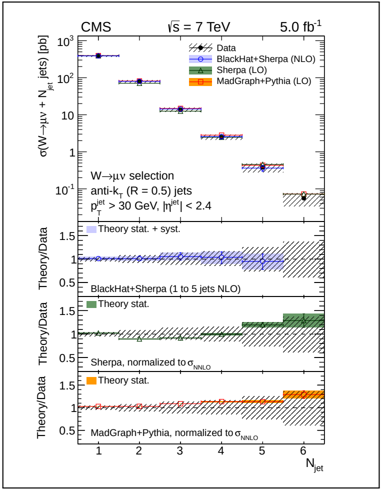
- Strategy to handle and cancel IR divergences
- Two-loop matrix elements

- Great steps towards understanding mechanisms to compute multi-scale master Feynman integrals, including insights into functional forms and numerical procedures, over the last few years
- Also new efficient tools developed for multi-loop integral reduction
- Integrand reduction techniques have shown a lot of power to tackle complicated amplitudes. Here we focus on the numerical unitarity method

How many actual scales in the considered processes?!

LHC Measurements of $W + n$ Jets Production

CMS [arXiv:1406.7533 [hep-ex]]



- Multiple $W + n$-jet studies carried out by CMS and ATLAS
- The more objects in the final state (leptons and jets) the more structure that can be explored
- Plenty of physical scales, posing a major challenge for theory studies

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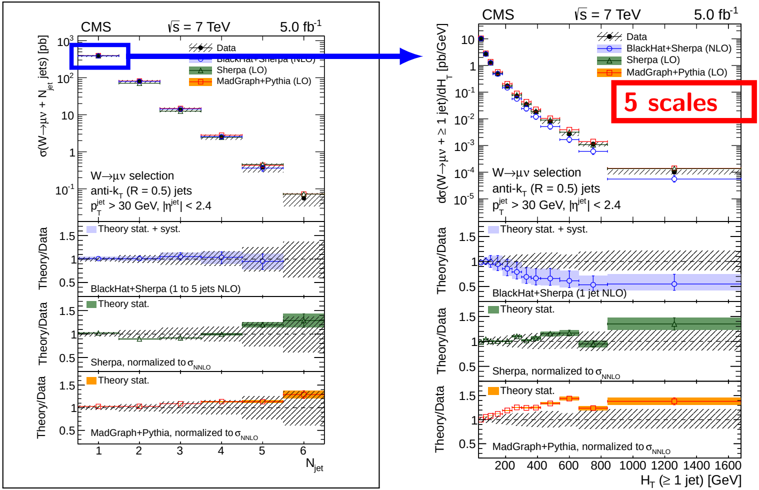

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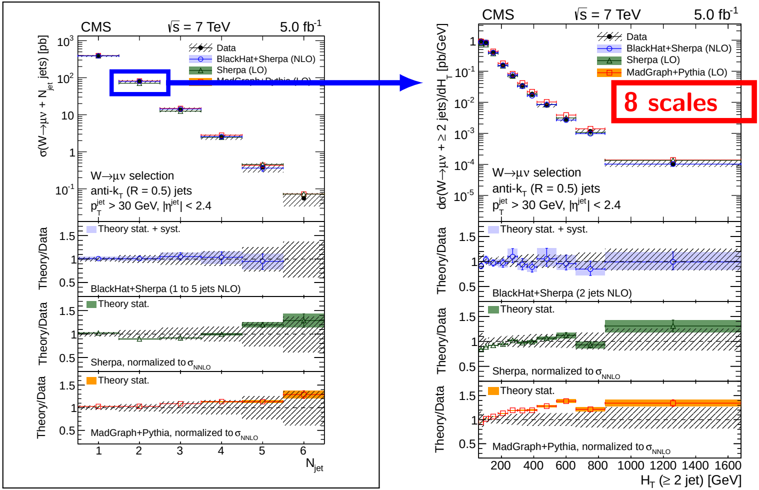

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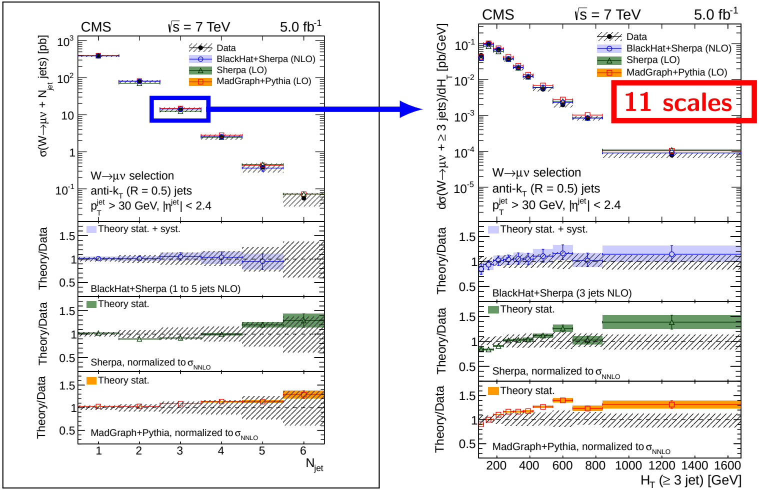

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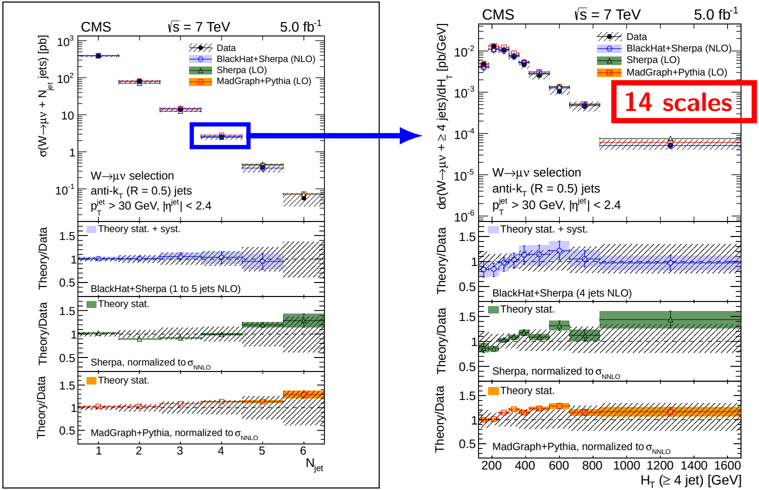
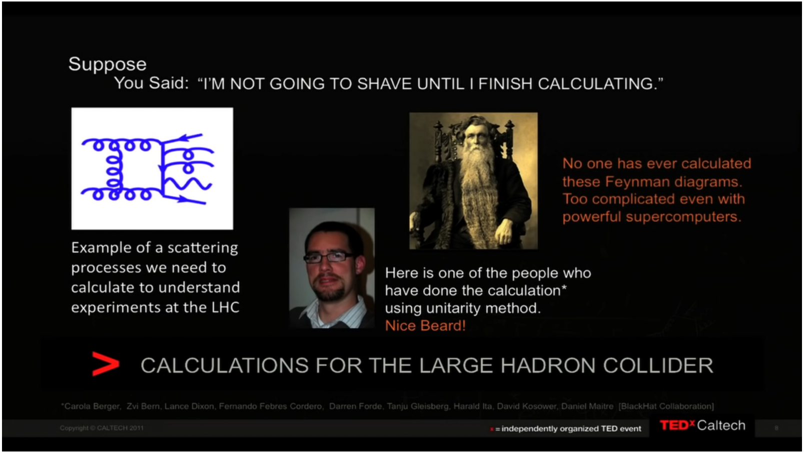


A Story of Long Beards — and Scales!



From Zvi Bern's [2011 TEDx]

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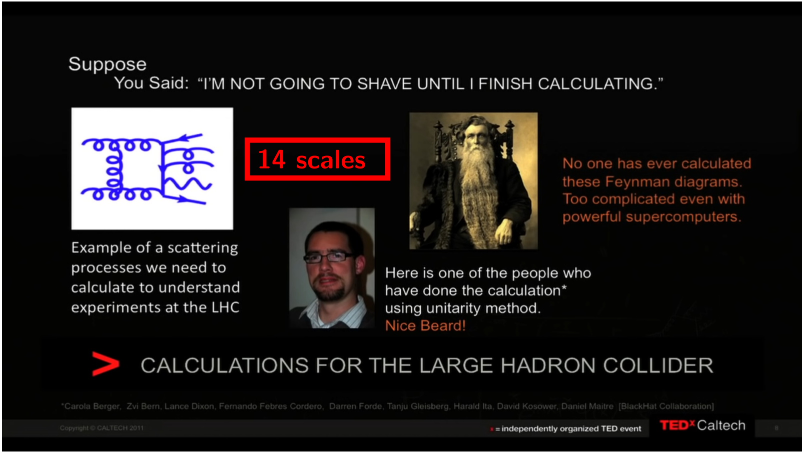

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Three-jet production at hadron colliders

Theoretical Studies in QCD of Jet Production

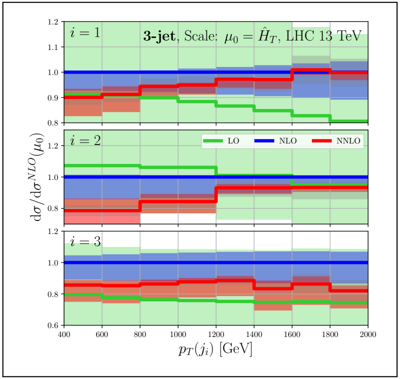
Quantitatively reliable predictions for multi-jet production processes at hadron colliders require the inclusion of NLO QCD corrections

- [Ellis, Kunszt, Soper, 92, ...]: Single inclusive and 2-jet @ NLO QCD
- [Giele, Glover, Kosower, hep-ph/9302225, ...]: Single inclusive and 2-jet @ NLO QCD
- [Nagy, [hep-ph/0110315], [hep-ph/0307268]]: 3-jet @ NLO QCD; `NLOJet++`. See also [Kilgore, Giele, hep-ph/0009193]
- [Bern, Dixon, FFC, et al., arXiv:1112.3940]: 4-jet @ NLO QCD; `BlackHat`
- [Badger, Biedermann, Uwer, Yundin, arXiv:1309.6585]: 5-jet @ NLO QCD; `NJet`
- [Dittmaier, Huss, Speckner, arXiv:1210.0438], [Frederix, Frixione, Hirschi, Pagani, Shao, Zaro, arXiv:1612.06548] and [Reyer, Shönherr, Schumann, arXiv:1902.01763]: 2- and 3-jet @ NLO EW
- [Currie, Glover, Pires, arXiv:1611.01460]: Single inclusive @ NNLO QCD
- [Currie, Gehrmann-De Ridder, Gehrmann, Glover, Huss, Pires, arXiv:1705.10271]: 2-jet @ NNLO QCD
- [Czakon, van Hameren, Mitov, Poncelet, arXiv:1907.12911]: Single inclusive @ NNLO QCD, with full color

And...

Three-Jet Production at NNLO QCD

[Czakon, Mitov, Poncelet, arXiv:2106.05331]



- Very recently completed!
- A proof of principle for generic $2 \to 3$ calculations
- Exciting precise QCD phenomenology ahead, for example enabling α_s measurement at high energies!
- Uses our double-virtual matrix elements, which are computed in the leading-color approximation
- Expected size of subleading-color corrections under 1%

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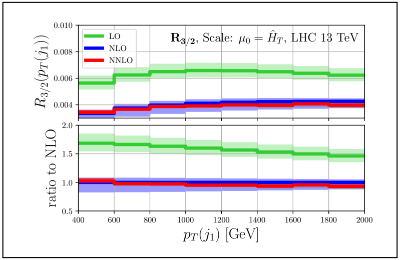

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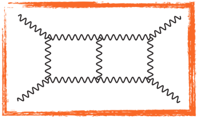
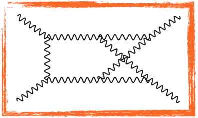
Graviton-graviton Scattering: A '*Toy*' Example

[Abreu, FFC, Ita, Jaquier, Page, Ruf, Sotnikov, arXiv:2002.12374]





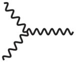
- Not related to collider phenomenology but, treated as an EFT, it can showcase the strengths of the multi-loop numerical unitarity method
- Of interest for classical gravitational applications, as already shown in the computation of classical deflection angles in Einstein gravity [Bern, Ita, Parra-Martinez, Ruf, arXiv:2002.02459]
- Showing the robustness of our computational framework CARAVEL, testing non-planar, colorless calculations with different particle content (as compared to the SM)

Target Amplitudes

$$\mathcal{L} = \mathcal{L}_{\mathsf{EH}} + \mathcal{L}_{\mathsf{GB}} + \mathcal{L}_{\mathsf{R}^3} + \ldots$$

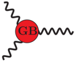
[Weinberg], ['t Hooft, Veltman], [Goroff, Sagnotti], [Donogue], ...

$$\mathcal{L}_{\mathrm{EH}} = -\frac{2}{\kappa^2}\sqrt{|g|}R$$



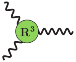
$\mathcal{O}(\kappa)$

$$\mathcal{L}_{\mathrm{GB}} = \frac{\mathcal{C}_{\mathrm{GB}}}{(4\pi)^2}\sqrt{|g|}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$$

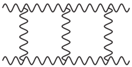


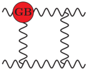
$\mathcal{O}(\kappa^3)$

$$\mathcal{L}_{\mathrm{R}^3} = \frac{\mathcal{C}_{\mathrm{R}^3}}{(4\pi)^4}\left(\frac{\kappa}{2}\right)^2\sqrt{|g|}R_{\alpha\beta}{}^{\mu\nu}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}$$



$\mathcal{O}(\kappa^5)$

$\mathcal{A}^{(2)} =$  $+$ 

$+$ 

$+$ 

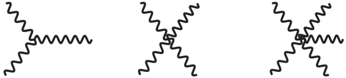
$\mathcal{O}(\kappa^6)$

Only three helicity configurations necessary:

$++++,\ -+++,\ --++$

Main Challenges

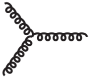
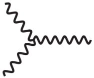
- EH Feynman rules are complicated



Terms: $\mathcal{O}(100)$ $\mathcal{O}(1000)$ $\mathcal{O}(10000)$

Feynman-diagram based calculation out of question
$\Rightarrow$ (Generalised) Unitarity

- EH interactions have high power-counting (QCD^2)

 $\sim \mathcal{O}(\ell)$ vs  $\sim \mathcal{O}(\ell^2)$

Complicated integrand
$\Rightarrow$ analytics from numerics

Plays to the strengths of
Two-loop numerical unitarity

Caravel

Two-Loop Numerical Unitarity

Decompose $\mathcal{A}$ in terms of *master* integrals:

$$\mathcal{A}^{(L)} = \sum_{\Gamma\in\Delta}\sum_{i\in M_\Gamma} c_{\Gamma,i}\,\mathcal{I}_{\Gamma,i}$$

All 4-point 2-loop integrals known [Anastasiou, Smirnov, Tausk, Tejeda-Yeomans, Veretin]

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$$\mathcal{A}^{(L)}(\ell_l) = \sum_{\Gamma\in\Delta} \sum_{k\in Q_\Gamma} c_{\Gamma,k} \frac{m_{\Gamma,k}(\ell_l)}{\prod_{j\in P_\Gamma} \rho_j(\ell_l)}$$

Functions $\mathcal{Q}_\Gamma = \{m_{\Gamma,k}(\ell_l) | k \in Q_\Gamma\}$ parametrize every possible integrand (up to a given power of loop momenta).

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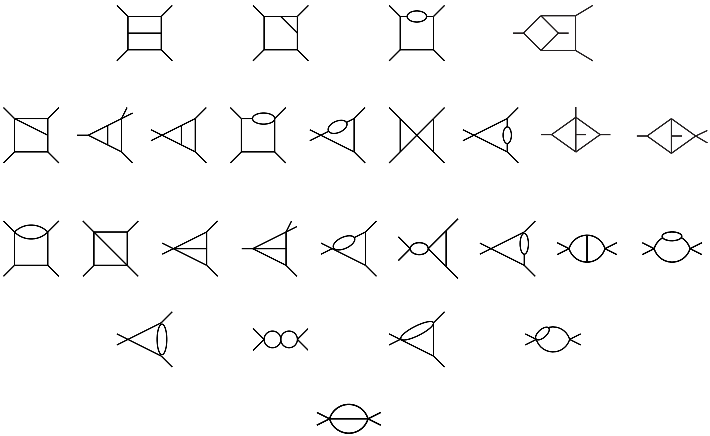
Functions $\mathcal{Q}_\Gamma = \{m_{\Gamma,k}(\ell_l)|k\in Q_\Gamma\}$ parametrize every possible integrand (up to a given power of loop momenta). E.g.:

- **Tensor Basis**: construct $\mathcal{Q}$ from monomials of loop momenta (parameters). Easy to build for general integrands, tough to relate to master integrals. Easy to extract function-space dimension
- **Master-Surface Basis**: a clever choice of parametrization makes mapping to master integrals straightforward [Ita, arXiv:1510.05626]. Break $Q_\Gamma = M_\Gamma \cup S_\Gamma$, where S_Γ integrate to zero and M_Γ correspond to master integrands

The Four-Graviton Hierarchy



All propagator structures ($\Gamma \in \Delta$) necessary for graviton-graviton scattering at two loops

Master/Surface Decompositions

[Ita '15; Abreu, FFC, Ita, Page, Zeng '17]

Consider the integration by parts (IBP) relation on Γ

$$0 = \int \prod_i d^D \ell_i \, \frac{\partial}{\partial \ell_j^\nu} \left[\frac{u_j^\nu}{\prod_{k \in P_\Gamma} \rho_k} \right]$$

making it *unitarity compatible* (controlling the propagator structure) [Gluza, Kadja, Kosower '10; Schabinger '11]

$$u_j^\nu \frac{\partial}{\partial \ell_j^\nu} \rho_k = f_k \rho_k$$

Write ansatz for u_j^ν expanded in external and loop momenta, and find solution to the polynomial equations using the CAS Singular

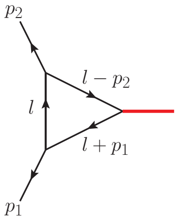
Build a full set of surface terms and fill the rest of the space with master integrands

Related [Boehm, Georgoudis, Larsen, Schulze, Zhang '16 - '19] [Agarwal, von Manteuffel '19]

A 1-loop Example for Surface Terms: Part 1

Consider the 1-loop 1-mass triangle with

$$\rho_1 = (\ell + p_1)^2, \quad \rho_2 = \ell^2, \quad \rho_3 = (\ell - p_2)^2$$



and we construct $u^\nu \partial/\partial \ell^\nu$ by parametrizing

$$u^\nu = u_1^{\text{ext}} p_1^\nu + u_2^{\text{ext}} p_2^\nu + u^{\text{loop}} \ell^\nu$$

We then get the syzygy equation (polynomial equation):

$$\left(u_1^{\text{ext}} p_1^\nu + u_2^{\text{ext}} p_2^\nu + u^{\text{loop}} \ell^\nu\right) \frac{\partial}{\partial \ell^\nu} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} - \begin{pmatrix} f_1 \rho_1 \\ f_2 \rho_2 \\ f_3 \rho_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can then show that we have an IBP-generating vector, with constrained propagator structure:

$$u^\nu \frac{\partial}{\partial \ell^\nu} = \left[(\rho_3 - \rho_2) p_1^\nu + (\rho_1 + \rho_2) p_2^\nu + (-s + 2\rho_3 - 2\rho_2) \ell^\nu\right] \frac{\partial}{\partial \ell^\nu}$$

A 1-loop Example for Surface Terms: Part 2

Now we have the surface term:

$$0 = \int d^D\ell \frac{\partial}{\partial l^\nu} \frac{u^\nu}{\rho_1\rho_2\rho_3} = \int d^D l \frac{1}{\rho_1\rho_2\rho_3}\left[-(D-4)s-2(D-3)\rho_2+2(D-3)\rho_3\right]$$

The scalar triangle integrand can be replaced by a surface term, though commonly it is kept, leading to a corresponding "master" integral in OPP reduction.

The IBP relation between the triangle and the $s = (p_1 + p_2)^2$ bubble is:

$$-(D-4)sI_{\mathrm{tri}} - 2(D-3)I_{\textsf{s-bub}} = 0$$

Similar manipulations can be carried out at two loops. More complicated *syzygy* equations (polynomial relations) need to be solved → SINGULAR. Surface terms appear as relatively compact

Surface Terms Factory

Solutions to u^ν_j are power-counting independent. When parametrizing a given numerator of a $\Gamma \in \Delta$ we need to consider the required power-counting for the theory at hand.

But we can *industrially* produce surface terms by considering polynomials $t_r(\ell_l)$, and then considering the vector $t_r(\ell_l)u^\nu_j$:

$$m_{\Gamma,(r,s)} = u^\nu_j \frac{\partial t_r(\ell_l)}{\partial \ell^\nu_i} + t_r(\ell_l)\left(\frac{\partial u^\nu_j}{\partial \ell^\nu_i} - \sum_{k \in P_\Gamma} f^s_k\right)$$

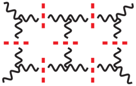
A four-graviton amplitude calculation in **Einstein gravity** structurally the same as a four-gluon amplitude calculation in **QCD**!
Though numerically much more demanding...

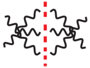
Computing Integrand Coefficients

[Bern, Dixon, Dunbar, Kosower] [Britto, Cachazo, Feng]

- In on-shell configurations of ℓ_l, the integrand factorizes

$$\sum_{\text{states}} \prod_{i \in T_\Gamma} \mathcal{A}_i^{\text{tree}}(\ell_l^\Gamma) = \sum_{\substack{\Gamma' \geq \Gamma \\ k \in Q_{\Gamma'}}} \frac{c_{\Gamma',k} \; m_{\Gamma',k}(\ell_l^\Gamma)}{\prod_{j \in (P_{\Gamma'}/P_\Gamma)} \rho_j(\ell_l^\Gamma)}$$

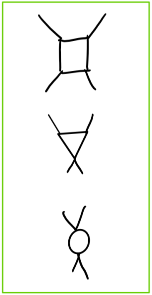


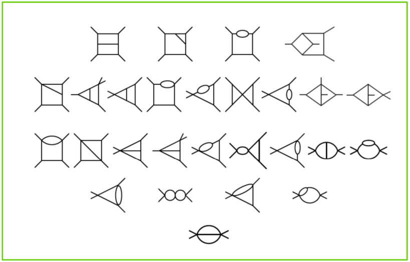


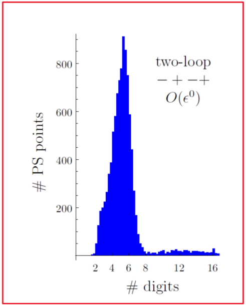
- Need efficient computation of (products of) tree-level amplitudes
  - Off-shell recursions [Berends, Giele '88], [Draggiotis, Kleiss, Papadopoulos '02 …] [Cheung, Remmen '17]
  - D_s-dimensional state sum, $D_s = 6, \ldots, 10$
- Never construct analytic integrand, numerics for every phase-space point: key, integrand would have $\mathcal{O}(10^{18})$ terms!

Numerical Stability:

e.g. 4-gluon amplitudes







Function spaces with $\mathcal{O}(10/50)$ dimensions

Function spaces with $\mathcal{O}(100/1000)$ dimensions

* Relative precision of two-loop 4-gluon amp numerical calculation
* High-precision floating point arithmetic a remedy

[Abreu, FFC, Ita, Jaquier, Page, Zeng, '17]

MODULAR ALGEBRA: [von Manteuffel, Schabinger, 2014]

* Integral reduction can be performed exactly in CAS if kinematical info is RATIONAL ($x_i \in \mathbb{Q}^n$)
* Nevertheless, RATIONAL computer algebra reflects the numerical complexity of corresponding ANALYTIC STRUCTURE (COMPUTATIONAL ALGORITHM)

$$\underbrace{x_i}_{\text{"simple" input}} \longrightarrow \underbrace{f(x_i)}_{\text{"complicated" function}} \longrightarrow \underbrace{\frac{p}{q}}_{\text{even if result is "simple"}}$$

$\Rightarrow$ "HEAVY" calculation

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⇒ “HEAVY” calculation

Avoid all numerical-stability issues, and more...

INTEGRAL COEFFS AS FUNCTIONS of ε:

(INTEGRAND'S ANSATZ)

$$A(\ell_e) = \sum_{\Gamma, i} c_{\Gamma,i} \frac{m_{\Gamma,i}(\ell_e)}{\prod_{k\in\Gamma} \rho_k(\ell_e)} \longrightarrow$$

$c_{\Gamma,i}$ are function of x_k & $D=4-2\varepsilon$

Indeed $c_{\Gamma,i}$ appear as rational function of ε

$$c_{\Gamma,i} = \frac{\sum_j f_j(x_k)\, \varepsilon^{j+N}}{\sum_j q_j\, \varepsilon^{j+M}}$$

} STRUCTURE NOT KNOWN A PRIORI !

ε dependence comes from the structure of $m_{\Gamma,i}(\ell_e)$ and through linear algebra ("subtraction" procedure)

Extracting Functional Form from Numerics

THIELE'S INTERPOLATION FORMULA:

Every rational function can be written as a continued fraction

$$f(x) = \frac{\sum_{r=0}^{R} n_r x^r}{\sum_{r'=0}^{R'} d_r x^{r'}} = a_0 + \cfrac{x - y_0}{a_1 + \cfrac{x - y_1}{a_2 + \cfrac{x - y_2}{\cdots + \cfrac{x - y_{N-1}}{a_N}}}}$$

* Determine a_i by evaluating $f(y_i)$ (y_i random)
* Stop when $f(y_{i+1})$ matches interpolated value (+ extra checks)
* Through only field operations recover rational function

(FF's result can be lifted to $\mathbb{Q}$)

See also [Peraro, arXiv:1608.01902] for multi-variate case
[Peraro, `FiniteFlow`, arXiv:1905.08019]
[Klappert, Klein, Lange, `Firefly`, arXiv:2004.01463]

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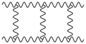
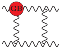
This same idea can be employed for the analytic reconstruction of the kinematic $x = t/s$ dependence of 4-pt amplitudes!

Gravity Results: 4-Graviton Amplitudes

- Computed the three independent helicities for
  - ✓ EH gravity 
  - ✓ Tree-level R^3 

  - ✓ Tree-level GB-GB 

  - ✓ One-loop GB 

- Checks
  - ✓ Remainders are finite, correct symmetry, no spurious poles
  - ✓ 1-loop amplitudes [Dunbar, Norridge 95], [Bern, Cheung, Chi, Davies, Dixon, Nohle, unpub.]
  - ✓ GB tree and 1-loop: $++++$ and $--++$ [Bern, Cheung, Chi, Davies, Dixon, Nohle, 15, unpub.]
  - ✓ R^3 tree [Bern, Cheung, Chi, Davies, Dixon, Nohle, 15], [Dunbar, Jehu, Perkins, 17]
  - ✓ 2-loop: $++++$ [Bern, Cheung, Chi, Davies, Dixon, Nohle, unpub.], [Dunbar, Jehu, Perkins, 17]

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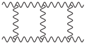
  - ✓ EH gravity 
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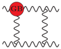
  - ✓ One-loop GB 

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  - ✓ 2-loop: $++++$ [Bern, Cheung, Chi, Davies, Dixon, Nohle, unpub.], [Dunbar, Jehu, Perkins, 17]

4-graviton amps complex numerically, but simple analytic structure (univariate). $\mathcal{O}(100k)$ CPU hours to obtain all 3 independent amplitudes!

PRECISE JET STUDIES @ the LHC
Data sets @ (HL-)LHC, NNLO QCD, Multi scale processes

NUMERICAL UNITARITY @ 2 LOOPS
Gravity '*toy*' example, Integrand ansatz, Analytics from numerics

3-JET 2-LOOP AMPLITUDES
Caravel for numerics, Analytic structure, LHC Phenomenology

A Brief History of 5-Parton Amplitudes

- [Badger, Frellegsvig, Zhang, '13]: Planar all-plus gluon amplitude with numerical master integrals
- [Badger, Mogull, Ochirov, O'Connell, '15]: Full-color all-plus gluon amplitude with numerical master integrals
- [Gehrmann, Henn, Lo Presti, '15]: Analytic form of the planar all-plus gluon amplitude. See also [Dunbar, Perkins, '16]
- [Badger, Brønnum-Hansen, Hartanto, Peraro, '17] and [Abreu, FFC, Ita, Page, Zeng, '17]: Numerical evaluation of all planar 5-gluon amplitudes
- [Badger, et al., '18] and [Abreu, FFC, Ita, Page, Sotnikov, '18]: Numerical evaluation of all planar 5-parton amplitudes
- [Boels, Jin, Luo, '18] and [Chawdhry, Lim, Mitov, '18]: Analytic form of IBP tables for massless 5-particle amplitudes
- [Badger, Brønnum-Hansen, Hartanto, Peraro, '18]: Analytic form of the single-minus 5-gluon amplitude
- [Abreu, Dormans, FFC, Ita, Page, '18]: Analytic form of all 5-gluon amplitudes in the Euclidean region
- [Abreu, Dormans, FFC, Ita, Page, Sotnikov, '19]: Analytic form of all 5-parton amplitudes in the Euclidean region
- [Abreu, FFC, Ita, Page, Sotnikov, '21]: Analytic form of double-virtual contributions to 3-jet production at the LHC

The CARAVEL Framework

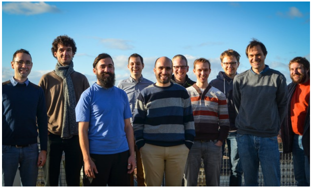
A framework to explore multi-loop multi-leg scattering amplitudes in the SM and beyond

- A modular C++17 library implementing the multi-loop numerical unitarity method

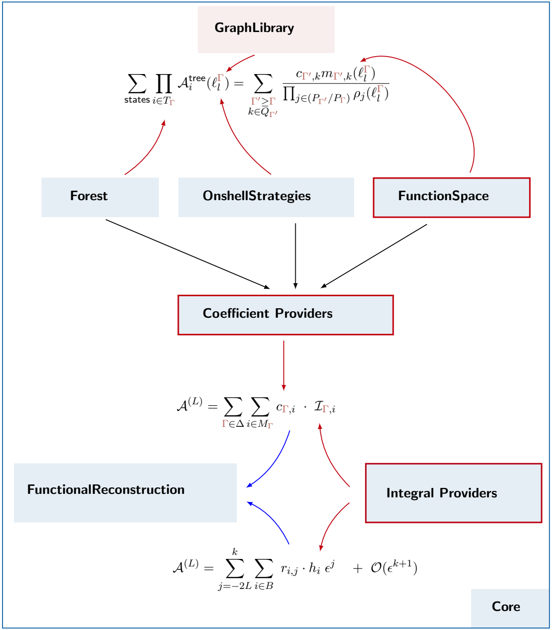
  [Abreu, Dormans, FFC, Ita, Kraus, Page, Pascual, Ruf, Sotnikov, arXiv:2009.11957]

  Caravel

  - Numerics in (high-precision) floating-point, rational and modular arithmetic
  - Generic design for calculations in QFT, e.g. in the SM and in Einstein gravity
  - Algebraic tools for semi-analytical calculations in C++
  - Publicly available!



The CARAVEL's Framework



Includes general tools for:

- D-dimensional kinematics
- graph isomorphism techniques
- tree-level and multi-loop cut calculations
- Master-Surface integral decomposition
- on-shell phase-space parametrizations
- Feynam integral handling
- Functional reconstruction tools
- ...

Caravel @ GitLab:

https://gitlab.com/caravel-public/caravel

Analytic Structure of Massless 5-particle Amplitudes (I)

IR structure:

$$A_R^{(1)} = \underbrace{\mathbb{I}^{(1)} A_R^{(0)}}_{\frac{1}{\varepsilon^2},\ \frac{1}{\varepsilon}\ \text{poles}} + \mathcal{O}(\varepsilon^0)$$

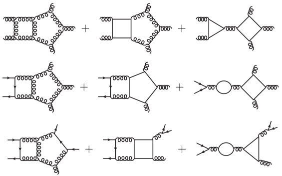
$$A_R^{(2)} = \underbrace{\mathbb{I}^{(1)} A_R^{(1)} + \mathbb{I}^{(2)} A_R^{(0)}}_{\frac{1}{\varepsilon^4},\ \dots,\ \frac{1}{\varepsilon}\ \text{poles}} + \mathcal{O}(\varepsilon^0)$$

Define Remainder:

$$\mathcal{R}^{(1)} = A_R^{(1)} - \mathbb{I}^{(1)} A_R^{(0)} + \mathcal{O}(\varepsilon)$$

$$\mathcal{R}^{(2)} = A_R^{(2)} - \mathbb{I}^{(1)} A_R^{(1)} - \mathbb{I}^{(2)} A_R^{(0)} + \mathcal{O}(\varepsilon)$$

We compute 33 independent remainders corresponding to all helicity configurations in:



$gg \to ggg$

$q\bar{q} \to ggg$ (+ crossing)

$q\bar{q} \to Q\bar{Q}g$ (+ crossing, distinct flavor)

Analytic Structure of Massless 5-particle Amplitudes (II)

Phase space parametrized : $s_{12}, s_{23}, s_{34}, s_{45}, s_{51}$

Parity odd $\quad tr_5 = 4i\,\varepsilon(k_1,k_2,k_3,k_4)$

↳ alternative to momentum twistor variables [Hodges, '13]

$$\longrightarrow \quad \mathcal{R}^{(L)}(\{k_i\}) = \sum_k r_k(\{k_i\})\, f_k(W_j(\{k_i\}))$$

coeff

Special func, "pentagon funcs" [Gehrmann, Henn, Lo Presti, '15]

And :
$$r_k(\{k_i\}) = \underbrace{r_k^{+}(s_{ij})}_{\text{Parity even}} + \underbrace{tr_5\, r_k^{-}(s_{ij})}_{\text{Parity odd}}$$

$r_k^{\pm}(s_{ij}) \longrightarrow$ Rational function of invariants!

Analytic Structure of Massless 5-particle Amplitudes (III)

By physical constraints :

$$r^{\pm}(s_{ij}) = \frac{\overbrace{n^{\pm}(s_{ij})}^{\text{Polynomial}}}{\prod_k \underbrace{W_k^{\alpha_k}(s_{ij})}_{\text{Special function's arguments (Alphabet letter)}}}$$

Determining $\prod_k W_k^{\alpha_k}(s_{ij})$ can be achieved by univariate reconstruction in curves $s_{ij}(\lambda)$ and polynomial division !

$$\boxed{\text{Multivariate reconstruction reduced to determination of the polynomials } n_k^{\pm}(s_{ij})}$$

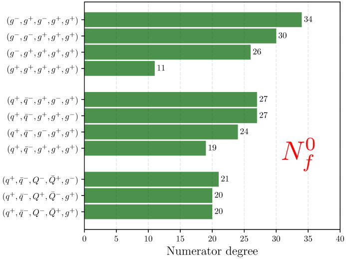
→ Simplify by multivariate partial fractions ! RELATED TO MULTIVARIATEAPART [Heller, von MANTEUFFEL]

Functional Reconstruction

Number of expected ansatz term: $\binom{N+m}{m}$

N: polynomial degree

m: # of variables



Max amount evals: 94,696 PSPs

→ for 5-gluon $-+-++$

All simplifications allow to extract the 33 amplitudes with modest computational resources ~ 200k CPU hours

Final product for pheno: All double-virtual contributions in 9MB of compressed data

FivePointAmplitudes++ for Phenomenology

Numerical stability of double virtual function $H(s_{ij})$:

$$K(\vec{s}) \equiv \frac{\min_k \{W_k(s_{ij})\}}{\max_k \{W_k(s_{ij})\}}$$

↓

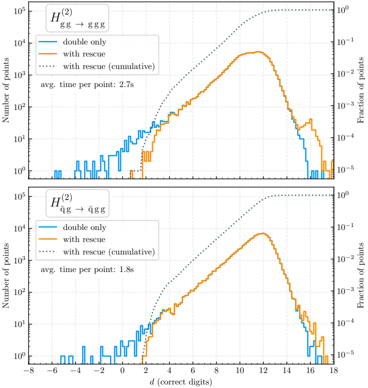
Proxy for potential instabilities

$$s_{ij}^{(\delta)} = s_{ij} + \text{'next F.P.'}$$

And then introduce the check:

$$\Delta(s_{ij}) = \left| 1 - \frac{H(s_{ij}^{(\delta)})}{H(s_{ij})} \right|$$

IF FAIL: double-double evaluation!



EVAL TIMES per PSP:

5 gluon	1.6 sec	12%
2 quark 3 gluon	1.0 sec	6%
4 quark 1 gluon	1.0 sec	3%

Solid Two-Loop Multi-Scale Progress

By now many 5-point two-loop hard functions for pheno in the literature!

- [Abreu, Page, Pascual, Sotnikov, arXiv:2010.15834] $\gamma\gamma\gamma$ (LC)
- [Chawdhry, Czakon, Mitov, Poncelet, arXiv:2012.13553] $\gamma\gamma\gamma$ (LC)
- [Agarwal, Buccioni, von Manteuffel, Tancredi, arXiv:2102.01820] $\gamma\gamma j$ (LC)
- [Badger, Hartanto, Zoia, arXiv:2102.02516] $Wb\bar{b}$ (LC)
- [Abreu, FFC, Ita, Page, Sotnikov, arXiv:2102.13609] jjj (LC)
- [Chawdhry, Czakon, Mitov, Poncelet, arXiv:2103.04319] $\gamma\gamma j$ (LC)
- [Agarwal, Buccioni, von Manteuffel, Tancredi, arXiv:2105.04585] $\gamma\gamma j$ (FC)
- [Badger, Hartanto, Kryś, Zoia, arXiv:2107.14733] $Hb\bar{b}$ (LC)

Outlook

- We presented the leading-color two-loop matrix elements for three-jet production at the LHC
- A first look at the impact of the NNLO QCD corrections by [Czakon, Mitov, Poncelet] appeared recently
- Numerical unitarity, which was critical in the automation of NLO QCD calculations, has approven robust in the calculation of two-loop scattering amplitudes. But critically, functional reconstruction techniques have been necessary in dealing with processes with few scales
- Steady progress points to matching of needs for precision phenomenology of $2 \to 3$ processes! Look forward to future $V + 2j$, $H + 2j$, $t\bar{t} + H$, $VV'j$ analyses!

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Thanks!