PHY831 - Subject Exam August 29 2012, 6pm - 9pm

Answer all questions (total points available is 100). Time for exam - 3 hours

Name:

$$\int_{-\infty}^{\infty} dx\ e^{-ax^2+bx} = (\frac{\pi}{a})^{1/2} e^{\frac{b^2}{4a}}; \quad \int_0^{\infty} dx\ x^n e^{-ax^2} = \frac{1}{2a^{(n+1)/2}}\Gamma(\frac{n+1}{2}) \tag{1}$$

$$\int \frac{dx}{(1+x^2)^{1/2}} = Sinh^{-1}(x); \quad \int_0^{\infty} \frac{x^{s-1}dx}{e^x - 1} = \Gamma(s)\zeta(s), \tag{2}$$

where $\Gamma(s) = (s-1)!$ for s a positive integer, and $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) = 1.202...$, $\zeta(4) = \pi^4/90$.

$$\left(\frac{\partial x}{\partial y}\right)_z = 1/\left(\frac{\partial y}{\partial x}\right)_z; \quad \left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial w}\right)_z \left(\frac{\partial w}{\partial y}\right)_z; \tag{3}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1\ ; \quad \left(\frac{\partial A}{\partial x}\right)_z = \left(\frac{\partial A}{\partial x}\right)_y + \left(\frac{\partial A}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z; \tag{4}$$

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_{V,N} = T\left(\frac{\partial S}{\partial T}\right)_{V,N}; C_P = \left(\frac{\partial H}{\partial T}\right)_{P,N} = T\left(\frac{\partial S}{\partial T}\right)_{P,N}, \tag{5}$$

$$\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N}; \quad \kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N}; \quad \alpha_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{P,N} \tag{6}$$

$$\frac{U}{L^d} = \frac{c_d}{h^d}\int_0^{\infty} dp\ p^{d-1}(cp^s)\frac{ze^{-\beta cp^s}}{1-ze^{-\beta cp^s}} = \frac{d}{s}\frac{k_BT}{\lambda^d}g_{1+d/s}(z); \quad P = \frac{s}{d}\frac{U}{L^d} \tag{7}$$

$$I(r,\nu\to\infty) = \int_0^{\infty} dx\ x^{r-1}\frac{ze^{-x}}{1+ze^{-x}} = \frac{1}{r}\int_0^{\infty} dx\ x^r \frac{e^{x-\nu}}{(e^{x-\nu}+1)^2} = \frac{1}{r}[\nu^r + \frac{\pi^2}{6}r(r-1)\nu^{r-2} + ...] \tag{8}$$

$$H = -\frac{1}{2}\sum_{ij} J_{ij}S_iS_j - \sum_i h_iS_i \quad \to \quad (for\ spin \pm 1)\ \ Z_{MF} = 2^N e^{-\frac{1}{2}\beta\sum_{ij}J_{ij}m_im_j}\prod_{i=1}^{N}(cosh(\beta\sum_j J_{ij}m_j + \beta h_i)). \tag{9}$$

$$Z_{vdw} = \frac{q^N}{N!\lambda^{3N}}; \quad \text{where} \quad q = (V-Nb)e^{aN/(Vk_BT)}; \quad \frac{Pv}{k_BT} = \sum_{l=1}^{\infty} a_l(T)(\frac{\lambda^3}{v})^{l-1} \quad (virial\ expansion) \tag{10}$$

$$g_{GL} = \int dV\left[\frac{1}{2m}|(-i\hbar\nabla - qA)\psi(\vec{r})|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4 + \frac{B^2}{2\mu_0} + \frac{\mu_0H^2}{2} - B\cdot H\right]. \tag{11}$$

$$\frac{\delta g}{\delta\psi^*} = a(T)\psi + b(T)|\psi|^2\psi + \frac{1}{2m}(-i\hbar\nabla - qA)^2\psi = 0; \tag{12}$$

$$\frac{\delta g}{\delta A} = 0 \quad\to\quad j_s = \frac{-iq\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^2}{m}A|\psi|^2 = \frac{q}{m}|\psi|^2(\hbar\nabla S - qA) = q|\psi|^2v_s \tag{13}$$

$$H_{MF} - \mu N = \sum_{\vec{k}\sigma}(\epsilon_{\vec{k}\sigma} - \mu)\ a^\dagger_{\vec{k}\sigma}a_{\vec{k}\sigma} - \sum_{\vec{k}}(\Delta_{\vec{k}}a^\dagger_{\vec{k}\uparrow}a^\dagger_{-\vec{k},\downarrow} + \Delta^*_{\vec{k}}a_{-\vec{k}\downarrow}a_{\vec{k}\uparrow} - b^*_{\vec{k}}\Delta_{\vec{k}}); \tag{14}$$

$$H_{MF} - \mu N = \sum_{\vec{k}}(\epsilon_{\vec{k}} - \mu - E_{\vec{k}} + \Delta_{\vec{k}}b^*_{\vec{k}}) + \sum_{\vec{k}} E_{\vec{k}}(\gamma^\dagger_{\vec{k}\uparrow}\gamma_{\vec{k}\uparrow} + \gamma^\dagger_{\vec{k}\downarrow}\gamma_{\vec{k}\downarrow}), \tag{15}$$

$$\Delta_{\vec{k}} = -\sum_{\vec{l}} V_{\vec{k}\vec{l}}\,b_{\vec{l}}, \quad b_{\vec{k}} = < a_{-\vec{k}\downarrow}a_{\vec{k}\uparrow} > = \frac{\Delta_{\vec{k}}}{2E_{\vec{k}}}(1 - 2f(E_{\vec{k}})); \quad E_{\vec{k}} = ((\epsilon_{\vec{k}} - \mu)^2 + |\Delta_{\vec{k}}|^2)^{1/2} \tag{16}$$

Problem 1. (16 points)
Consider a magnetic system with Hamiltonian,

$$H = -\frac{1}{2}\sum_{ij} J_{ij} S_i S_j - h \sum_i S_i \tag{17}$$

where J_{ij} is the exchange constant, $S_i = \pm 1$ is the spin variable and h is the applied field.
(i) (5 points) Using the canonical ensemble show that the magnetization M is given by,

$$M = \sum_i < S_i >= \frac{1}{\beta}\frac{\partial ln Z}{\partial h} \tag{18}$$

where the brackets $< O >$ denote an average over the canonical ensemble, and $\beta = 1/(k_B T)$
(ii) (7 points) Show that the magnetic susceptibility $\chi = \partial M/\partial h$, is given by,

$$k_B T \chi =< (\sum_i S_i)^2 > - < (\sum_i S_i) >^2 = \sum_{ij} C_{ij}; \tag{19}$$

where $C_{ij} =< S_i S_j > - < S_i >< S_j >$ is the pair correlation function.
(iii) (4 points) Discuss the physical meaning of the "Central limit theorem". Explain why the results of your calculations in (i) and (ii) are in agreement with this theorem.

Problem 2. (16 points)
Consider an ideal monatomic Fermi gas of N particles of mass m in a cube of volume $V = L^3$. In this calculation you can ignore the spin degree of freedom so we treat "spinless" Fermions.

(i) (6 points) In the ground state of a non-relativistic gas, find an expression for the pressure as a function of the number density. Also find an expression of the ground state pressure of the gas as a function of the Fermi Energy of the gas.

(ii) (6 points) Repeat the calculation of part (i) for an ultrarelativistic Fermi gas.

(iii) (4 points) Explain the differences between the behavior of the pressure of the Bose, Fermi and classical gases in the low and high temperature limits. Assume the same values of particle number and volume (N, V) for the three cases. A qualititative discussion is sufficient.

Problem 3. (16 points)
The van der Waals equation of state (EOS) is given by,

$$p = \frac{k_B T}{v - b} - \frac{a}{v^2} \tag{20}$$

where p is pressure, $v = V/N$ is volume per particle, T is temperature, k_B is Boltzmann's constant and a, b are model parameters.

(i) (3 points) Explain the physical origin of the parameters a and b.

(ii) (3 points) Find the second virial coefficient of the van der Waals gas.

(iii) (6 points) Keeping terms up to the second virial coefficient, plot isotherms for the equation of state of the gas as a function of temperature. Also sketch isotherms based on Eq. (20). On your graph indicate the critical point and illustrate how the Maxwell construction is used to find the gas and liquid densities in the co-existence regime.

(iv) (4 points) Derive expressions for the critical point p_c, v_c, T_c of the van der Waals gas, in terms of the model parameters a, b.

Problem 4. (16 points)

(i) (8 points) Starting with the expressions for the ideal Bose gas given on the cover sheet (or otherwise), prove that a $d-$*dimensional* Bose gas with dispersion relation $\epsilon_p = Ap^s$, where A is a constant and p is the momentum, obeys the relation,

$$p = \frac{s}{d}\frac{U}{L^d} \tag{21}$$

Where p is pressure, U is energy density, d is dimension and L is the linear dimension of a box of volume L^d in d dimensions.

(ii) (4 points) From this expression, deduce relations that are correct for an ideal Bose gas in three dimensions for two cases: (a) A non-relativistic Bose gas, and (b) An ultrarelativistic Bose gas

(iii) (2 points) Does the relation (40) hold in the Bose condensed phase of an idea Bose gas? Explain your reasoning.

(iv) (2 points) What is the relation between pressure and energy density for the ideal monatomic non-relativistic classical gas in d dimensions. Show your reasoning.

Problem 5. (18 points)

(i) (8 points) (a) Sketch the field-temperature phase diagram for a type II superconductor indicating the Meissner, mixed and normal phases; (b) Sketch the behavior of the superconducting gap, Δ, as a function of temperature, at zero applied magnetic field, for an s-wave BCS superconductor. (c) Write down a mathematical expression for the scaling behavior of the gap near the critical temperature, T_c; (d) For an s-wave BCS superconductor, sketch the density of states of the quasiparticle excitations from the BCS ground state as a function of energy, near the Fermi energy, ϵ_F.

(ii) (4 points) Within London theory, which is valid in the extreme type II limit, the Helmholtz free energy per unit length of an isolated flux quantum in a type II superconductor is approximately,

$$f_1 \approx \frac{\phi_0^2}{4\pi\mu_0\lambda^2} Ln(\frac{\lambda}{\xi}) \tag{22}$$

Using this result find an expression for the lower critical field H_{c1} of a type II superconductor. Here ϕ_0 is the flux quantum, λ is the penetration depth, ξ is the coherence length, μ_0 is the permeability.

(iii) (6 points) By linearizing the G-L equation, find an expression for the upper critical field of the superconductor, H_{c2}, in terms of the parameters in the G-L theory. To write the result in its usual form you can use $\xi^2 = \hbar^2/(2m|a|)$, $\phi_0 = h/q = h/2e$. (hint: the eigenvalues of a Cooper pair in a magnetic field are, $\epsilon_{n,k_z} = (n + 1/2)\hbar\omega_c + \hbar^2 k_z^2/2m$)

Problem 6. (18 points)

A spin half Ising model with four spin interactions on a square lattice has Hamiltonian,

$$H = -\sum_{ijkl\ in\ square} J S_i S_j S_k S_l \tag{23}$$

where the sum is over the smallest squares on an infinite square lattice, the interaction is ferromagnetic $J > 0$ and $S_i = \pm 1$. Each elementary square is counted only once in the sum.

(i) (6 points) Using a leading order expansion in the fluctuations (i.e. write $S_i = m_i + (S_i - m_i)$ and expand to leading order in the fluctuations), find the mean field Hamiltonian for this problem (Here $m_i =< S_i >$ is the magnetization at site i).

(ii) (6 points) Using the mean field Hamiltonian and assuming a homogeneous state where $m_i = m$, find an expression for the mean field Helmholtz free energy, and the mean field equation for this problem.

(iii) (6 points) Taking $J = 1$, sketch the behavior of the solutions to the mean field equation as a function of temperature. Does the non-trivial solution move continuously toward the $m = 0$ solution as the temperature increases? Is the behavior of the order parameter at the critical temperature discontinuous or continuous? Do you expect the correlation length to diverge at the critical point in this problem?