Statistical Mechanics Subject Exam, September 8, 2010

SECRET STUDENT NUMBER: STUDNUMBER

FUN FACTS TO KNOW AND TELL

$$\int_0^\infty dx\, \frac{x^{n-1}}{e^x - 1} = \Gamma(n)\zeta(n), \qquad \int_0^\infty dx\, \frac{x^{n-1}}{e^x + 1} = \Gamma(n)\zeta(n)\left[1 - (1/2)^{n-1}\right],$$

$$\zeta(n) \equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!,$$

$$\zeta(3/2) = 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90},$$

$$\int_{-\infty}^{\infty} dx\, e^{-x^2/2} = \sqrt{2\pi}, \qquad \int_0^\infty dx\, x^n e^{-x} = n!$$

LONG ANSWER SECTION

1. (15 pts) Consider the equation of state,

$$P(\rho, T) = \rho T + a\frac{\rho^2}{\rho_0}e^{-\rho/\rho_0}.$$

Solve for the critical density ρ_c and critical temperature T_c in terms of ρ_0 and a.

Extra work space for #1

2. (15 pts) The partition function for a one-dimensional spin system in a magnetic field B is

$$Z = \mathrm{Tr}\exp\left(-\beta H_0 + \beta\mu_b B\int dx\ m(x)\right).$$

After some calculations, one finds the spin-spin correlation function for zero field to be,

$$\langle (m(x=0) - \langle m\rangle)(m(x) - \langle m\rangle)\rangle = A(T)e^{-|x|/\ell}.$$

Calculate the susceptibility,

$$\chi \equiv \frac{d\langle m\rangle}{dB},$$

and express the answer in terms of T, A, ℓ and μ_b.

Extra work space for #2

3. Consider a two-dimensional system of spinless non-interacting non-relativistic particles of mass m at a temperature T and chemical potential μ. Assuming the particles are non-degenerate, i.e., $\mu << -T$, in terms of m, μ and T find closed expressions for

   (a) (5 pts) the density per unit area ρ

   (b) (5 pts) the energy per unit area ϵ

   (c) (5 pts) the force per unit length pushing against a boundary confining the gas

Extra work space for #3

SHORT ANSWER SECTION

4. (2 pts each) Three identical spin-zero bosons occupy two single-particle energy levels ϵ and $-\epsilon$.

   (a) What is the average energy when $T = 0$? ____________

   (b) What is the entropy when $T = 0$? ____________

   (c) What is the average energy when $T >> \epsilon$? ____________

   (d) What is the entropy when $T >> \epsilon$? ____________

5. (3 pts) A solution of fixed number is kept at atmospheric pressure and room temperature in an open flask. When it adjusts its chemical compositions to approach thermodynamic equilibrium, which of the below is true? (choose the single most correct answer)

   (a) The net Gibbs free energy of the solution is minimized

   (b) The net entropy of the solution is maximized

   (c) The net Helmholtz free energy of the solution is minimized

   (d) (a) and (b)

   (e) all of the above

6. (3 pts each) Consider a TWO-dimensional array of N coupled oscillators confined to the $x-y$ plane. The oscillators are allowed to move only in the x and y directions. Both transverse and longitudinal waves move with velocity c_s. Let C/N refer to the specific heat per oscillator.

   (a) As $T \to 0$, the specific heat from phonons behaves as $C \sim T^n$. What is n? ____________

   (b) What is C/N as $T \to \infty$? ____________

7. (2 pts each) Consider a Fermi gas of non-interacting particles at low temperature. For each case below, the specific heat behaves as $\sim T^n$ for small T. In each case find n.

   (a) For a two-dimensional gas of massive non-relativistic particles, $n =$______

   (b) For a three-dimensional gas of massless particles, $n =$______

8. (1 pt each) For each of the following discern whether one expects Goldstone Bosons to accompany the phase transition. Label each case with either "YES" or "NO".

   (a) A liquid gas phase transition. _________

   (b) A Ginzburg-Landau model where the free-energy density is given by the form,

   $$f(\phi) = \frac{m(T)^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4,$$

   where ϕ is a real field. _________

   (c) A Ginzburg-Landau model where the free-energy density is given by the form,

   $$f(\phi) = \frac{m(T)^2}{2}|\phi|^2 + \frac{\lambda}{4!}|\phi|^4,$$

   where ϕ is a complex field. _________

   (d) A Ginzburg-Landau model where the free-energy density is given by the form,

   $$f(\phi) = \frac{m(T)^2}{2}(\phi_x^2 + \phi_y^2) + \frac{\lambda}{4!}(\phi_x^2 + \phi_y^2)^2,$$

   where ϕ_x and ϕ_y are real fields. _________

9. (2 pts each) Label each of the following as true or false

   - Two systems with different microscopic Hamiltonians cannot have the same critical exponents
   - For mean field models of magnetization transitions, the critical exponents are independent of the dimensionality of the system
   - Two phase transitions of the same universality class must have the same critical exponents

10. (6 pts) Consider a **two-dimensional** Ising model, where each spin can have $\sigma = \pm 1$, and nearest neighbor spins experience an attractive interaction, $H_{nn} = -J \sum_{ij} \sigma_i \sigma_j$. Each spin also experiences an interaction with an external field, $H_B = -\mu B \sum_i \sigma_i$. Plot $\langle\sigma\rangle$ as a function of the temperature T (qualitatively) for two cases:

    (a) zero magnetic field

    (b) a fixed positive magnetic field B