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THEORY OF NONLINEAR OSCILLATOR INTERACTING WITH A MEDIUM

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Abstract

The relaxation and fluctuations in a nonlinear subsystem interacting with a medium are considered. A concrete analysis is carried out for a Duffing oscillator with a potential energy

$$U(q) = \tfrac{1}{2}\omega_0^2 q^2 + \tfrac{1}{4}\gamma q^4$$

in a wide range of the nonlinearity parameter γ and at various types of interaction with the medium. In the classical theory, the dynamic equations of the oscillator and the medium are reduced, after eliminating the medium variables, to a nonlinear stochastic equation for the subsystem under consideration. A special method is developed that allows us to solve this equation and to describe the evolution of the oscillator, and to find in explicit form the time correlation function of the coordinates $Q(t)$ and its spectral representation $Q(\omega)$. Even at relatively weak nonlinearity, the time dependence of $Q(t)$ is not exponential, and the distribution $Q(\omega)$ is substantially non-Lorentzian and asymmetric. Solution of the quantum kinetic equation makes it possible to express $Q(t)$ in terms of elementary functions also. This has made it possible to investigate the $Q(\omega)$-distribution fine structure that can manifest itself in the quantum case. The results of the investigation of the spectral distributions are used to analyze the spectra of the infrared absorption by local or quasilocal vibrations near defects in crystals.

Besides the fluctuations about the equilibrium positions, the relatively infrequent fluctuations between equilibrium states of a nonlinear nonequilibrium subsystem are considered (using the path-integral method). The probabilities of the transitions between stable states of a Duffing oscillator in an external resonant field are determined in explicit form. The external-field absorption peculiar features, including the anomalously narrow absorption peak of the weak additional field in the region of a smeared kinetic phase transition, where the stable states have nearly equal populations, are considered. The character of the fluctuations near bifurcation points of marginal and spinodal type is investigated. The dynamics of a nonlinear oscillator is considered in a strong nonresonant external field that produces additional relaxation channels, and the possibility of heating or cooling the oscillator by the field is demonstrated.

§1 Introduction

The relaxation of a subsystem interacting with a medium is one of the classical problems of statistical physics. It was apparently first considered within the framework of the theory of Brownian motion [1, 2]. Several simple models were investigated, including a harmonic oscillator in a highly viscous medium.

Two approaches are possible to the analysis of Brownian motion and of other fluctuation phenomena in vibrational subsystems. One is semiphenomenological and is based on the description of the interaction with the medium with the aid of friction forces and random forces having various statistical properties. In the simplest most widely used model of Gaussian δ-correlated random forces, the problem reduces to the solution of the Einstein–Fokker–Planck (EFP) equation or of the corresponding Langevin equation.

Another approach is based on a microscopic description of the entire system (subsystem + thermostat) with the aid of a Hamiltonian. In a number of cases it is possible to obtain in a consistent way here, by eliminating the dynamic variables of the medium, relatively simple equations that contain only the variables of the subsystem, such as an equation for its classical distribution function or density matrix. This occurs as a rule if the interaction of the subsystem with the medium is so weak that the characteristic relaxation times of the subsystem exceed the characteristic reciprocal width of the thermostat spectrum. Solution of the equation for the distribution function makes it possible to obtain the subsystem time correlation functions that describe the statistical properties of its motion, as well as to describe the variation of the subsystem distribution in time for given initial conditions.

Actually, in cases when the characteristic frequencies of the medium greatly exceed the frequencies of the subsystem, the second of the indicated approaches can substantiate the description of the motion of a classical subsystem with the aid of the EFP equation or the Langevin equation, and allows us to calculate the parameters in these equations. For an arbitrary ratio of the frequencies (but at a weak interaction between the subsystem and the medium), the microscopic approach was realized initially for one of the simplest types of

subsystems, the classical harmonic oscillator [3]. A detailed quantum theory of a harmonic oscillator that interacts with a thermostat was also developed [4].

The investigation of a harmonic oscillator is greatly simplified by the fact that the system is described by linear equations. In this case the random oscillator motion induced by the fluctuations in the medium is separated from the regular motion, and the time correlation functions of the dynamic variables of the system decay exponentially with time in a large time interval. As a result, the complete solution of the problem can be obtained in simple explicit form.

In fact real oscillatory systems are nonlinear to some degree or another. Even a small nonlinearity can qualitatively alter the character of the subsystem relaxation. In terms of classical theory, the nonlinearity causes a dependence of the frequency and of the friction coefficient of a nonlinear oscillator on the oscillation amplitude. As a result of this dependence, the regular and random motions are intermixed, so that the damping acquires a complicated non-exponential character. In terms of quantum theory, the problem of the relaxation of a weakly nonlinear oscillator is made complicated by the fact that the energy spectrum of the subsystem constitutes an infinite set of almost equidistant levels, i.e., even if the interaction with the thermostat is weak, and the quantum kinetic equation is applicable, it cannot be separated into equations for an aggregate of two-level systems. (From this point of view, the simplicity of the problem of relaxation of a linear oscillator whose interaction with the medium is linear in the oscillator coordinate is due to the special degeneracy of this problem; see section 4.3.) Notwithstanding the indicated complexities of the problem of a nonlinear oscillator that interacts with a medium, it is possible to obtain in a number of cases simple explicit expressions for the time correlation functions of the oscillator in a wide range of the nonlinearity parameter both in the classical theory (Sec. 2) and in the quantum theory (Sec. 4).

A characteristic property of classical nonlinear systems is the possibility of having several stable equilibrium states. If the subsystem is in a static potential field, the probability of its transition between minima of the potential due to interaction with the thermostat is proportional to $\exp(-\Delta U/T)$, where ΔU is the height of the potential

barrier. Several stable states can arise, however, also in the case of nonpotential motion of the subsystem or when the subsystem moves in an external periodic field. In particular, two stable states (and one unstable one) arise in a certain range of parameters when a resonant external force acts on a nonlinear oscillator. The determination of the probabilities of transitions between stable states in this case calls for a special treatment. If the transition probabilities are low enough, and the random force that causes them has a Gaussian distribution, it is convenient to calculate these probabilities by the path-integral method. It permits the problem to be reduced to an investigation of the extremal trajectory of a certain auxiliary dynamic system. It turns out that the transition probability depends exponentially on the reciprocal intensity of the random force (i.e., on T^{-1}, if the random force is due to the interaction of the subsystem with the thermostat). The transition probability and some related problems are considered in Sec. 3.

At certain values of the parameters (at bifurcation points of marginal or spinodal type, respectively) one or two stable states and one unstable state of the system merge. The anomalously large fluctuations that arise in this region are also analyzed in Sec. 3.

Besides the onset of several stable states, an external alternating field can lead also to effects of another type, due to the appearance of additional relaxation channels. As shown in Sec. 5, in the case of an oscillator this can cause strong heating or cooling, as well as "runaway" of the oscillator.

Important examples of nonlinear oscillators that interact with a medium are local and quasilocal vibrations near defects in crystals. In this case the characteristic features of the time correlation functions or of their spectral representations for the nonlinear vibrations can be directly investigated in experiment, since the spectral densities of the cross sections for absorption and Raman scattering of light, as well as for inelastic scattering of neutrons, are proportional to the spectral representations of the correlators. Applications of the results of the theory of nonlinear oscillators that interact with the medium to optics are discussed in Sec. 6.

In the present section we shall examine qualitatively questions concerning the relaxation and fluctuations of a harmonic oscillator and the effects due to nonlinearity.

1.1 Harmonic Oscillator Interacting with a Medium

Brownian motion of a harmonic oscillator can be described with the aid of the Langevin stochastic equation (see, e.g., [5])

$$\ddot{q} + 2\Gamma\dot{q} + \omega_0^2 q = f_L(t),$$
$$\langle f_L(t) f_L(t')\rangle = 4\Gamma T\delta(t - t'). \tag{1.1}$$

Here q is the normal coordinate of the oscillator, ω_0 is its eigenfrequency (in the absence of friction), and T is the temperature of the medium in energy units. The random force $f_L(t)$ exerted on the oscillator by the medium is Gaussian and δ-correlated, and presents white noise. In the simplest model it is due to collisions of the vibrating particle with the atoms of the medium. Equation (1.1) is applicable also if the oscillator is acted upon not by fluctuations of an equilibrium thermostat, but by a white noise of arbitrary nature; in this case the parameter $4\Gamma T$ must be replaced by the noise intensity $2\mathscr{B}\Gamma$.

Equation (1.1) is valid if ω_0 and Γ are small compared with the characteristic frequencies of the medium, i.e., if the subsystem motion is slow compared with the motion of the particles of the medium. In the case when the characteristic frequencies of the medium are of the same order of magnitude as ω_0, Eq. (1.1) no longer holds. If, however, the interaction with the medium is weak enough, so that the friction coefficient is small:

$$\Gamma \ll \omega_0, \tag{1.2}$$

the stochastic motion of the oscillator can also be described as Brownian motion, by changing from the normal coordinate and momentum to "slow" auxiliary variables that vary little over time $\sim\omega_0^{-1}$ and characterize an oscillator motion averaged over the time $\sim\omega_0^{-1}$. This can be illustrated with simple dynamic models of the medium and of its interaction with the oscillator, using the aforementioned microscopic approach.

Apparently, the simplest of these models is one in which the medium is described by a set of non-interacting vibrations of a quasicontinuous spectrum with frequencies ω_k and with coordinates

q_k (k is the number of the vibration), and the interaction of the singled-out oscillator with the medium is of the form [3]

$$H_i = \sum_k \epsilon_k q_k . \tag{1.3}$$

The interaction parameters ϵ_k are proportional to $N^{-1/2}$ (N is the number of vibrational degrees of freedom of the medium) and, in addition, contain a small constant ϵ that ensures satisfaction of the condition (1.2).

By eliminating the vibrations of the continuous spectrum (for details see Sec. 2.1), the equation of motion of the oscillator in the time region $|t| \gg \omega_0^{-1}$ can be represented in the form of a complex Langevin equation

$$\begin{gathered} \dot{u}_1 = (-\Gamma \operatorname{sgn} t + iP)u_1 + f_1(t), \\ \langle f_1(t) f_1^*(t') \rangle = \frac{\Gamma T}{\omega_0^2} \delta(t - t'), \qquad \Gamma, |P| \ll \omega_0, \quad |t| \gg \omega_0^{-1}. \end{gathered} \tag{1.4}$$

Here $u_1 \equiv u_1(t)$ is the complex amplitude of an oscillator (u_1 varies slowly over the time $\sim\omega_0^{-1}$):

$$q(t) = u_1 \exp(i\omega_0 t) + \text{c.c.}, \qquad \dot{q}(t) = -i\omega_0 (u_1 \exp(i\omega_0 t) - \text{c.c.}). \tag{1.5}$$

The parameters Γ and P in Eq. (1.4) are quadratic in ϵ_k (explicit expressions (2.15) for Γ and P are given in Sec. 2). The random-force correlator $\langle f_1(t) f_1^*(t') \rangle$ in Eq. (1.4) is strictly speaking different from zero in the finite interval $|t - t'| \lesssim \omega_0^{-1}$. In the "slow" time scale, however, $f_1(t)$ corresponds, accurate to terms $\sim\Gamma/\omega_0$, to a Gaussian δ-correlated process, with Re $f_1(t)$ and Im $f_1(t)$ statistically independent. When the condition (1.2) is satisfied, Eq. (1.4) follows directly from Eqs. (1.1) and (1.5) (excluding the term $-iPu_1$, which determines the renormalization of the frequency ω_0 owing to the interaction of the oscillator with the medium); in this case

$$f_1(t) = (2i\omega_0)^{-1} \exp(-i\omega_0 t) f_L(t).$$

The condition (1.2) is assumed satisfied here and elsewhere.

The linear equations (1.4) can be easily solved and an explicit time dependence of the oscillator coordinates can be obtained:

$$\begin{gathered} u_1(t) = \exp(\Gamma|t| + iPt) u_1(0) + \xi(t), \qquad |t| \gg \omega_0^{-1}, \\ \xi(t) = \exp(-\Gamma|t| + iPt) \int_0^t dt_1 \exp(\Gamma|t_1| - iPt_1) f_1(t_1). \end{gathered} \tag{1.6}$$

It can be seen from Eqs. (1.6) and (1.5) that, in the considered linear model, the oscillator displacement is a linear superposition of a regular and a random part, described respectively by the first and second term in the expression for $u_1(t)$. The regular part decays exponentially with time.

Important characteristics of an oscillator that interacts with a medium are the time correlation function of its coordinates

$$Q(t) = \langle q(t) q(0) \rangle \tag{1.7}$$

($\langle \ldots \rangle$ denotes statistical averaging), and its spectral representation

$$Q(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \exp(i\omega t) Q(t). \tag{1.8}$$

In the model considered, as follows from Eqs. (1.6) and (1.5), the time correlation function $Q(t)$ decays exponentially (like $\exp(-\Gamma|t|)$), and the spectral representation $Q(\omega)$ is described near the maximum by a Lorentz function

$$Q(\omega) = \frac{T}{2\pi\omega_0^2} \frac{\Gamma}{(\omega - \omega_0 - P)^2 + \Gamma^2}. \tag{1.9}$$

When averaging over $q(0)$, we have used here the relation $\langle q^2(0) \rangle \approx T/\omega_0^2$, which follows from the equipartition law (and is obtained directly from Eqs. (1.4) and (1.5) as a stationary solution for $\langle q^2 \rangle$).

In quantum theory, the relaxation of a harmonic oscillator was considered mainly for models in which the interaction with the medium is linear in q and is weak (see [4, 6, 7] and the reviews [8–10]). For such models, the equation for the oscillator creation operator $\hat{a}^+$ in the interaction representation, is similar to Eq. (1.4) for $u_1(t)$ [10], but now the random force is an operator. Therefore, just

as in the classical theory, the time correlation function $Q(t)$ for long times decays exponentially ($Q(t) \propto \exp(-\Gamma|t|)$), and the peak of $Q(\omega)$ is described by Eq. (1.9), in which T/ω_0 is replaced by $\bar{n}+1$, where $\bar{n}$ is the Planck number of the oscillator. This result obtained for $Q(\omega)$ by an asymptotic method with the terms $\sim\Gamma/\omega_0$ neglected, is confirmed by exact calculation for the model (1.3) of the interaction with the medium (see Appendix A).

Besides the direct solution of the equations of motion, the problem of relaxation of a classical harmonic oscillator was considered by solving the EFP equation (see [5]). In quantum theory, the analog of the EFP-equation is the quantum kinetic equation. It was analyzed in detail for the case of the harmonic oscillator in [11, 12].

1.2 Simple Models of Fluctuating Nonlinear Vibrational Subsystems

The investigation of the relaxation becomes much more difficult if the oscillator is nonlinear. The reason is that in the general case it is impossible to solve the nonlinear equations of motion in the presence of a random external force, and hence to average over the realizations of this force. As already noted, even a relatively weak nonlinearity can lead to strong deviations from the exponential damping of the correlation function $Q(t)$; the spectral distribution $Q(\omega)$ becomes accordingly much more complicated.

Besides the change in the character of the random motion because of the nonlinearity, the presence of even a weak random force can in its turn alter qualitatively the character of the motion of the nonlinear system. Thus, if the system has several stable states (corresponding, e.g., to different amplitudes of the constrained vibrations), it would stay in the absence of random forces for an infinitely long time in one of these states, whereas fluctuations cause transitions between the states and a distribution over the states is formed.

An analysis of the relaxation and of the fluctuations in nonlinear vibrational systems is not only of general interest but is also of importance for various applications. These include the problem of fluctuations in autogenerators [13], lasers and masers [9, 10, 14], and a number of other problems of radiophysics and nonlinear optics, the problem of light-absorption spectra and neutron scattering by local and quasilocal vibrations of impurities in crystals [7, 15, 16] (see also

Sec. 6 below), a large number of problems of nonlinear mechanics, acoustics, and hydrodynamics (see, e.g., [14, 17, 18], and others). A number of approximate methods were developed in this connection for the investigation of random nonlinear vibrations. The most widely used among them are the perturbation method, in which the nonlinearity is regarded as weak and is taken into account by perturbation theory, and the equivalent-linearization method, in which the nonlinear system is approximated by a linear one having certain optimally chosen parameters. The reliability of these methods can be verified by the investigation of quite simple but nontrivial exactly solvable models. These models are as a rule also of their own interest, since their analysis allows to reveal certain common properties of nonlinear systems.

One of the most widely studied nonlinear vibrational systems is the Van der Pol oscillator. The random vibrations of this oscillator are described by the equation

$$\ddot{q} - 2\Gamma'\dot{q}(\mu_P - q^2) + \omega_0^2 q = f_L(t), \qquad \Gamma' \ll \omega_0. \tag{1.10}$$

The model (1.10) (with insignificant modification of the nonlinear term) is used to describe fluctuations in single-mode generators of various types [13, 10]. It is important that the nonlinear term is not conservative in this case, i.e., the nonlinearity concerns the effective friction force while the potential restoring force is linear. At the same time, the force $f_L(t)$ presents white noise, and it is assumed that its intensity, just as in the case of linear friction (1.1), is independent of the oscillator amplitude. (It should be noted that in the microscopic model that results in an equation of motion corresponding to Eq. (1.10) (with a nonlinear friction force), the force $f_L(t)$ is found to depend on the coordinate and momentum; the phenomenological model (1.10) with force $f_L(t)$ independent of q and $\dot{q}$ does not describe an oscillator that interacts with a system that is in thermodynamic equilibrium.)

If in Eq. (1.10) the nonlinear friction is small, $\Gamma'\langle q^2\rangle \ll \omega_0$, it is possible to separate in the corresponding EFP equation the variables, namely the amplitude and the phase of the oscillator. This equation can be solved numerically [10]. As a result, it becomes possible to obtain the eigenfunctions and the eigenvalues [10] and thus investigate fully the relaxation of the Van der Pol oscillator.

The simplest but nontrivial model of a nonlinear oscillator with conservative nonlinearity is the Duffing oscillator. Its Hamiltonian in the absence of interaction with the medium takes the form

$$H_0 = \tfrac{1}{2}(p^2 + \omega_0^2 q^2) + \tfrac{1}{4}\gamma q^4 \tag{1.11}$$

(p is the oscillator momentum), and the Brownian motion is described by the equation

$$\ddot{q} + 2\Gamma\dot{q} + \omega_0^2 q + \gamma q^3 = f_L(t), \qquad \langle f_L(t) f_L(t')\rangle = 4\Gamma T\delta(t - t'). \tag{1.12}$$

Just as in the case of Eq. (1.1) for the linear oscillator, Eq. (1.12) describes also the dynamics of the Duffing oscillator in the field of an arbitrary white noise, if $4\Gamma T$ is replaced by the white-noise intensity.

The problem of the Duffing oscillator in the case of weak coupling ($\Gamma \ll \omega_0$) can be solved analytically with asymptotic accuracy in both the classical [19, 20] and in the quantum theory [21] (see Secs. 2 and 4). Thus, with this as an example, it is possible to describe completely the relaxation and the fluctuations in the nonlinear subsystem.

1.3 *Qualitative Analysis of the Dynamics of a Duffing Oscillator that Interacts with a Medium*

It can be easily seen, by changing over in Eq. (1.12) to the dimensionless variables $\omega_0 t$ and $\omega_0 T^{-1/2} q$, that the dynamics of the oscillator is determined by two dimensionless parameters, α and $\mathfrak{b}$, or α and Γ/ω_0, where

$$\alpha = \frac{3\gamma T}{8\omega_0^3 \Gamma}, \qquad \mathfrak{b} = \frac{\gamma T}{\omega_0^4}. \tag{1.13}$$

The meaning of these parameters becomes clear if it is recognized that the frequency of the nonlinear oscillator depends on its amplitude. As a result of the interaction with the medium (or of the action of a stationary random force), a certain distribution of the oscillator in amplitude is developed, and in the limit of weak nonlinearity ($\gamma \to 0$),

as already noted, $\langle q^2\rangle = T/\omega_0^2$. It is seen from Eq. (1.12) that the mean squared frequency is

$$\omega_0^2 + \tfrac{3}{4}\gamma\langle q^2\rangle = \omega_0^2(1 + \tfrac{3}{4}\mathfrak{b}),$$

i.e., $\mathfrak{b}$ characterizes the relative shift of the square of the oscillator frequency.

The quantity T/ω_0^2 determines not only the mean squared amplitude of the oscillations, but also the width of the distribution in amplitude. Directly connected with the latter is the width $\delta\omega$ of the oscillator distribution in the frequency. Obviously,

$$\delta\omega \sim |\gamma| T/\omega_0^3.$$

The frequency spread $\delta\omega$ due to the nonlinearity must be set in comparison with the oscillator-frequency uncertainty Γ due to the oscillator damping. The relation between these quantities is characterized precisely by the parameter α. It is therefore just this parameter which determines which of the frequency smearing mechanisms (and consequently which of the spectral-distribution broadening mechanisms) predominates.

At $|\alpha| \ll 1$ the broadening of $Q(\omega)$ due to the damping predominates and the shape of peak of $Q(\omega)$ is close to Lorentzian. It is precisely in this region (and not in the entire region $|\mathfrak{b}| \ll 1$) that the nonlinearity, as shown in [22, 23], can be treated by perturbation theory. Accordingly, the results of [24, 25] are applicable here.

In the region $|\alpha| \gg 1$, the dominant mechanism is the spectrum broadening due to modulation of the nonlinear-oscillator frequency by the amplitude fluctuations (modulation broadening). In the limit $\Gamma \to 0$, the spectral distribution in the vicinity of the peak can be calculated by assuming the oscillator motion to be quasiconservative and by averaging the coordinate correlator, obtained at $\Gamma = 0$ and $f_L(t) = 0$, over the oscillator energy.

The most complicated is the investigation of the case $|\alpha| \sim 1$, when both broadening mechanisms make comparable contributions. Solution of the problem in this region makes it possible to trace how the relaxation of the initial oscillator state becomes essentially nonexponential with increasing nonlinearity, and how the shape of peak of the distribution $Q(\omega)$ becomes non-Lorentzian.

The large change of the spectral distribution $Q(\omega)$ as a result of even relatively weak nonlinearity ($|\mathfrak{b}| \ll 1$) can be easily understood also in terms of quantum theory. To this end we express the Hamiltonian (1.11) in terms of the creation and annihilation operators $\hat{a}^+$ and $\hat{a}$:

$$H_0 = \hbar\omega_0\hat{n} + \tfrac{1}{2}\hbar V\hat{n}^2, \qquad \hat{n} = \hat{a}^+\hat{a}, \quad V = \tfrac{3}{4}\hbar\gamma/\omega_0^2. \tag{1.14}$$

We have discarded in Eq. (1.14) terms of the type $V\hat{a}^4, V\hat{a}^3\hat{a}^+, \ldots$. If the condition

$$|V|(\bar{n}+1) \ll \omega_0, \qquad \bar{n} = \left(\exp\frac{\hbar\omega_0}{T} - 1\right)^{-1} \tag{1.15}$$

is satisfied, these terms lead to small corrections $\sim\hbar V^2/\omega_0$ to the energy. (In the limit of large $T(\hbar\omega_0)^{-1}$ the condition (1.15) goes over into the inequality $|\mathfrak{b}| \ll 1$.) In Eq. (1.14), ω_0 differs from the ω_0 in Eq. (1.11) by $V/2$. In quantum theory we shall use hereafter the frequency definition (1.14) (in the classical limit $\hbar \to 0$ the expressions for ω_0 and $\omega_0 + V/2$ are identical).

The energy spectrum of an oscillator with a Hamiltonian (1.14) is shown in Fig. 1. It presents a set of nonequidistant levels. The light-absorption spectrum of the oscillator with such energy levels (neglecting the violation of the selection rules on account of the nonlinearity, i.e., assuming that transitions are allowed only between neighboring levels) consists of lines at the frequencies

$$\omega(n) = \omega_0 + V(n - \tfrac{1}{2}), \qquad n = 1, 2, \ldots \tag{1.16}$$

(see Fig. 1). At a finite but sufficiently small interaction with a medium, each line broadens, in accord with the Weisskopf–Wigner theory [26], into a narrow peak having a width that is proportional to the damping parameter Γ and depends on the number n. With increasing Γ, these peaks overlap, the structure of the spectrum smears, and a single smooth spectrum is developed. The form of the function $Q(\omega)$ (which describes the oscillator light-absorption spectrum) turns out to be essentially non-Lorentzian in the case $|V| \gtrsim \Gamma$,

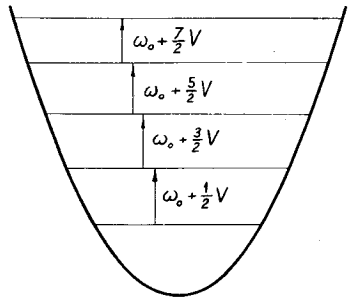

Figure 1 Energy-level scheme for nonlinear oscillator.

and the width of the peak greatly exceeds the reciprocal oscillator lifetime Γ.

Study of the spectral distribution of a nonlinear oscillator makes it possible to investigate a peculiar phenomenon, namely, the interference of transitions. This phenomenon takes place in systems in which either the energy spectrum constitutes a set of almost equidistant levels, or else the levels are degenerate, so that light of a given frequency can cause transitions between different states. Because of the interference between the transitions, the Weisskopf–Wigner theory of the spectrum shape for the nondegenerate case [26] is inapplicable to such systems. In particular, it is inapplicable to the harmonic oscillator (the harmonic-oscillator "paradox," see [11]).

The nonlinearity lifts the specific degeneracy, the equidistant character of the spectrum, which is a feature of the harmonic oscillator, and thus suppresses the interference of the transitions. It has been pointed out above that for $|V| \gg \Gamma$, the spectral distribution of the oscillator presents a set of fine-structure lines at frequencies $\omega(n)$, and the shape of an individual line is described by the Weisskopf–Wigner theory. With decreasing $|V|/\Gamma$ the role of the interference of the transitions increases and a smooth spectral distribution is formed gradually, wherein the individual lines are not simply superimposed on one another, but the shape of each of them is distorted (see Sec. 4.3). A description of this process makes it possible to analyze the relaxation in a case intermediate between two opposite cases that have been investigated in detail: that of a two-level system (or a multilevel system with an essentially nonequidistant spectrum) and that of a harmonic oscillator.

1.4 *Features of the Dynamics of a Nonlinear Oscillator in an External Periodic Field*

The behavior of a weakly nonlinear oscillator interacting with a medium in an external field

$$h(t) = h\cos\omega t \tag{1.17}$$

depends essentially on the ratio of the oscillator eigenfrequency $\approx\omega_0$ and the field frequency ω. Under resonance conditions ($|\omega - \omega_0| \ll \omega_0$) in a certain interval of the values of the parameters h and $\omega - \omega_0$, the amplitude A of the constrained oscillations of a nonlinear oscillator can take on three stationary values (see Fig. 2). The largest and smallest of them are stable with respect to small fluctuations, and the intermediate value (shown dashed in Fig. 2) is unstable (see, e.g., [27, 28]). The possibility of the existence of two stable states is due to the dependence of the effective frequency of the nonlinear oscillator

$$\omega_{\text{eff}} = \omega_0 + 3\gamma A^2(8\omega_0)^{-1}$$

on the amplitude (as a result, we can have both a stable state with

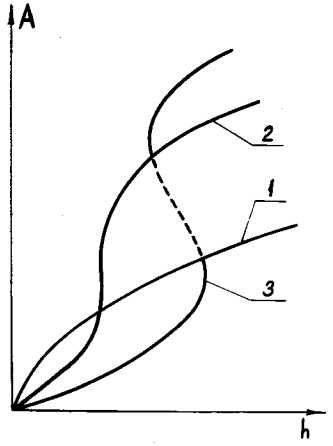

Figure 2 Schematic dependence of the amplitude A of the forced oscillations of a nonlinear Duffing oscillator on the amplitude h of an external periodic force. Curves 1 to 3 correspond to different frequency detunings $\omega - \omega_0$. Curve 2 corresponds to the critical value of $\omega - \omega_0$ starting with which the plot of A *vs.* h becomes S-shaped. The unstable stationary states are shown dashed.

large amplitude, in which ω_{eff} is close to ω ($(\omega - \omega_0)/\gamma > 0$) and the resonance condition is fulfilled self-consistently, and a stable state with low amplitude and accordingly with considerably larger difference $\omega - \omega_{\text{eff}}$).

Besides the small fluctuations, which do not take the oscillator out of the immediate vicinity of the stable state, there are possible (but hardly probable) sufficiently large fluctuations that lead to transitions between the states. The transition probability W, as indicated above, depends exponentially on the reciprocal of the random-force intensity $\mathscr{B}$:

$$W \propto \exp(-S/\mathscr{B}) \tag{1.18}$$

where S plays the role of the activation energy [29] (for details see Sec. 3). The probabilities w_i of finding a system in one of two states $i = 1, 2$ are inversely proportional to the probabilities of transition from this state. Since the values of S_1 and S_2 for the transitions $1 \to 2$ and $2 \to 1$ are generally speaking different and at small $\mathscr{B}$ their difference $|S_1 - S_2| \gg \mathscr{B}$, the values of w_1 and w_2 differ by many times in almost the entire range of the parameter h and ω. Only at a certain ratio of the parameters h and ω, corresponding to the dashed line of Fig. 3, do the values of w_1 and w_2 become equal. Owing to the

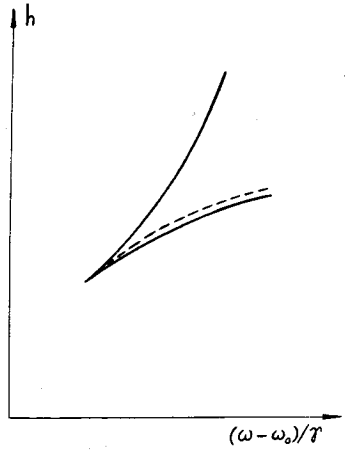

Figure 3 Schematic dependence of the external-field amplitude h, at which two stable states appear, on the relative frequency detuning $(\omega - \omega_0)/\gamma$. The line on which the transition probabilities between the states are equal is shown dashed.

strong exponential dependence (1.18) of W on the parameters, the passage through the dashed line with changing ω or h should be perceived as causing a smeared first-order phase transition in the subsystem. In such a transition, the amplitude of the constrained vibrations changes sharply. Also produced in the transition region are characteristic peculiar features of the absorption of both the field $h(t)$ (1.17) that produces the vibrational state of the oscillator, and of the weak trial field. In particular a very narrow peak appears in the frequency dependence of the absorption coefficient of the trial field.

Even in the parameter range where the nonlinear oscillator has only one stable state (outside the region bounded by the curves of Fig. 3), a strong external field influences substantially the oscillator eigenoscillations excited by a resonant trial force. The picture of the oscillator motion in an external field turns out to be qualitatively different if the system is far from resonance, so that the field frequency ω differs substantially from ω_0 (and from frequencies that are multiples of ω_0), i.e.,

$$\Gamma \ll \omega, |\omega - \omega_0|,$$

$$\frac{\gamma}{\omega_0}\langle q^2\rangle \ll \omega, \omega_0, |\omega - \omega_0|.$$

If these conditions are satisfied and thus the intrinsic oscillator nonlinearity is small, and if the interaction with a medium is linear (cf. Eq. (1.3)), then the forced oscillations of the oscillator at the frequency ω have practically no effect on the eigenoscillations at frequency ω_0.

The situation changes if the interaction of the medium with the oscillator is nonlinear in its coordinate. Let, e.g., the medium constitute a set of quasicontinuous-spectrum vibrations, and let the Hamiltonian of the interaction contain besides the term (1.3) also the term

$$H_i^{(2)} = \sum_k \epsilon_k^{(2)} q^2 q_k \,. \tag{1.19}$$

In the presence of an external field, q^2 contains the doubled product of the regular term $q_{\text{reg}}(t) \propto \cos\omega t$ oscillating with the field frequency, and a term q_0 that describes the random eigenoscillations. The corresponding terms in (1.19) ($\propto q_0 q_k \cos\omega t$) give rise to parametric resonance [27] for the eigenoscillations of the singled-out oscillator and for those continuous-spectrum vibrations for which $\omega_k \approx |\omega \pm \omega_0|$. In

the case considered, the resonance condition is satisfied at once for a very large number of vibrations of the quasicontinuous spectrum. Since $\epsilon_k^{(2)} \propto N^{-1/2}$, each of them "builds up" very weakly, whereas the buildup of the considered singled-out oscillator at the eigenfrequency ω_0 can be strong [30] (see Sec. 5 for more details).

In this case, in fact, the field produces a new relaxation (or excitation) channel for the oscillator. This can be easily understood in quantum language. The relaxation described by the Hamiltonian (1.3) corresponds to decay processes in which the transition between the oscillator levels is accompanied by creation or annihilation of a vibration of the medium at a frequency $\omega_k \approx \omega_0$. As a result, in the absence of an external field the oscillator is imposed to the same energy distribution as that of the vibrations of the medium having a frequency $\omega_k \approx \omega_0$, i.e., a Gibbs distribution. In an external field, when account is taken of the interaction (1.19), a transition between the oscillator levels can be accompanied by creation or annihilation of vibrations of the medium, having a frequency $\omega_k \approx |\omega \pm \omega_0|$. If the amplitude of the external force is large enough, the induced decay processes can become dominant. As a result, the stationary distribution of the oscillator is greatly altered, and approaches to the distribution of the vibrations of the medium, which take part in the decay process, i.e., to a Gibbs distribution with temperature $T_h = T \cdot \omega_0/(\omega_0 \pm \omega)$. The oscillator can thus be strongly heated or cooled, depending on the ratio of the frequencies ω and ω_0. At $\omega_k = \omega - \omega_0$, i.e., when the oscillator transition to a higher energy level is accompanied not by annihilation but by creation of a vibration of the medium, the distribution of the oscillator in a certain energy range presents a Gibbs distribution with negative temperature. It turns out that in the latter case, when account is taken of the nonlinear friction, the classical equation of motion for the oscillations at the frequency ω_0 takes the same form as in the case of a Van der Pol oscillator (1.10) (with $\mu_P > 0$).

§2 Classical Theory of a Duffing Oscillator Interacting with a Medium

In this section we consider the relaxation and calculate the time correlation functions and their spectral distributions for a weakly

damped Duffing oscillator. The most complete analysis of these questions can be carried out within the framework of various microscopic models. To investigate the oscillator-relaxation specific features that are due to the nonlinearity of the restoring force (to the internal anharmonicity), it is expedient to consider first a very simple model. We assume correspondingly that the elementary excitations in the medium are vibrations that do not interact with one another and have a quasicontinuous spectrum of the frequencies ω_k. The Hamiltonian of such a medium is of the form

$$H_m = \frac{1}{2}\sum_k (p_k^2 + \omega_k^2 q_k^2). \tag{2.1}$$

The interaction of the considered oscillator with the medium will be assumed linear both in the oscillator coordinate q and in the coordinates q_k (cf. Eq. (1.3)). The total Hamiltonian of the system is then

$$H = \mathscr{H}_0 + H_i, \qquad \mathscr{H}_0 = H_0 + H_m,$$
$$H_0 = \frac{1}{2}(p^2 + \omega_0^2 q^2) + \frac{1}{4}\gamma q^4, \qquad H_i = \sum_k \epsilon_k q q_k \tag{2.2}$$

(it is shown in Sec. 4 and in Appendix F that the results obtained in the present section are valid both for other models of the medium and for a more general type of interaction; it is important only that $H_i \propto q$).

The dynamics of a Duffing oscillator in the absence of interaction with a medium depend on the signs of the parameters ω_0^2 and γ, and on the relation between them. At $\omega_0^2 > 0$ and $\gamma > 0$, the potential energy of the oscillator has one minimum, and the motion is finite at all energies. At $\omega_0^2 > 0$ and $\gamma < 0$ the motion is finite only if the oscillator is in the vicinity of the point $q = 0$, and its energy is less than the height of the potential "hump" $\omega_0^4/4|\gamma|$. When the interaction with the medium is taken into account, the motion in the case $\gamma < 0$ becomes, strictly speaking, infinite. If, however, the characteristic energy obtained by the oscillator due to the interaction (it is obviously equal to the temperature T) is small compared with $\omega_0^4/|\gamma|$, i.e.,

$$|\mathfrak{b}| \ll 1, \qquad \mathfrak{b} = \frac{\gamma T}{\omega_0^4}. \tag{2.3}$$

the oscillator will vibrate in the vicinity of the point $q = 0$ for an exponentially long time, and the problem of fluctuations under such oscillations becomes of physical interest.

It was indicated earlier (in Sec. 1.3) that even if the condition (2.3) is satisfied, and thus the nonlinearity is small, it can influence the relaxation quite substantially. The reason is that the oscillator energy distribution brought about by the interaction with the medium leads, on account of the nonlinearity, to a distribution in frequency. Although the width $\delta\omega$ of the latter is small, when condition (2.3) is satisfied, compared with ω_0 ($\delta\omega \sim |\gamma| T\omega_0^{-3} \ll \omega_0$), it can exceed noticeably the frequency uncertainty Γ due to the damping (which is small to the extent of smallness of the interaction with the medium).

The analysis of the relaxation of a weakly damped ($\Gamma \ll \omega_0$) oscillator becomes most complicated in the region in which both frequency "smearing" mechanisms make contributions of the same order,

$$|\alpha| \sim 1, \qquad \alpha = \frac{3\gamma T}{8\omega_0^3 \Gamma} \tag{2.4}$$

($\delta\omega \sim |\alpha|\Gamma$). To solve the problem in the region (2.4), a special mathematic method was developed [19] and is expounded in Secs. 2.1 and 2.2 below. It makes it possible to describe in a unified manner how the initial state of the oscillator is "forgotten" both in the microscopic model (2.2) and in the Brownian-motion model [20]. It makes it also possible to analyze the time correlation functions of the oscillator and their spectral distributions in the parameter region (2.3) (see Sec. 2.3). Appendix C (see also [20]) contains the solution of the EFP equation for a nonlinear oscillator in the case $|\mathfrak{b}| \ll 1$.

In the region $|\alpha| \gg 1$, the nonlinearity of the oscillator is relatively large and in the calculation of the peak of the spectral distribution $Q(\omega)$ we can neglect the frequency uncertainty due to the damping. This allows us to use in the calculation another (simpler) method [20]. The corresponding results of the calculation of $Q(\omega)$ with the damping neglected are presented in Sec. 2.4. They cover not only the region of small $|\mathfrak{b}|$ (2.3), but also the region of arbitrary positive $\mathfrak{b}$.

Obviously, if $\mathfrak{b} > 0$ and $\Gamma = 0$, the function $Q(\omega)$ vanishes at $\omega < \omega_0$, since ω_0 is the lowest of the oscillator frequencies. The situation changes if $\omega_0^2 < 0$ (but $\gamma > 0$). As seen from Eq. (2.2), the oscillator potential has in this case (neglecting the interaction) two minima separated by a "hump." Near the "hump" the oscillator

moves very slowly. As a result, the spectral distribution acquires a tail in the region $\omega < |\omega_0|$. As shown in Sec. 2.4, the tail is exponential (at $\Gamma = 0$). This form of the distribution on the tail is common to systems with local potential maxima.

2.1 *Elimination of the Continuous-Spectrum Vibrations and Differential Equation for the Slow Variables of the Oscillator at* $|\mathfrak{b}| \ll 1$

If the conditions

$$\Gamma \ll \omega_0, \qquad |\mathfrak{b}| \ll 1 \tag{2.5}$$

are satisfied, the oscillator motion is split into "fast" with characteristic time

$$t_c = \max(\omega_0^{-1}, t_{\text{cor}}) \tag{2.6}$$

and "slow" with characteristic times $\sim\Gamma^{-1}$ and $(\delta\omega)^{-1}$. Parameter t_{cor} of Eq. (2.6) characterizes the decay time of the correlators of the medium coupled to the oscillator. In the microscopic model (2.1), in which the medium is regarded as a set of harmonic oscillators with a continuous spectrum, t_{cor} corresponds to the characteristic period of medium vibrations. In the phenomenological description, t_{cor} corresponds to the characteristic random-force correlation time in the Langevin equation (1.12) ($t_{\text{cor}} = 0$ in the δ-correlated force approximation).

The "fast" motion depends substantially on the model of the medium, on the type of the random force, and others. On the other hand, the "slow" motion is described, as shown below, by equations of quite general form which contain a small number of parameters, and it is precisely in the "slow" motion that the specific features of the relaxation of a weakly nonlinear oscillator manifest themselves.

2.1.1 Elimination of the Continuous-Spectrum Vibrations We consider first the oscillator motion using the microscopic model (2.2). The dynamics of a system with Hamiltonian (2.2) is described by the

equations of motion

$$\begin{aligned} \ddot{q} + \omega_0^2 q + \gamma q^3 &= -\sum_k \epsilon_k q_k, \\ \ddot{q}_k + \omega_k^2 q_k &= -\epsilon_k q. \end{aligned} \tag{2.7}$$

Obviously, a complete dynamic description of the system, which reduces to the solution of $N + 1$ equations for N oscillators of the medium and the singled-out oscillator, is impossible. It is possible, however, to present a statistical description of the motion of the singled-out oscillator. To this end we assume that at the initial instant of time the system of vibrations of the continuous spectrum is in a state of thermodynamic equilibrium, i.e., it is described by a Gibbs distribution, while the state of the singled-out oscillator is strictly determinate. As a result of the interaction with the thermostat, the motion of the singled-out oscillator becomes randomized in the course of time, the initial state is "forgotten," and after a sufficiently long time interval the oscillator arrives at a state of thermodynamic equilibrium with the medium and is likewise described by a Gibbs distribution function.

To carry out the statistical description of the subsystem it is convenient to eliminate the dynamic variables of the continuous-spectrum vibrations from the $N + 1$ equations of motion (2.7), and thus to reduce the problem to a single equation that contains only the coordinate of the singled-out oscillator and a certain random function [3]. To this end, we solve Eqs. (2.7) for q_k and change over thereby from the dynamic variables $q_k(t)$ to random variables, namely the initial amplitudes A_k and phases φ_k of the oscillators of the medium:

$$q_k(t) = A_k \cos(\omega_k t + \varphi_k) - \frac{\epsilon_k}{\omega_k} \int_0^t dt_1\, q(t_1) \sin \omega_k (t - t_1). \tag{2.8}$$

It follows, from the assumption that the oscillators of the medium have initially a Gibbs distribution, that the amplitude distribution is of the form

$$w(\ldots A_k^2 \ldots) = \prod_k \frac{\omega_k^2}{2T} \exp\left(-\frac{\omega_k^2 A_k^2}{2T}\right), \tag{2.9}$$

and the phases φ_k are uniformly distributed in the interval $(0, 2\pi)$.

Substituting Eq. (2.8) in the right-hand side of the first equation of (2.7), we obtain for the singled-out oscillator a nonlinear stochastic integro-differential equation

$$\ddot{q} + \omega_0^2 q + \gamma q^3 = f(t) + L[q]. \tag{2.10}$$

The function

$$f(t) = -\sum_k \epsilon_k A_k \cos(\omega_k t + \tag{2.11}$$

describes here the random force exerted on the oscillator by the medium, while the functional

$$L[q] = \int_0^t dt_1 \sum_k \frac{\epsilon_k^2}{\omega_k} q(t - t_1) \sin \omega_k t_1 \tag{2.12}$$

describes the retarded action of the oscillator "on itself" (retarded self-action) due to the interaction with the medium, i.e., it describes in fact the friction force. Equation (2.10) does not contain the dynamic variables of the medium.

2.1.2 Transition to Slowly Changing Variables The stochastic equation (2.10) is most complicated, first because of the nonlinearity, and second because of the presence of the integral term $L[q]$. It can be considerably simplified because processes with greatly differing characteristic times take place in the system, namely fast vibrational processes with a characteristic period t_c (2.6), and slow relaxation processes with characteristic times Γ^{-1} and $(\delta\omega)^{-1}$. The damping Γ of the singled-out oscillator is due here to the term $L[q]$ in Eq. (2.10), and can be seen from Eq. (2.12) to be sufficiently small in the considered case of weak interaction (small ϵ_k).

The presence of two greatly differing times permits solution of Eq. (2.10) at relatively long times $|t| \gg t_c$ by using the known averaging method of nonlinear mechanics (see, e.g., [28, 31]). The method consists of separating and investigating the slowly varying amplitude of the oscillator and the slowly varying part of its phase. It is possible then to take into account the perturbation with asymptotic accuracy, i.e., the solution is not obtained in the form of a perturbation-theory

series in the parameters $\Gamma|t|$ and $\Gamma|\alpha t|$, but is valid for arbitrary $\Gamma|t|(1 + |\alpha|)$.

To this end, we transform from the fast oscillating functions $q(t)$ and $\dot{q}(t)$ to the smooth complex functions $u_1(t)$ and $u_2(t) = u_1^*(t)$:

$$q = u_1 e^{i\omega_0 t} + u_2 e^{-i\omega_0 t}, \qquad \dot{q} = i\omega_0 (u_1 e^{i\omega_0 t} - u_2 e^{-i\omega_0 t})$$

(cf. Eq. (1.5)), and use the identity that follows from these expressions:

$$\ddot{q} + \omega_0^2 q = 2i\omega_0 \dot{u}_1 e^{i\omega_0 t} = -2i\omega_0 \dot{u}_2 e^{-i\omega_0 t}$$

Substituting this identity in Eq. (2.10), we rewrite the latter in the form

$$\dot{u}_1 = -\frac{3\gamma}{2i\omega_0} y u_1 + f_1(t) + \Lambda_1[u_1],$$

$$y = u_1 u_2 = |u_1|^2, \qquad f_1(t) = \frac{1}{2i\omega_0} e^{-i\omega_0 t} f(t), \tag{2.13}$$

$$\Lambda_1[u_1] = \frac{1}{2i\omega_0} \int_0^t dt_1 \sum_k \frac{\epsilon_k^2}{\omega_k} \sin(\omega_k t_1) \times \left[e^{-i\omega_0 t_1} u_1(t - t_1) + e^{2i\omega_0 t} e^{i\omega_0 t_1} u_1^*(t - t_1) \right]$$

(the equation for $u_2(t)$ is the complex conjugate of (2.13)).

In the spirit of the averaging method, we have discarded in Eq. (2.13) the terms

$$\propto \gamma u_2^3 e^{-4i\omega_0 t}, \qquad \propto \gamma u_1^3 e^{2i\omega_0 t}, \qquad \propto \gamma u_1 u_2^2 e^{-2i\omega_0 t},$$

which are proportional not only to the small nonlinearity parameter γ, but also to fast oscillating factors of the type $\exp(\pm 2i\omega_0 t)$. These terms manifest themselves only at short times $|t| \lesssim t_c$. In the investigated region of long times $|t| \gg t_c$ they are of no importance, (see Appendix B). It is shown in Appendix B that in the region $|t| \gg t_c$ we can neglect the retardation in the functional $\Lambda_1[u_1]$ and taking (B.4)

into account, rewrite Eq. (2.13) in the form

$$\dot{u}_1 = -\frac{3\gamma}{2i\omega_0} y u_1 + (-\Gamma \operatorname{sgn} t + iP)u_1 + f_1(t), \qquad |t| \gg t_c, \quad (2.14)$$

where the parameters Γ and P are defined by the expressions

$$\Gamma = \frac{\pi}{4} g_\epsilon(\omega_0), \qquad P = \frac{1}{2\omega_0} \text{v.p.} \int \frac{\omega^2 g_\epsilon(\omega)}{\omega^2 - \omega_0^2} d\omega,$$
$$g_\epsilon(\omega)\, d\omega = \sum_{\substack{k \\ \omega \leqslant \omega_k \leqslant \omega + d\omega}} \epsilon_k^2/\omega_k^2 . \quad (2.15)$$

Thus, replacement of the variables (1.5) standard for the averaging method has made it possible to change from the second-order integro-differential equation (2.10) to a system of first-order differential equations for the smooth functions $u_1(t)$ and $u_2(t)$.

The damping of the singled-out oscillator and the renormalization of its frequency, that are determined by the parameters Γ and P respectively, can be regarded as the result of the scattering by vibrations of the medium. It is seen from Eq. (B.3) that the characteristic duration of the scattering process is equal to the parameter t_c introduced in (2.6). The condition for the validity of Eq. (2.14)

$$\Gamma t_c \ll 1$$

means that the scattering probability is considerably smaller than the reciprocal scattering duration.

It is shown in Appendix C that the force $f_1(t)$ in Eq. (2.14) constitutes, in the scale of the "slow" time $t \gg t_c$, a δ-correlated Gaussian random process, and thus Eq. (2.14) is a Langevin equation and $u_1(t)$ is a Markov process.

2.2 *Statistical Properties of Random Motion of an Oscillator and Averaging Over the Initial State of the Medium*

2.2.1 Stochastic Integral Equation Although Eq. (2.14) is much simpler than the initial equation (2.10), it is still complicated enough,

since it contains in the right-hand side the nonlinear term

$$-\frac{3\gamma}{2i\omega_0} y u_1 \equiv -\frac{3\gamma}{2i\omega_0} u_1^2 u_1^* .$$

To solve this equation we use a certain artificial device [19]. Namely, we regard formally the term with yu_1 as linear, assuming y to be a known function, and write u_1 in the usual form of a solution of a first-order differential equation:

$$u_1(t)e^{-iPt} = u_1(0)\exp\big[F(t) - \Gamma|t|\big] + \exp\big[F(t)\big]\xi(t), \quad (2.16)$$

where

$$\xi(t) = \sum_k \big(\xi_{k1}(t)e^{i\varphi_k} + \xi_{k2}(t)e^{-i\varphi_k}\big),$$
$$\xi_{k1}(t) = i\frac{\epsilon_k A_k}{4\omega_0} e^{-\Gamma|t|} \int_0^t dt_1 \exp\big[i(\omega_k - \tilde{\omega}_0)t_1 + \Gamma|t_1| - F(t_1)\big], \quad (2.17)$$
$$\xi_{k2}(t) = i\frac{\epsilon_k A_k}{4\omega_0} e^{-\Gamma|t|} \int_0^t dt_1 \exp\big[-i(\omega_k - \tilde{\omega}_0)t_1 + \Gamma|t_1| - F(t_1)\big],$$
$$F(t) = \frac{3i\gamma}{2\omega_0} \int_0^t dt_1\, y(t_1), \qquad \tilde{\omega}_0 = \omega_0 + P.$$

Expression (2.16) contains $y = |u_1|^2$ in the right-hand side and is, of course, not a solution of the differential equation (2.14), but merely an integral equation obtained from it. At first glance, the latter is even more complicated than the initial equation (2.14). Actually, however, it is more convenient for averaging in the asymptotic time region $|t| \gg t_c$.

2.2.2 Probability Distribution of the Random Process $\xi(t)$

a. Distribution of Auxiliary Random Process $\tilde{\xi}(t)$ Expression (2.16) for $u_1(t)$ is convenient because the process $\xi(t)$ for long times is asymptotically approximately Gaussian. It is easy to verify the distribution of $\xi(t)$ would be Gaussian in the statistical limit $N \to \infty$ if it were possible to put $F(t) = 0$ in Eq. (2.17). Actually the quantities

$\tilde{\xi}_{k\lambda}(t)$ ($\lambda = 1,2$; the expressions for $\tilde{\xi}_{k\lambda}(t)$ and for $\tilde{\xi}(t)$ are obtained by putting $F(t_1) = 0$ in Eqs. (2.17) for $\xi_{k\lambda}(t)$ and $\xi(t)$) do not depend on the phases φ_k. Since φ_k with different k are independent and their distribution is uniform from 0 to 2π, only the mean values of products containing an even number of factors of the type $\tilde{\xi}(t)$ and $\tilde{\xi}^*(t)$ differ from zero. The parameters ϵ_k of the interaction of the oscillator with the vibrations of the medium are proportional to $N^{-1/2}$, and consequently $\tilde{\xi}_{k\lambda}(t)$ are also proportional to $N^{-1/2}$. Therefore sums of the type

$$\sum_k \tilde{\xi}_{k1}(t)\tilde{\xi}^*_{k1}(t'), \qquad \sum_k \tilde{\xi}_{k2}(t)\tilde{\xi}^*_{k2}(t'), \qquad \sum_k \tilde{\xi}_{k1}(t)\tilde{\xi}_{k2}(t') \quad (2.18)$$

are finite as $N \to \infty$ (these sums result from averaging the products $\xi(t)\xi^*(t')$ and $\tilde{\xi}(t)\tilde{\xi}(t')$ over φ_k), while the sums containing a larger number of factors $\tilde{\xi}_{k\lambda}$ with identical k vanish in the limit $N \to \infty$. Thus, as a result of averaging over φ_k the mean value of the product of the type $\tilde{\xi}(t_1) \ldots \xi^*(t_n)$ breaks up into products of sums of the type (2.18), i.e., into a sum of products of different pair correlators. This property of the mean values is obviously preserved also under a subsequent averaging over the amplitudes A_k, since the averaging for different k is carried out independently. The separation of the average product into a sum of products of pair correlators is a known property of Gaussian random processes [32], i.e., the distribution of $\tilde{\xi}(t)$ is Gaussian.

b. Gaussian Distribution of the Random Process $\xi(t)$ at Long Times The function $\xi(t)$ (2.17), which determines the dynamics of the nonlinear oscillator, is actually more complicated than $\tilde{\xi}(t)$, inasmuch as the definition (2.17) of $\xi_{k\lambda}(t)$ contains the quantity $F(t)$, which itself is a functional. As a result, the quantities $\xi_{k\lambda}$ and the phases φ_k are strictly speaking not statistically independent, and the distribution of $\xi(t)$ is not Gaussian. However, if we confine ourselves to the region of sufficiently long times $|t| \gg t_c$, the presence of the factor $\exp[-F(t_1)]$ in the definition (2.17) for $\xi_{k\lambda}$ does not influence any of the calculated mean values asymptotically, neglecting the small corrections $\sim t_c/|t|$ or $\sim \epsilon^2$ (ϵ denotes a small parameter contained in all ϵ_k and determining the weakness of the interaction with the medium). Therefore, neglecting the indicated small corrections it is possible to regard the distribution of $\xi(t)$ as Gaussian.

Indeed, consider first the paired correlation function

$$\langle \xi(t)\xi^*(t') \rangle_m = \frac{1}{16\omega_0^2} \Big\langle \sum_{k,k'} \sum_{\lambda,\lambda'=1,2} \epsilon_k \epsilon_{k'} A_k A_{k'} \exp\big[-\Gamma(|t| + |t'|)\big] \times \int_0^t dt_1 \int_0^{t'} dt_1' \exp\big[i(\omega_{k\lambda} - \tilde{\omega}_0)t_1 - i(\omega_{k'\lambda'} - \tilde{\omega}_0)t_1' + i(\varphi_{k\lambda} - \varphi_{k'\lambda'}) + \Gamma|t_1| + \Gamma|t_1'| - F(t_1) + F(t_1')\big] \Big\rangle_m . \quad (2.19)$$

Here

$$\omega_{k1} = \omega_k , \qquad \omega_{k2} = -\omega_k , \qquad \varphi_{k1} = \varphi_k , \qquad \varphi_{k2} = -\varphi_k$$

and $\langle \ldots \rangle_m$ denotes averaging over the random initial amplitudes and phases A_k and φ_k of the medium vibrations. The main contribution to Eq. (2.19) is made by terms with $\lambda = \lambda'$ and $k = k'$. Neglecting the small corrections $\sim \epsilon^2$ (due to the presence of $\exp[F(t_1') - F(t_1)]$ in these terms), we can replace $\langle A_k^2 \rangle$ by $2T\omega_k^{-2}$ and transform, in accordance with Eq. (2.15), from summation over k with weight ϵ_k^2/ω_k^2 to integration with respect to ω with weight $g_\epsilon(\omega)$. It is important that the integral with respect to ω determines the δ-like function $\delta_1(t_1 - t_1')$:

$$\int_0^\infty d\omega \; g_\epsilon(\omega) \exp\big[i(\omega - \tilde{\omega}_0)(t_1 - t_1') \big] = 2\pi g_\epsilon(\tilde{\omega}_0)\delta_1(t_1 - t_1'). \quad (2.20)$$

At $|t_1 - t_1'| \gg t_c$, owing to the fast oscillating factor under the integral sign, this function is $\sim t_c|t_1 - t_1'|^{-1}$, while at $|t_1 - t_1'| \sim t_c$ it has a peak, and the integral of $\delta_1(t_1 - t_1')$ with respect to t_1 is equal to unity. Since the function $\delta_1(t_1 - t_1')$ enters in (2.19) under the integral over a long time interval (compared with the width $\sim t_c$ of the function δ_1), it can be replaced, neglecting the asymptotically small corrections $\sim t_c/|t|$,

$t_c/|t'|$, or $\sim\epsilon^2$, by simply the usual δ function $\delta(t_1 - t'_1)$. The random function $\exp[F(t'_1) - F(t_1)]$ therefore cancels in the integrand, and upon integration (with account taken of the relation between Γ and $g_\epsilon(\omega_0)$) we obtain

$$\langle \xi(t)\xi^*(t')\rangle = 2A(t,t'),$$

$$A(t,t') = \frac{T}{4\omega_0^2}\exp\big[-\Gamma(|t| + |t'|)\big]\big[\exp(2\Gamma|t_{\min}|) - 1\big], \quad (2.21)$$

$$|t_{\min}| = \min(|t|,|t'|) \qquad (\operatorname{sgn} t = \operatorname{sgn} t').$$

The discarded off-diagonal terms with $k \neq k'$ or $\lambda \neq \lambda'$ would be equal to zero upon averaging over φ_k if F were zero. Since according to Eq. (2.17) the expansion of $F(t)$ contains random quantities of the form $\exp[i(\varphi_{k_1\lambda_1} - \varphi_{k'_1\lambda'_1})]\xi_{k_1\lambda_1}(t_2)\xi^*_{k'_1\lambda'_1}(t'_2)$, the terms with $k \neq k'$ in Eq. (2.19) lead, in the considered case $F \neq 0$, to a nonzero result (e.g., if $k = k'_1$ and $k' = k_1$). However, integration with respect to the frequencies ω_k and $\omega_{k'}$, just as above, lead to the appearance of the δ functions (more accurately, the δ-like functions) $\delta(t_1 - t'_2)\delta(t'_1 - t_2)$ in the integrand, with t_2 or t'_2 always smaller than one of the variables t_1 or t'_1. Therefore the terms with $k \neq k'$ or $\lambda \neq \lambda'$ lead to small corrections, and they can indeed be neglected in the approximation considered, thereby justifying Eq. (2.21). It is easy to verify that in this approximation

$$\langle \xi(t)\xi(t')\rangle_m = \langle \xi^*(t)\xi^*(t')\rangle_m = 0. \quad (2.22)$$

It can be shown in the same manner that when the small corrections are neglected we can leave out the function $F(t)$ when calculating the mean value $\langle \xi(t_1)\ldots\xi^*(t_4)\rangle$, which determines the fourth moment of the distribution of $\xi(t)$, as well as the mean values of a larger number of factors that determine the higher moments. Indeed, if certain $k_i = k_j$ coincide in a $2p$-fold sum over k, the summation over these k_i, just as in (2.19), leads to the appearance of the δ function $\delta(t_i - t_j)$ and to cancellation of the factors $\exp[F(t_j) - F(t_i)]$, so that this part of the expression reduces to a $(2p-2)$-fold sum and can be treated by the method of mathematical induction. If, however, all the k_i are different then, as can be seen from the integral form (2.17) for $F(t)$, the largest of the arguments of the functions $\xi(t_i)$

always exceeds all the other times that enter in the arguments of the δ functions, i.e., in the calculation of the higher moments the presence of the functions $F(t)$ leads only to negligibly small corrections.

Thus, the presence of the functions $F(t)$ in Eq. (2.17) does not influence asymptotically the moments of the distribution of $\xi(t)$, and if the corrections $\sim\epsilon^2$ and $t_c|t_{\min}|^{-1}$ are neglected the distribution of $\xi(t)$, just as the distribution of $\tilde{\xi}(t)$, is Gaussian with correlators (2.21) and (2.22). This means that the probability density of the values of $\xi(t) = \xi'(t) + i\xi''(t)$, taken at different arbitrary instants of time $t_1, t_2, \ldots, t_m$, take the form

$$w(\ldots,\xi'(t_n),\xi''(t_n),\ldots) = (2\pi)^{-m}|A_{nn'}|^{-1} \times \exp\left\{-\frac{1}{2}\sum_{n,n'=1}^{m} A^{-1}_{nn'}\big[\xi'(t_n)\xi'(t_{n'}) + \xi''(t_n)\xi''(t_{n'})\big]\right\} \quad (2.23)$$

Here $A_{nn'} \equiv A(t_n,t_{n'})$ is defined in (2.21), $|A_{nn'}|$ is the determinant of the matrix $\|A_{nn'}\|$, and $A^{-1}_{nn'}$ are the matrix elements of the inverse matrix $\hat{A}^{-1}$.

c. Gaussian Distribution of the Random Process $\xi(t)$ in the Phenomenological Description of Brownian Motion of an Oscillator The statistical properties of the motion of the singled-out oscillator, which were obtained in the microscopic model, are valid also in the phenomenological model of Brownian motion of the oscillator (1.12). The Langevin equation (1.12) differs from Eq. (2.10) of the microscopic theory in that the operator $L[q]$, which describes the retarded action of the oscillator "on itself," is replaced by the friction force $-2\Gamma\dot{q}$, while the random force $f(t)$ is regarded as a δ-correlated Gaussian process. The transformation to slow variables in Eq. (1.12), which is perfectly analogous to the transformation of Eq. (2.10), leads to Eqs. (2.16), where $P = 0$ and

$$\xi(t) = -\frac{i}{2\omega_0}\int_0^t dt_1 \exp\big[-\Gamma(t - t_1) - F(t_1)\big]f(t_1)e^{-i\omega_0 t_1} \qquad (t > 0). \quad (2.24)$$

If it were possible to put $F(t_1) = 0$ in (2.24), the Gaussian form of

the distribution of the random function $\xi(t)$ would be the direct consequence of the Gaussian distribution of the force $f(t)$. However, since $f(t)$ is δ-correlated, the presence of the factor $\exp[-F(t_1)]$ in the integrand at $t \gg \omega_0^{-1}$ produces practically no changes in any of the moments of the distribution of $\xi(t)$, and consequently in the distribution itself. Indeed, since

$$\langle f(t)\exp\big[-F(t_1)\big]\rangle_m = 0 \qquad \text{at} \quad t_1 \leqslant t, \tag{2.25}$$

then, e.g., in the double integral with respect to $dt_1 dt_1'$, which determines, with allowance for Eq. (2.24), the pair correlator $\langle\xi(t)\xi^*(t')\rangle_m$, there appears the δ function $\delta(t_1 - t_1')$, so that the factors $\exp[-F(t_1)]$ and $\exp[F(t_1')]$ cancel out. The result is equation (2.21) for $\langle\xi(t)\xi^*(t')\rangle_m$ at $t > 0$ and $t' > 0$. The factors $\exp[\pm F(t_n)]$ cancel out similarly also in the expressions for the higher-order correlators.

In the case of the correlators $\langle\xi(t)\xi(t')\rangle_m$ and $\langle\xi^*(t)\xi^*(t')\rangle_m$ there remains, after averaging, the fast oscillating factor $\exp(\mp 2i\omega_0 \cdot t_1)$ under the integral sign. At $|t - t'| \gg \omega_0^{-1}$ these correlators are therefore proportional to the small parameter Γ/ω_0 and they can be set equal to zero in the assumed approximation, just as in Eq. (2.22).

In Appendix C we analyze the EFP equation that describes Brownian motion of an oscillator in the time scale $t \gg t_c$. This equation is valid both in the microscopic model (2.1), (2.2), and (2.14), and in the phenomenological model (1.12).

2.2.3 Solution of Stochastic Integral Equation To investigate the relaxation of the singled-out oscillator and to determine the time correlation function of the coordinates it is necessary to obtain the oscillator complex amplitude $u_1(t)$ (2.16) averaged over the vibrations of the medium (for a given initial value $u_1(0)$). This averaging reduces to averaging Eq. (2.16) over the random process $\xi(t) = \xi'(t) + i\xi''(t)$. It is convenient to carry out the latter by the path-integral method. From Eqs. (2.21)–(2.23) it can be seen that the Gaussian random processes $\xi'(t)$ and $\xi''(t)$ are statistically independent. In accordance with Eq. (2.23), the functional $\mathscr{P}[\xi', \xi'']$ which describes the probability distribution of these random functions takes the form [32]

$$\mathscr{P}[\xi', \xi''] = \exp\Big\{-\tfrac{1}{2}\int\int dt_1\, dt_2 \mathscr{A}(t_1, t_2)\big[\xi'(t_1)\xi'(t_2) + \xi''(t_1)\xi''(t_2)\big]\Big\}. \tag{2.26}$$

Here $\mathscr{A}(t, t')$ is the analog of the matrix $A_{nn'}^{-1}$ (see Eq. (2.23)) in the continual limit and is defined by the formula

$$\int dt_3 A(t_1, t_3)\mathscr{A}(t_3, t_2) = \delta(t_1 - t_2). \tag{2.27}$$

Since $A(t_1, t_2) = A(t_2, t_1)$, it follows that $\mathscr{A}(t_1, t_2) = \mathscr{A}(t_2, t_1)$. Although the distribution $\xi(t)$ is Gaussian, it is not δ-correlated, making the calculations difficult.

It is seen from Eq. (2.16) that to investigate the oscillator motion averaged over the random forces it is necessary to obtain the mean values of the functionals $\exp F(t)$ and $\xi(t) \cdot \exp F(t)$. This appears to be possible because the functional $F(t)$ is quadratic in ξ. We divide $F(t)$ into a regular part $F^{(0)}(t)$ that is independent of $\xi(t)$, and a random part $F^{(1)}(t)$:

$$F(t) = F^{(0)}(t) + F^{(1)}(t), \qquad F^{(0)}(t) = \frac{3i}{4\omega_0}\frac{\gamma}{\Gamma} r(0)(1 - e^{-2\Gamma t}), \tag{2.28}$$

$$F^{(1)}(t) = \int_0^t dt_1 \left\{ \mu(t_1)\mathrm{Re}\big[\exp(i\varphi(0))\xi^*(t_1)\big] + \nu|\xi(t_1)|^2\right\} \qquad (t \geqslant 0), \tag{2.29}$$

$$\mu(t) = 2\nu r^{1/2}(0)e^{-\Gamma t}, \qquad \nu = \frac{3i\gamma}{2\omega_0}.$$

Here $(1/2)r^{1/2}(0)$ and $\varphi(0)$ are the initial values of the amplitude and phase of the oscillator:

$$u_1(0) = r^{1/2}(0)\exp(i\varphi(0)). \tag{2.30}$$

We denote the result of averaging $\exp F^{(1)}(t)$ over $\xi(t)$ as $\exp \mathscr{F}(t)$. We get then, with allowance for Eq. (2.28),

$$\begin{aligned} \langle \exp F(t)\rangle_m &= \exp\big[F^{(0)}(t) + \mathscr{F}(t)\big], \\ \dot{\mathscr{F}}(t) &= \langle \dot{F}^{(1)}(t)\exp F^{(1)}(t)\rangle_m \big(\langle \exp F^{(1)}(t)\rangle_m\big)^{-1} \\ &\equiv \mu(t)\mathrm{Re}\,\overline{\xi^*(t)\exp(i\varphi(0))} + \nu\overline{|\xi(t)|^2}, \end{aligned} \tag{2.31}$$

where the bar denotes averaging over $\xi(t)$ with the probability functional

$$\tilde{\mathscr{P}}[\xi',\xi''] = \mathscr{P}[\xi',\xi'']\exp F^{(1)}(t).$$

According to Eqs. (2.26) and (2.28), this functional is Gaussian and takes the form

$$\tilde{\mathscr{P}}[\xi',\xi''] = \exp\Big\{-\frac{1}{2}\int\int dt_1\,dt_2\mathscr{B}(t_1,t_2)\big[\xi(t_1)-\mathscr{E}(t_1)\exp(i\varphi(0))\big] \times\big[\xi^*(t_2)-\mathscr{E}(t_2)\exp(-i\varphi(0))\big]\Big\}. \tag{2.32}$$

The functions $\mathscr{B}(t_1,t_2)$ and $\mathscr{E}(t_1)$ are defined here by the formulas

$$\mathscr{B}(t_1,t_2) = \mathscr{A}(t_1,t_2) - 2\nu\delta(t_1-t_2)\Theta(t-t_1), \tag{2.33}$$

$$\mathscr{E}(t_1) = \Theta(t-t_1)\int_0^t B(t_1,t_2)\mu(t_2)\,dt_2,$$

$$\Theta(x) = 1, \quad x>0$$

$$= 0, \quad x<0$$

$$\int B(t_1,t_3)\mathscr{B}(t_3,t_2)\,dt_3 = \delta(t_1-t_2). \tag{2.34}$$

From Eqs. (2.31)–(2.34) follows directly that

$$\dot{\mathscr{F}}(t) = \mu(t)\mathscr{E}(t) + 2\nu B(t,t) + \nu\mathscr{E}^2(t). \tag{2.35}$$

The functional (2.32) makes it possible to average just as easily the second term in the right-hand side of Eq. (2.16)

$$\langle\xi(t)\exp F(t)\rangle_m = \mathscr{E}(t)\exp(i\varphi(0))\exp\big[F^{(0)}(t)+\mathscr{F}(t)\big]. \tag{2.36}$$

The calculation of the mean values reduces thus to calculation of the functions $B(t,t_1)$ and $\mathscr{E}(t)$. To calculate $B(t,t_1)$ we substitute in

Eq. (2.34) the expression (2.33) for $\mathscr{B}(t_3,t_2)$:

$$\int X(t_3)\mathscr{A}(t_3,t_2)\,dt_3 - 2\nu X(t_2)\Theta(t-t_2) = \delta(t-t_2), \qquad X(t') \equiv B(t,t'). \tag{2.37}$$

Multiplying Eq. (2.37) by $A(t',t_2)$ and integrating with respect to t_2 we obtain for $X(t)$ the more convenient integral equation

$$X(t') - 2\nu\int_0^t dt_1 X(t_1)A(t',t_1) = A(t,t'). \tag{2.38}$$

Taking into account the expression (2.21) for $A(t',t'')$, we rewrite Eq. (2.38) in the form

$$X(t') - \frac{T}{\omega_0^2}\nu e^{-\Gamma t'}\int_0^{t'} dt_1\,\mathrm{sh}(\Gamma t_1)X(t_1) - \frac{T}{\omega_0^2}\nu\,\mathrm{sh}\,\Gamma t'\int_{t'}^t dt_1 e^{-\Gamma t_1}X(t_1) = \frac{T}{2\omega_0^2}e^{-\Gamma t}\mathrm{sh}\,\Gamma t'. \tag{2.39}$$

Differentiating this equation twice with respect to t' and subtracting the initial equation multiplied by Γ^2, we can reduce the integral equation (2.39) to the simple differential equation

$$\frac{d^2X(t')}{dt'^2} - a^2X(t') = 0,$$

where the complex parameter a is equal to

$$a = \left(\Gamma^2 - \frac{T}{\omega_0^2}\Gamma\nu\right)^{1/2} = \Gamma(1-4i\alpha)^{1/2} \qquad (\mathrm{Re}\,a>0) \tag{2.40}$$

(α is defined in (2.4)).

The constants of the solution of this equation depend on t. They can be determined from the requirement that the solution satisfy the initial equation (2.39). We then obtain for $X(t')$

$$X(t') = \frac{\Gamma T}{\omega_0^2}\big[a\,\mathrm{ch}\,at + \Gamma\,\mathrm{sh}\,at\big]^{-1}\mathrm{sh}\,at' \qquad (t'>0). \tag{2.41}$$

Substituting the obtained values of $X(t')$ for $B(t,t')$ in Eqs. (2.33) and (2.35) we can determine $\mathscr{E}(t)$ and $\mathscr{F}(t)$, and then $\mathscr{F}(t)$. We ultimately obtain, with allowance for Eqs. (2.16), (2.31), and (2.36),

$$\langle u_1(t)\rangle_m = u_1(0)a^2\exp\left[(\Gamma + iP)t\right](a\,\mathrm{ch}\,at + \Gamma\,\mathrm{sh}\,at)^{-2}$$

$$\times \exp\left(\frac{3i\gamma|u_1(0)|^2}{2\omega_0}\,\frac{\mathrm{sh}\,at}{a\,\mathrm{ch}\,at + \Gamma\,\mathrm{sh}\,at}\right). \qquad (2.42)$$

Thus, since the distribution of the random process $\xi(t)$ in the nonlinear integro-differential equation (2.16) is Gaussian (albeit not δ-correlated), and the functional $F(t)$ is quadratic in $\xi(t)$, the averaging in (2.16) can be carried out in explicit form.

2.3 *Analysis of Relaxation and Fluctuations of the Coordinates of a Weakly Nonlinear Duffing Oscillator*

2.3.1 Relaxation of Singled-out Oscillator Expression (2.42) for $\langle u_1(t)\rangle_m$ describes the relaxation of the nonlinear oscillator, i.e., its "forgetting" the initial value of the complex amplitude $u_1(0)$. This expression is valid for an arbitrary ratio of the oscillator frequency spread due to the internal anharmonicity and the frequency uncertainty due to the damping.

As seen from Eq. (2.42), in the case of a linear oscillator ($\gamma = 0$) the oscillation amplitude $2\langle u_1(t)\rangle_m$ decays exponentially with time, and there is no relaxation of the phase for the considered model of the interaction with the medium (linear in q). The oscillator nonlinearity causes the phase to have a complicated time dependence, and the relaxation of $u_1(t)$ becomes essentially non-exponential. Only at very long times does $\langle u_1(t)\rangle_m$ relax exponentially with time:

$$\langle u_1(t)\rangle_m = u_1(0)\,\frac{4a^2}{(a+\Gamma)^2}\exp\left[\frac{3i\gamma|u_1(0)|^2}{2\omega_0(a+\Gamma)}\right]\exp\left[-(2a - \Gamma - iP)t\right], \qquad (2.43)$$

$$t\,\mathrm{Re}\,a \gg 1.$$

We note that although the nonlinearity of the oscillator does not by itself lead to damping, the decrement in Eq. (2.43) does depend on γ.

It follows from Eq. (2.42) that as a result of the nonlinearity the initial value of the oscillator amplitude $4|u_1(0)|^2$ influences explicitly the time dependence of the relaxation. Since the parameter a in Eq. (2.42) is connected according to Eq. (2.40) with the intensity of the fluctuations in the medium ($\propto T$), the latter also influence the relaxation of the oscillator. The deviation of the relaxation from exponential sets in already in first order in the parameters $\gamma|u_1(0)|^2\omega_0^{-1}\Gamma^{-1}$ and α, which characterize the indicated effects:

$$\langle u_1(t)\rangle_m = u_1(0)\exp\left[-\Gamma t + i(4\alpha\Gamma + P)t\right]$$

$$\times\left[1 - 2i\left(\alpha - \frac{3\gamma|u_1(0)|^2}{8\omega_0\Gamma}\right)(1 - e^{-2\Gamma t})\right], \qquad (2.44)$$

$$|\alpha|, \frac{|\gamma||u_1(0)|^2}{\omega_0\Gamma} \ll 1.$$

It is shown in Appendix C that although the nonlinearity of the Duffing oscillator influences strongly the relaxation of the coordinates and of the momentum (i.e., $\langle u_1(t)\rangle$), at $|b| \ll 1$ the energy relaxation (i.e., the damping of $\langle|u_1(t)|^2\rangle_m$) is exponential, just as in the case of the harmonic oscillator.

2.3.2 Time Correlation Function and Its Spectral Distribution Expression (2.42) makes it easy to determine the time correlation functions of the coordinates and momenta of the oscillator. An oscillator that interacts weakly with a medium has a Gibbs stationary distribution. Upon satisfaction of the condition (2.3) that the nonlinearity be relatively small, its influence on the stationary distribution can be neglected. Neglecting the small nonlinearity and the weak interaction with the medium, the energy of the singled-out oscillator is

$$E = 2\omega_0^2|u_1|^2$$

(cf. Eqs. (1.5) and (2.2)), so that the stationary distribution of the quantities u_1' and u_1'' ($u_1 = u_1' + iu_1''$) is Gaussian

$$w_{\mathrm{st}}(u_1) = \frac{2\omega_0^2}{\pi T}\exp\left[-\frac{2\omega_0^2}{T}(u_1'^2 + u_1''^2)\right]. \qquad (2.45)$$

Calculation of the time correlation function of the coordinates of the oscillator $Q(t)$ (see Eq. (1.7)) reduces to averaging of expression (2.42) multiplied by $\exp(i\omega_0 t)q(0)$ over $u_1'(0)$ and $u_1''(0)$ with weight (2.45). As a result we get [19]

$$Q(t) = \langle q(t)q(0)\rangle = \frac{T}{\omega_0^2}\,\mathrm{Re}\big[\exp(i\tilde{\omega}_0|t|)\tilde{Q}(|t|)\big], \qquad \tilde{\omega}_0 = \omega_0 + P, \tag{2.46}$$

$$\tilde{Q}(t) = e^{\Gamma t}\psi^{-2}(t), \qquad \psi(t) = \mathrm{ch}\,at + \frac{\Gamma(1-2i\alpha)}{a}\,\mathrm{sh}\,at. \tag{2.47}$$

It is obvious from Eq. (2.47) that the correlation function $Q(t)$ is expressed in terms of elementary functions. The dependence of its "amplitude" $\tilde{Q}(t)$ on the dimensionless time Γt is determined by the single parameter α. As $\alpha \to 0$, $Q(t)$ decays like $\exp(-\Gamma|t|)$, and at $|\alpha| \gtrsim 1$ the decay of $Q(t)$ is substantially non-exponential.

Equations (2.46), (2.47), and (1.8) determine the spectral distribution $Q(\omega)$ in the form of an integral of an elementary function. Inasmuch as in the derivation of Eqs. (2.42) and (2.47) we used asymptotic methods in which we took into account in succession terms containing $\epsilon^2|t|$, but discarded corrections of order $\sim\epsilon^2$ which were not proportional to the long time t, these expressions are valid in the region of greatest interest, that of long times, and no longer hold at $|t| \lesssim t_c$ (where $Q(t)$ can be easily determined by ordinary perturbation theory). Expressions (1.8), (2.46), and (2.47) describe correspondingly $Q(\omega)$ at frequencies $|\omega - \tilde{\omega}_0| \ll t_c^{-1}$, i.e., in the region of the peak of the spectral distribution, but not on the far wings of the distribution $|\omega - \tilde{\omega}_0| \sim t_c^{-1}$ (where $Q(\omega)$ is determined with the aid of the usual (not asymptotic) perturbation theory).

$Q(\omega)$ for a nonlinear oscillator is described at finite α by a non-Lorentzian asymmetric curve. At $|\alpha| \ll 1$ the deviations of $Q(\omega)$ from a Lorentzian appear in the second order in α, while asymmetry appears in the third order; all that appears in first order in α is a shift of the curve by $4\alpha\Gamma$ (see Eq. (2.44)). With increasing α, the deviation of $Q(\omega)$ from a Lorentzian becomes more and more strongly pronounced. The $Q(\omega)$ dependence for each given α can be easily obtained by numerical integration in accordance with Eqs. (2.46),

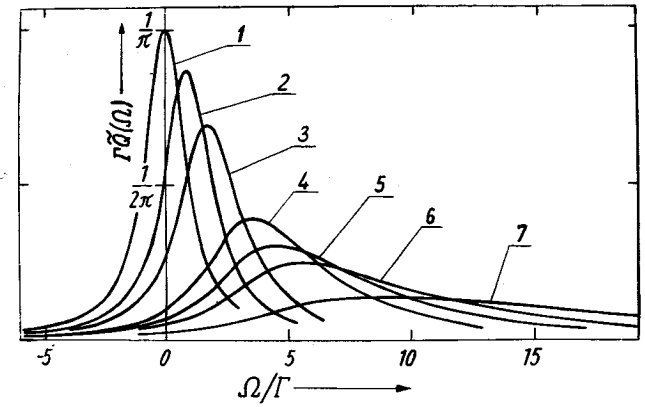

Figure 4 Spectral distribution $\tilde{Q}(\Omega)$ at different values of the dimensionless parameter α. Curves 1–7 correspond to the values $\alpha = 0$, 0.25, 0.5, 1.25, 1.75, 2.25, and 4.

(2.47), and (1.8). The results of such numerical calculations can be conveniently represented for the quantity

$$\tilde{Q}(\Omega) = \frac{2\omega_0^2}{T}\,Q(\omega) = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty \tilde{Q}^*(t)\exp(i\Omega t)\,dt, \qquad \Omega = \omega - \tilde{\omega}_0, \quad |\Omega| \ll \omega_0, \tag{2.48}$$

regarded as a function of the frequency detuning Ω. These results are given in Fig. 4, where plots of $\tilde{Q}(\Omega)$ are shown for different α. It is seen from this figure that with increasing α a gradual increase takes place in the width of the distribution (measured in units of Γ), its height is decreased (the total area of the distribution does not depend on α or on the temperature), its maximum shifts (towards lower frequencies at $\gamma < 0$), and the distribution becomes more and more asymmetrical. The dependence of the position of the maximum of the distribution Ω_m and of its integral width $\delta\omega_i = \tilde{Q}^{-1}(\Omega_m)$ on α are shown in Fig. 5. At $\alpha = 0$, the maximum of the Lorentz distribution is located at $\omega = \tilde{\omega}_0$ ($\Omega_m = 0$) and the integral width is equal to $\pi\Gamma$, while in the limit of large $|\alpha|$ we have

$$\Omega_m\,\Gamma^{-1} \approx 2\alpha + 0.17, \qquad \delta\omega_i\,\Gamma^{-1} \approx 2e|\alpha|(1 + 0.044|\alpha|^{-1}) \approx 5.44|\alpha|.$$

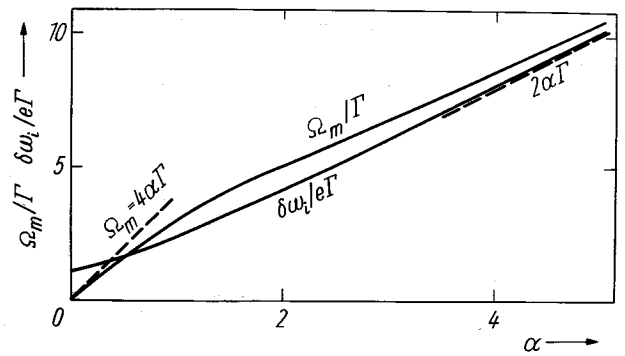

Figure 5 Dependences of the position of the maximum Ω_m and of the integral width $\delta\omega_i$ of the spectral distribution $\tilde{Q}(\Omega)$ on the parameter α. The dashed lines show the asymptotes for small and large α.

In the limiting case of strong nonlinearity, when $|\alpha| \gg 1$, we have according to Eq. (2.40) $a = 2i\sqrt{i}\,\alpha^{1/2}\Gamma$. The spectral distribution $\tilde{Q}(\Omega)$ in the region of the peak is described in this case by the strongly asymmetric function

$$\tilde{Q}(\Omega) = \frac{|\Omega|}{\Omega_0^2}\exp\left(-\frac{\Omega}{\Omega_0}\right)\Theta\left(\frac{\Omega}{\Omega_0}\right) \quad \text{at} \quad |\Omega| \gg \Gamma|\alpha|^{1/2},$$

$$\tilde{Q}(\Omega) = M|\alpha|^{-1/2} \quad \text{at} \quad \Omega = 0, \tag{2.49}$$

$$\Omega_0 = 2\alpha\Gamma = 3\gamma T/4\omega_0^3, \qquad M \sim 1,$$

where $\Theta(x)$ is the Heaviside function defined in Eq. (2.33).

More detailed analytic expressions for $\tilde{Q}(\Omega)$ at small and large α are given in [19].

2.4 *Spectral Distribution of the Time Correlation Function of a Strongly Nonlinear Duffing Oscillator*

The method used in Sec. 2.2 above is based essentially on the assumption (2.3), which means that the frequency straggling $\delta\omega$ connected with the nonlinearity of the weakly damped oscillator is small compared with the frequency ω_0, but is comparable with the

damping Γ. In the case of a strongly nonlinear oscillator, the relation between $\delta\omega$ and ω_0 can be arbitrary, and this method cannot be used. On the other hand, if the damping is not very strong, the condition

$$\Gamma \ll \delta\omega$$

i.e., $\alpha \gg 1$, is satisfied for a strongly nonlinear oscillator. In this case the broadening of the spectral distribution $Q(\omega)$ of the time correlation function of the coordinates is due mainly to the frequency straggling $\delta\omega$ (which is due in turn to the dependence of the oscillation frequency on the amplitude (or energy) of the oscillator), while the damping Γ leads only to small corrections. They can be neglected in the calculation of the peak of $Q(\omega)$ in the zeroth approximation (in the case of $|b| \ll 1$ this can be seen, e.g., from Eq. (2.49) by recognizing that $\alpha\Gamma$ is independent of Γ). It follows therefore, in particular, that the peak of $Q(\omega)$ is formed within a time $\sim(\delta\omega)^{-1}$ at $\delta\omega \gg \Gamma$.

The stationary distribution of the oscillator in energy, neglecting the interaction with the medium, is described by the expression

$$w_{st}(E) = Z^{-1}\exp\left(-\frac{E}{T}\right), \qquad Z = \int dq\,dp\exp\left(-\frac{E}{T}\right),$$
$$E = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 q^2 + \frac{1}{y}\gamma q^y. \tag{2.50}$$

Since the oscillator motion within a time $\sim(\delta\omega)^{-1}$ is quasi-conservative in the case $\Gamma \ll \delta\omega$, the time correlation function and its spectral distribution near the maximum can be calculated by first solving the problem of free motion of a nonlinear oscillator with a given energy, and then averaging over the energy (and phase) with weight (2.50).

2.4.1 Motion of Isolated Oscillator In the case $\gamma > 0$ (this condition is assumed satisfied below), the motion of the oscillator is finite and periodic. At $\omega_0^2 > 0$, the potential energy has one minimum, and at $\omega_0^2 < 0$, as shown in Fig. 6, the potential has two minima separated by a barrier. Near the peak, the barrier is parabolic (the results presented below for the spectral distribution pertaining to the case $\omega_0^2 < 0$ were obtained jointly with S. M. Soskin).

At positive energy E, both at $\omega_0^2 > 0$ and at $\omega_0^2 < 0$, the region of

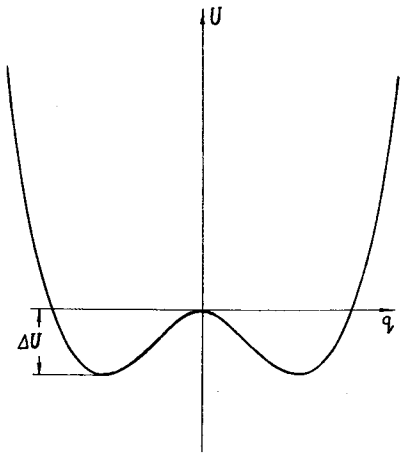

Figure 6 Duffing-oscillator potential $U = \frac{1}{2}\omega_0^2 q^2 + \frac{1}{4}\gamma q^4$ at $\omega_0^2 < 0$. The height of the potential barrier between the minima is $\Delta U = |E_m| = \frac{1}{4}\omega_0^4/\gamma$.

the oscillator motion is singly connected and the equation of motion

$$\ddot{q} + \omega_0^2 q + \gamma q^3 = 0 \qquad (\gamma \geqslant 0) \tag{2.51}$$

has the solution

$$q(t) = \frac{\sqrt{2}\,|\omega_0|}{\sqrt{\gamma}}\, b_E k \operatorname{cn}\left(\frac{2K}{\pi}\varphi\right), \qquad \varphi = \frac{\pi}{2K} b_E |\omega_0| t + \varphi(0), \tag{2.52}$$

$$k^2 = m = \frac{1}{2} - \frac{\omega_0^2}{2|\omega_0|^2 b_E^2}, \qquad b_E^2 = \left(1 + \frac{4\gamma E}{\omega_0^4}\right)^{1/2}, \qquad E > 0.$$

Here $\operatorname{cn} u \equiv \operatorname{cn}(u \mid m)$ is the Jacobi elliptic cosine, k is the modulus (m is a parameter), $K \equiv K(k)$ is a complete elliptic integral of the first kind, and $\varphi(0)$ is the initial phase. The period of the function $\operatorname{cn} u$ is $4K$ [33]. Therefore the frequency of the nonlinear oscillator is, according to Eq. (2.52),

$$\omega(E) = \frac{\pi}{2K} b_E |\omega_0| \qquad (E > 0). \tag{2.53}$$

At $\omega_0^2 < 0$ and $E < 0$ an oscillator can vibrate, at one and the same energy, in the left or right potential well (see Fig. 6). Equation (2.51)

has therefore in this region two solutions:

$$q_i(t) = (-1)^i \frac{|\omega_0|}{\sqrt{\gamma}} \left(1 + b_E^2\right)^{1/2} \operatorname{dn}\left(\frac{K}{\pi}\varphi_i\right), \quad \varphi_i = \omega(E)t + \varphi_i(0), \quad i = 1, 2,$$

(2.54)

$$k^2 = m = \frac{2b_E^2}{1 + b_E^2}, \quad \omega(E) = \frac{\pi}{\sqrt{2}\,K} |\omega_0| \left(1 + b_E^2\right)^{1/2}, \quad \omega_0^2 < 0, E < 0,$$

where $\operatorname{dn} u \equiv \operatorname{dn}(u \mid m)$ is the delta amplitude, b_E^2 is defined in Eq. (2.52), and $\varphi_i(0)$ are the initial values of the phases φ_i.

It can be seen from Eqs. (2.52)–(2.54) that at $\omega_0^2 > 0$ the frequency ω_0 is equal to the minimum frequency of the oscillator (it corresponds to $E = 0$). At $\omega_0^2 < 0$ the frequency $\omega(E)$ tends to zero logarithmically as $E \to 0$:

$$\omega(E) \approx \pi |\omega_0| \left(\ln \frac{16\omega_0^4}{\gamma E}\right)^{-1} \qquad (E > 0)$$

$$\approx 2\pi |\omega_0| \left(\ln \frac{16\omega_0^4}{\gamma |E|}\right)^{-1} \qquad (E < 0), \ \frac{\gamma |E|}{\omega_0^4} \ll 1, \ \omega_0^2 < 0. \tag{2.55}$$

That the period of the oscillations become infinite as $E \to 0$ at $\omega_0^2 < 0$ is obvious from Fig. 6: as $E \to 0$ the classical particle "sticks" to the point $q = 0$.

2.4.2 Time Correlation Function of the Coordinates in the Absence of Damping Neglecting damping, the correlator $Q(t)$ is

$$Q(t) = \int q(t) q(0) w_{st}(E)\, dq(0)\, dp(0), \tag{2.56}$$

where $w_{st}(E)$ and $q(t)$ are defined by Eqs. (2.50), (2.52), and (2.54). To calculate the integral (2.56) it is convenient to transform to new canonical variables, namely the action I and the phase φ, and then to the energy and phase, with account taken of the relations

$$dq\, dp = dI\, d\varphi, \qquad dI = \omega^{-1}(E)\, dE \tag{2.57}$$

(the action depends only on the energy). The averaging over the initial coordinates and momenta in Eq. (2.56) reduces then to integration with respect to E and with respect to the initial phase $\varphi(0)$ that enters in Eqs. (2.52) and (2.54) for $q(t)$ (in the region $E<0$ it is necessary to integrate with respect to $\varphi_1(0)$ and $\varphi_2(0)$). It is easy to average over $\varphi(0)$ by using the expansion of the functions $\mathrm{cn}(2K\varphi/\pi)$ and $\mathrm{dn}(K\varphi/\pi)$ in a series in the Jacobi parameter q_J (see, e.g., [33]):

$$\mathrm{cn}\left(\frac{2K}{\pi}\varphi\right)=\frac{2\pi}{kK}\sum_{n=0}^{\infty}q_y^{n+\frac{1}{2}}\left(1+q_y^{2n+1}\right)^{-1}\cos(2n+1)\varphi,$$

$$\mathrm{dn}\left(\frac{K}{\pi}\varphi\right)=\frac{\pi}{2K}+\frac{2\pi}{K}\sum_{n=1}^{\infty}q_y^{n}\left(1+q_y^{2n}\right)^{-1}\cos n\varphi, \qquad (2.58)$$

$$q_y=q_y(k)=\exp\left[-\pi K'(k)/K(k)\right], \qquad K'(k)=K\left(\sqrt{1-k^2}\right).$$

Substituting Eqs. (2.52), (2.54), and (2.58) into Eq. (2.56) and changing over to integration with respect to E and $\varphi(0)$ we obtain as a result of integration with respect to $\varphi(0)$, taking Eqs. (2.50), (2.53), and (2.57) into account,

$$Q(t)=Q^{(1)}(t)+Q^{(2)}(t)\Theta(-\omega_0^2),$$

$$Q^{(1)}(t)=\frac{32\pi}{\gamma}Z^{-1}\int_0^{\infty}dE\,\omega(E)\exp\left(-\frac{E}{T}\right)$$

$$\times\sum_{n=0}^{\infty}\frac{q_y^{2n+1}}{\left(1+q_y^{2n+1}\right)^2}\cos\left[(2n+1)\omega(E)t\right],$$

$$Q^{(2)}(t)=\frac{2\pi}{\gamma}Z^{-1}\int_{E_m}^{0}dE\,\omega(E)\exp\left(-\frac{E}{T}\right)$$

$$\times\left\{1+8\sum_{n=1}^{\infty}\frac{q_y^{2n}}{\left(1+q_y^{2n}\right)^2}\cos\left[n\omega(E)t\right]\right\}, \qquad (2.59)$$

$$Z=Z^{(1)}+Z^{(2)}, \qquad Z^{(1)}=2\pi\int_0^{\infty}\frac{dE}{\omega(E)}\exp\left(-\frac{E}{T}\right),$$

$$Z^{(2)}=4\pi\Theta(-\omega_0^2)\int_{E_m}^{0}dE\,\omega^{-1}(E)\exp\left(-\frac{E}{T}\right), \qquad E_m=-\frac{1}{4}\frac{\omega_0^4}{\gamma}.$$

The terms $Q^{(1)}(t)$ and $Z^{(1)}$ are determined by the region of positive energies of the oscillator and contribute to the coordinate correlator and to the partition function of the oscillator both at $\omega_0^2>0$ and at $\omega_0^2<0$. The terms $Q^{(2)}(t)$ and $Z^{(2)}$ are determined by the region of negative energies (E_m is the minimum value of the energy) and contribute only at $\omega_0^2<0$. We note that when no account is taken of the relaxation and the fluctuations, an oscillator with energy $E<0$ does not go over from one potential minimum to another, and oscillates about one of two equilibrium positions

$$q_m=\pm\gamma^{-1/2}|\omega_0|.$$

This leads to the appearance in $Q^{(2)}(t)$ of a term independent of the time. When the transitions are taken into account, it will decay with time.

It is seen from Eq. (2.52) that $k^2<1/2$ at $\omega_0^2>0$, and consequently $q_J<\exp(-2\pi)\approx 2\cdot 10^{-3}$. The series (2.59) therefore converges very rapidly in this case and in practice we need retain in $Q^{(1)}(t)$ only the first term (if we are not interested in the spectrum in the region $\omega\gg\omega(E=T)$). In the case $\omega_0^2<0$ the series for $Q(t)$ converges rapidly in the energy regions $E\gg|E_m|$ and $E-E_m\ll|E_m|$. In the region of small $|E|$ the series converges slowly. In this region, the frequencies $\omega(E)$ are low and the relaxation therefore turns out to be substantial.

2.4.3 Spectral Distribution of Duffing Oscillator with Single-well Potential ($\omega_0^2>0$) The spectral distribution of the time correlation function of the coordinates at $\omega_0^2>0$ takes according to Eq. (2.59) the form

$$Q(\omega)=Q^{(1)}(\omega), \qquad \omega_0^2>0,$$

$$Q^{(1)}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,e^{i\omega t}Q^{(1)}(t)=\frac{16\pi\omega}{\gamma}Z^{-1}\Theta(\omega-\omega_0)\sum_{n=0}^{\infty}\chi_n(\omega)$$

$$\approx\frac{16\pi\omega}{\gamma}Z^{-1}\Theta(\omega-\omega_0)\chi_0(\omega), \qquad (2.60)$$

$$\chi_n(\omega)=(2n+1)^{-2}\exp\left(-\frac{E_n}{T}\right)\left(\frac{d\omega(E_n)}{dE_n}\right)^{-1}\frac{q_y^{2n+1}(k_n)}{\left[1+q_y^{2n+1}(k_n)\right]^2},$$

where $k_n \equiv k(E_n)$ and E_n is defined by the equation

$$\omega(E_n) = \frac{\omega}{2n+1}. \tag{2.61}$$

According to Eq. (2.60), the function $Q(\omega)$ is expressed in terms of elliptic integrals. It is easy to obtain for $Q(\omega)$ explicit expressions in the limiting cases of small and large values of the parameter $\mathfrak{b}$ (2.3) ($\mathfrak{b} > 0$). The case $\mathfrak{b} \ll 1$ corresponds to an oscillator whose anharmonic part of the potential energy $\gamma q^4/4$ is small in the actual energy region $E \sim T$ compared with the harmonic part, while the case $\mathfrak{b} \gg 1$ corresponds to an oscillator whose harmonic part is negligibly small compared with $\gamma q^4/4$.

At $\mathfrak{b} \ll 1$ in the region $E \lesssim T$ we have $b_E^2 \approx 1$ and $k^2 \ll 1$. Expression (2.60) reduces then to the formula (2.49) obtained above by another method. At $\mathfrak{b} \gg 1$ and $\omega^4 \gg \omega_0^4$ we have according to Eq. (2.52) $b_E^2 \gg 1$ and $k^2 \approx 1/2$, and when account is taken of Eqs. (2.53) and (2.61) the expression (2.60) takes in the region of the maximum (where only the term with $n = 0$ need be retained in (2.60)) the form

$$Q(\omega) = \frac{\zeta_1 \omega_0}{\gamma \mathfrak{b}^{3/4}} \left(\frac{\omega}{\omega_0}\right)^4 \exp\left[-\frac{\zeta_2}{\mathfrak{b}}\left(\frac{\omega}{\omega_0}\right)^4\right] \qquad (\omega^4 \gg \omega_0^4,\ \mathfrak{b} \gg 1), \tag{2.62}$$

$$\zeta_1 = \frac{16\sqrt{2}}{\pi^3 \Gamma(\frac{3}{4})} \operatorname{ch}^{-2}\left(\frac{\pi}{2}\right) K^3\left(\frac{1}{\sqrt{2}}\right) \approx 0.60, \qquad \zeta_2 = \frac{4}{\pi^4} K^4\left(\frac{1}{\sqrt{2}}\right) \approx 0.49.$$

The position ω_m of the maximum of this distribution and its integral width $\delta\omega_i$ are given by

$$\omega_m = \left(\frac{\mathfrak{b}}{\zeta_2}\right)^{1/4} \omega_0 \approx 1.2\mathfrak{b}^{1/4}\omega_0, \qquad \delta\omega_i \approx 0.74\mathfrak{b}^{1/4}\omega_0. \tag{2.63}$$

From Eqs. (2.63) and (2.49) it can be seen that the shift of the maximum of the spectral distribution and its broadening are of the same order of magnitude at $|\alpha| \gg 1$, i.e. when they are both due to nonlinearity.

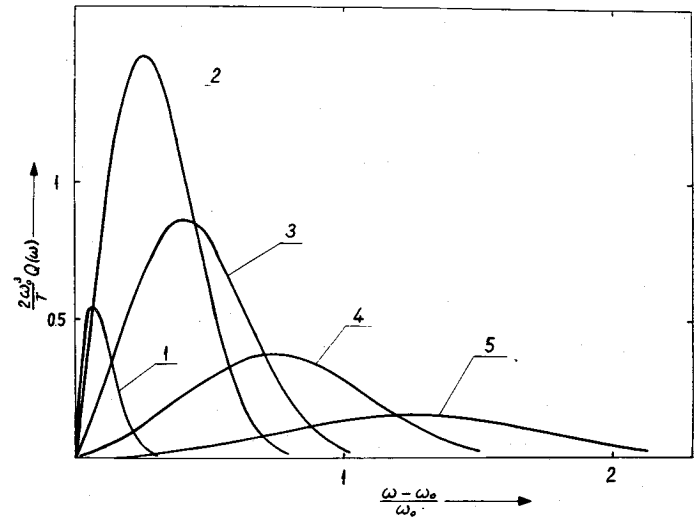

Figure 7 Spectral distribution $Q(\omega)$ for a Duffing oscillator with a single-well potential, neglecting damping. Curves 1–5 correspond to values $\mathfrak{b} = 0.1$, 0.5, 1, 3, and 10. The ordinates of curve 1 are decreased by a factor of ten.

At arbitrary values of $\mathfrak{b}$, the function $Q(\omega)$ can be easily obtained numerically. Plots of this function for several values of $\mathfrak{b}$ are shown in Fig. 7. It is seen that with increasing $\mathfrak{b}$ the half-width of the distribution increases, and its maximum shifts towards higher frequencies.

It should be noted that in the calculation of the spectral distribution it is correct to neglect the damping only in the frequency region where $Q(\omega)$ does not become negligibly small. The presence of even small damping causes the increase of the distribution $Q(\omega)$ on the wings. In particular, in the phenomenological model (1.12) it falls off at $\mathfrak{b} \lesssim 1$ like $\Gamma T(\omega^2 - \omega_m^2)^{-2}$ ($|\omega - \omega_m| \gg \delta\omega_i$). The results are therefore valid only in the region where this quantity is small compared with expression (2.60).

2.4.4 Specific Features of the Spectral Distribution of a Duffing Oscillator with a Two-Well Potential ($\omega_0^2 < 0$) in the Low-Frequency Range

In the region of temperatures considerably exceeding the height ΔU of the potential barrier (see Fig. 6), i.e., at $\mathfrak{b} \gg 1$, the maximum of the spectral distribution $Q(\omega)$ results from the motion in the region of relatively high energies, where the form of the potential near the

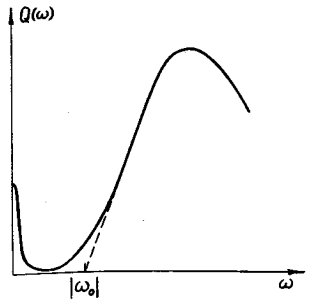

Figure 8 Schematic representation of the spectral distribution $Q(\omega)$ for a Duffing oscillator with a two-well potential at $T \gg \Delta U$. The dashed line shows the edge of the distribution for an oscillator with a single-well potential.

minimum does not manifest itself. Therefore the resultant broad spectral distribution near the maximum (see Fig. 8) at $\omega_0^2 < 0$ is described by the same formulas (2.62) as in the case of the single-well potential ($\omega_0^2 > 0$) considered above. At low temperatures $T \ll \Delta U$ (i.e., $\mathfrak{b} \ll 1$) the maximum of $Q(\omega)$ is connected with the motion of the oscillator near one minimum or the other. Near the minimum the potential becomes weakly nonlinear, and $Q(\omega)$ in the region of the maximum is described by Eqs. (1.8), (2.47) and (2.48) for the single-well Duffing oscillator (interacting with the medium), but with renormalized parameters

$$\omega_0' = \sqrt{2}\,|\omega_0|, \qquad \gamma' = -4\gamma \qquad (\gamma T \omega_0^{-4} \ll 1)$$

(the renormalization of γ is connected with the term $\propto (q - q_m)^3$ in the potential; see [19]).

The qualitative difference between the spectral distributions $Q(\omega)$ for two-well and one-well potentials manifests itself at low frequencies. Localization of an oscillator with energy $E < 0$ near one minimum or the other leads, in accordance with Eq. (2.59), to the appearance of a peak at zero frequency (see Fig. 8). Neglecting the interaction with the medium the peak is δ-like (see Eq. (2.59)). The transitions over the barrier caused by this interaction broaden the peak. The width of the peak is proportional to the relaxation parameter Γ, and at low temperatures $T \ll \Delta U$ contains an additional small factor $\exp(-\Delta U/T)$.

Another significant feature of the behavior of $Q(\omega)$ for a two-well potential manifests itself in the region of the low-frequency wing of the broad distribution. In the case of a single-well potential, in accordance with Eq. (2.60), if the damping is neglected the spectral distribution tends to zero at $\omega - \omega_0 \to 0$ in accordance with the linear law indicated by the dashed line in Fig. 8. In the case of the two-well potential, however, the motion of a particle with near-zero energy in the region of the local maximum of the potential ($U(0) = 0$; see Fig. 6) is slow, and the period of the oscillations is large. As a result, a low-frequency tail appears in the spectral distribution at $\omega < |\omega_0|$.

To calculate the spectral distribution in the region $\Gamma \ll \omega < |\omega_0|$ we can neglect, as before, the damping and use expression (2.59) for the time correlation function. As a result we get

$$Q(\omega) = Q^{(1)}(\omega) + Q^{(2)}(\omega), \qquad \omega_0^2 < 0,$$

$$Q^{(1)}(\omega) = \left(\frac{8\pi}{\gamma}\right)^2 \frac{|\omega_0|^5}{\omega \operatorname{ch}^2(\pi\omega/2|\omega_0|)} Z^{-1} \times \sum_{n=0}^{\infty} e^{-E_n/T} \exp\left[-\frac{\pi|\omega_0|}{\omega}(2n+1)\right] \Theta(E_n),$$

$$Q^{(2)}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t} Q^{(2)}(t) = \left(\frac{8\pi}{\gamma}\right)^2 \frac{|\omega_0|^5}{\omega \operatorname{ch}^2(\pi\omega/2|\omega_0|)} Z^{-1} \times \sum_{n=1}^{\infty} e^{-E_n/T} \exp\left(-\frac{2\pi|\omega_0| n}{\omega}\right) \Theta(-E_n), \tag{2.64}$$

$$\exp(\pi|\omega_0|/\omega) \gg 1, \qquad \omega \gg \Gamma.$$

The energies E_n are determined here from the condition (2.61) at $E_n > 0$ and from the condition $\omega(E_n) = \omega/n$ at $E_n < 0$. From Eqs. (2.55) and (2.61) it can be seen that at $\mathfrak{b} \gg \exp(-\pi|\omega_0|/\omega)$ the energies E_n are small compared with T, and accurate to corrections

that are exponentially small in $\pi|\omega_0|/\omega$ we have

$$Q(\omega)=\left(\frac{8\pi}{\gamma}\right)^2\frac{|\omega_0|^5}{\omega\,\mathrm{ch}^2(\pi\omega/|\omega_0|)}Z^{-1}\exp\left(-\frac{\pi|\omega_0|}{\omega}\right). \quad (2.65)$$

According to Eq. (2.65), neglecting damping, the spectral distribution $Q(\omega)$ decreases exponentially with decreasing frequency. The characteristic frequency that determines the rate of the decrease is equal to $|\omega_0|$ and does not depend on the nonlinearity parameter γ. The parameter $|\omega_0|^2$ at $\omega_0^2<0$ is equal to the curvature of the potential at the local-maximum point (see Fig. 6). It is precisely the shape of the potential hump which governs the form of the distribution (2.65).

It can be shown that for an arbitrary oscillator whose potential $U(q)$ has a local maximum, where $U(q)=-(1/2)|\omega_0|^2q^2$ (the coordinate q is reckoned from the position of the maximum), the spectral distribution on the low-frequency tail of the broad peak is proportional to $\exp(-\pi|\omega_0|/\omega)$. In the general case, the cyclic frequency $\omega(E)$ of oscillations with energy $E>0$ (the energy is reckoned from the value of $U(0)$) is equal to

$$\omega(E)=\pi\left\{\int_{q_1}^{q_2}dq\left[2E-2U(q)\right]^{-1/2}\right\}^{-1}, \quad (2.66)$$

where q_1 and q_2 are the turning points that limit the motion; $U(q_i)=E$ $(i=1,2)$; $q_1<0$, and $q_2>0$. At small energies the main contribution to the integral (2.66) is made by the region $|q|\lesssim E^{1/2}/|\omega_0|$ $(|q_{1,2}|\gg E^{1/2}/|\omega_0|)$, where the integrand is large. Integration yields

$$\omega(E)\approx\pi|\omega_0|\ln^{-1}(C/E), \qquad C\sim|\omega_0|^2q_i^2, \quad E\to0 \quad (E>0). \quad (2.67)$$

The time correlation function of the coordinates of an arbitrary vibrating particle placed in a thermostat, neglecting relaxation, can be represented in a form similar to (2.59):

$$Q(t)=\int dE\exp(-E/T)\sum_{n=0}^{\infty}\chi_n(E)\cos n\omega(E)t, \quad (2.68)$$

where the functions $\chi_n(E)$ are determined by the details of the potential. We take the Fourier transform of Eq. (2.68) with respect to time and transform from integration with respect to E to integration with respect to $\omega(E)$. At small E the integrand acquires then a factor

$$\frac{dE}{d\omega(E)}\approx\pi|\omega_0|C\omega^{-2}(E)\exp\left[-\pi|\omega_0|/\omega(E)\right], \qquad E\to0.$$

This factor makes the spectral distribution in the region of small ω proportional to $\exp(-\pi|\omega_0|/\omega)$ (at $\pi|\omega_0|>\omega$ we can neglect the terms $\propto\exp(-n\pi|\omega_0|/\omega)$ with $n>1$ in $Q(\omega)$, which stem from the terms with $n>1$ in (2.68)). It is easy to verify that at small ω the contribution made to $Q(\omega)$ by the region $E<0$ is small (just as for the Duffing oscillator).

Thus, the exponential decrease of the spectral distribution is the result of the presence of a local parabolic maximum of the potential. In the region where

$$\exp(-2\pi|\omega_0|/\omega)\lesssim\Gamma/|\omega_0| \qquad (\text{at } b\sim1)$$

the calculation must be carried out with allowance for the relaxation processes. The latter lead, in particular, to the appearance of a wing of the broad peak (see Fig. 8). The criterion presented means that Eq. (2.65) is valid so long as the value of $Q(\omega)$ determined by it exceeds the contribution made to $Q(\omega)$ by the relaxation processes.

2.5 *Discussion of the Results of the Classical Theory of the Nonlinear Duffing Oscillator*

The foregoing analytic solution of the problem of a Duffing oscillator interacting with a medium or acted upon by white noise makes it possible to reveal explicitly the distinguishing features, connected with the nonlinearity of the system, of the relaxation and of the fluctuations. The relaxation of the oscillator at $|b|\ll1$ is described by the simple formula (2.42), from which it can be seen that even at a relatively weak nonlinearity the damping of the initial values of the coordinate and of the momentum is not exponential and depends substantially on the initial conditions. Under the same circumstances, the shape of the spectral distribution of the time correlation function

$Q(\omega)$ becomes greatly distorted compared with the Lorentz distribution for the harmonic oscillator (see Eq. (2.48) and Fig. 4). In particular, at a relatively strong nonlinearity ($|\alpha| \gg 1$) as can be seen from Eq. (2.49), the spectral distribution decreases exponentially with frequency on one side of the maximum of the peak, and vanishes linearly on the other side (in the limit $\Gamma = 0$).

The results of the present section cover a large range of the two dimensionless parameters α and $\mathfrak{b}$ (or Γ/ω_0 and $\mathfrak{b}$), which characterize the Duffing oscillator:

$$0 \leqslant |\mathfrak{b}| \ll |\alpha| < \infty \quad \text{or} \quad 0 \leqslant |\mathfrak{b}| < \infty, \qquad \Gamma \ll |\omega_0|. \tag{2.69}$$

The results are valid both at $\alpha > 0$ and at $\alpha < 0$ in the region $|\mathfrak{b}| \ll 1$, but only at positive γ (i.e., $\mathfrak{b} > 0$) at $|\mathfrak{b}| \gtrsim 1$. In the region $\mathfrak{b} \gtrsim 1$ the spectral distribution was obtained for the case $\omega_0^2 < 0$ as well as for the case $\omega_0^2 > 0$. The region $\alpha \sim \mathfrak{b} \sim 1$ cannot be described with the aid of the employed methods. Some results for this region were obtained numerically [34] and also by expanding the spectral distributions in continued fractions [35].

It is of interest to compare the spectral distributions $Q(\omega)$ presented above with the distributions obtained by approximate methods. It is easy to verify that in the region $|\alpha| \ll 1$, where the results of [22–25] are valid, the expressions given in Sec. 2.3 agree with these results. As can be seen from Eqs. (2.40) and (2.47), in this region the expansion parameter is α. In the region $|\alpha| \ll 1$ one can use also the equivalent-linearization method [17], in which the nonlinear oscillator is replaced by a linear one whose effective frequency ω_{eff} is defined as the square root of the normalized second moment of the spectral distribution $Q(\omega)$.

In the region $|\mathfrak{b}| \ll 1$ but $|\alpha| \gtrsim 1$, expansions in powers of the nonlinearity parameter γ (or in α) are no longer valid. At $|\alpha| \gtrsim 1$ the broadening of the spectral distribution as a result of the nonlinearity is of the same order as the shift of the distribution, and hence the effective frequency ω_{eff} can differ significantly from the frequency ω_m of the maximum of the distribution $Q(\omega)$ (e.g., at $|\mathfrak{b}| \ll 1$ and $|\alpha| \gg 1$ the ratio $(\omega_{\text{eff}} - \omega_0)/(\omega_m - \omega_0) = 2$, and at $\mathfrak{b} \gg 1$ we have $\omega_{\text{eff}}/\omega_m \approx 1.4$). The spectral distribution obtained from the exact solution differs substantially in this region from the distribution obtained by the equivalent-linearization method.

§3 Theory of Fluctuation Transitions Between Stable States of a Nonlinear Oscillator

3.1 Picture of the Fluctuations in Nonlinear Systems with Several Stable States

Nonlinear systems frequently have several stationary states or limit cycles which are stable with respect to small fluctuations. Random external forces due to interactions with the medium or of some other origin produce transitions between these states. Since large fluctuations are needed for the transitions, the probabilities of the latter in the case of weak random forces is low. An example of a system with several stationary states is the Duffing oscillator at $\omega_0^2 < 0$ (see Fig. 6). Stable equilibrium positions correspond to each of the two minima of the potential in this case. The Duffing oscillator has also two stable stationary states in the case when $\omega_0^2 > 0$, but the oscillator is acted upon by an external resonant field whose amplitude satisfies certain relations indicated in Sec. 3.2 below (see Fig. 2).

The problem of calculating the transitions, caused by interaction with the medium, between the states separated by a potential barrier is one of the classical problems of diffusion theory. For Brownian motion of a particle in a static potential field, it was considered by Kramers [36]. In this section we present and illustrate, with the Duffing oscillator as the example, results pertaining to the calculation of the probability of a transition between stable states of a system in a case when the motion is not potential or when the potential depends periodically on the time.

The phase portrait of the simplest nonlinear system which has one degree of freedom is possessed of two stable stationary states (in particular, of a Duffing oscillator), is shown in Fig. 9. The dashed line in the figure is the separatrix between the attraction regions of the foci f_1 and f_2; located on this line is the saddle-point s. In the absence of a random force, the system, located at the initial instant at some general-position point, will approach, after a characteristic relaxation time t_r, that focus (or node) in whose attraction region it was initially located (examples of phase trajectories are shown in Fig. 9). In the presence of a weak random force, the system will move, with overwhelming probability, practically along the same trajectory. On ap-

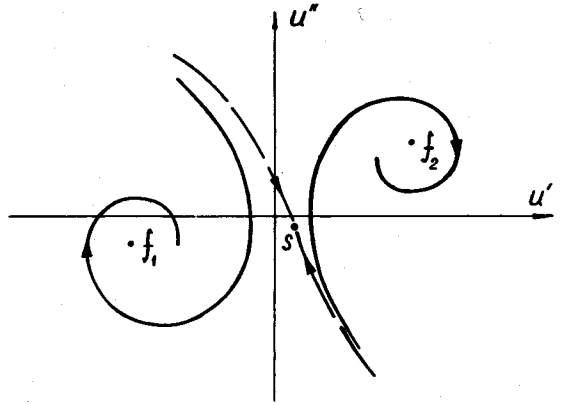

Figure 9 Phase portrait of a nonlinear dynamic system with two stable states of the focus type. The dashed line is the separatrix; s—saddle point; f_1, f_2—foci. The arrows indicate the direction of the motion.

proaching the focus, the system will stay near it for a long time, greatly exceeding t_r, and will undergo small fluctuations. Ultimately it will experience a sufficiently large fluctuation, as a result of which the phase trajectory crosses the separatrix (with overwhelming probability, near the saddle; see below). After this, the system approaches another focus over a time $\sim t_r$ and then fluctuates near it. This just means a transition to a new stable state.

In this section we consider the probability W of such transitions in the limiting case of a small random force, when $W \ll t_r^{-1}$ (this condition is necessary for the concept of the stationary transition probability to be meaningful). Obviously, the transitions $1 \to 2$ and $2 \to 1$ have different probabilities. Their ratio determines the stationary distribution over the states. In the case of Brownian motion in an external static potential, the distribution over the states is determined by the Boltzmann formula. In other cases, however, the calculation of the distribution over the states is more complicated. For weak random forces it reduces in fact to the solution of the transition-probability problem.

The analysis presented below for the system whose phase portrait is given in Fig. 9 can be generalized in elementary fashion to include the case of dynamic systems with higher dimensionality, as well as systems having stationary states of other types or limit cycles corresponding to periodic motion.

It is known that greatest interest in the problem of the passage of a diffusing particle over a potential barrier, at a sufficiently low transition probability W, attaches to the calculation of the argument of the exponential, and it frequently suffices to estimate only the order of magnitude of the pre-exponential factor. Respectively, a probability of a transition between states of a nonequilibrium system whose motion represents a Markov process is calculated below with logarithmic accuracy. To do it we, following the Feynman's approach [32] to fluctuations in dynamical subsystems, represent W in the form of a certain path integral, which is calculated by the saddle-point method [29]. The probability of reaching the phase-space point far from points corresponding to stable states was investigated in a mathematical paper [37] by another method for a certain type of Markov systems. The functional-integration method makes it possible to obtain in simple manner the results given in [37] and to generalize them to include the case of Markov processes of more general type, as well as the case of non-Markov processes. The path-integral formulation of the problem of fluctuations was developed in [38] for the systems described by the master equation of the Kramers–Moyal type. The approach [38] was somewhat different from the Feynman's one used below, and the probability of large non-Gaussian fluctuations causing transitions between stable states was not considered in [38].

Using certain additional relations that follow from the physical picture of the fluctuation transition, the problem of calculating W can be made closed. This approach is illustrated for the concrete case of transitions between stable states of a Duffing oscillator in an external field, where the problem of calculating W can be completely solved in explicit form [29].

Transitions between stable states of the Duffing oscillator in an external field are considered below in Secs. 3.2–3.4. In Sec. 3.2 using the same approach as in Sec. 2.1 we obtain the equations of motion for the slow variables of such an oscillator interacting with a medium. In Sec. 3.3 is obtained an explicit expression for the probability of the transition between the stable states of a subsystem; this expression is later applied in Sec. 3.4 to the Duffing-oscillator problem, with account taken of the concrete form of the equations of motion of the latter.

For Markov processes, the value of W can be calculated also with the aid of the Einstein–Fokker–Planck (EFP) equation (see, e.g., [10, 39]). This method is convenient if the random process is one-

dimensional, but for multidimensional processes its realization is usually difficult. In a number of cases the EFP equation is convenient for the analysis of a stationary distribution over stable states (see, e.g., [40], as well as the reviews [10, 14, 39] and references cited there).

3.1.1 Fluctuations Near Bifurcation Points Fluctuations in a nonlinear system acquire peculiar features if the system is near bifurcation points, i.e., if its parameters are close to values at which the topological picture of the trajectories on the phase plane changes, for example, new equilibrium positions or limit cycles appear or vanish. The problem of fluctuations near bifurcation points was first considered for the systems with limit cycles. In a number of papers (see, e.g., [10, 13, 39, 41]) the case of the Van der Pol oscillator was investigated in detail. The solution of this problem is greatly simplified by the fact that in the EFP equation for the corresponding Markov process the variables (amplitude and phase) are separable and the equation reduces to a one-dimensional one. This circumstance has made it possible, in particular, to investigate in detail fluctuations in the region of bifurcation corresponding to soft excitation of a limit cycle (see in particular [41]). For certain concrete systems, fluctuations were also considered near bifurcation points of another type (see, e.g., [42, 43]).

Besides the bifurcation points, at which the roots $\lambda_{1,2}$ of the characteristic equation for a dynamic subsystem pass through the imaginary axis and a limit cycle is excited, as in the case of the Van der Pol oscillator, there are also quite general bifurcation points at which $\lambda_1 = 0$ ($\lambda_i \neq 0$ at $i \neq 1$) and mutual annihilation or creation of two singular points takes place in phase space, such as coalescence of a node and a saddle (marginal points).

In the parameter range where $\lambda_1 \ll \lambda_2, \lambda_3, \ldots$ one of the motions in the system becomes slow (a soft mode sets in [14]). This results in the increase of fluctuations. The smallness of λ_1 makes it possible to use an adiabatic approximation for the description of the fluctuations and reduce, generally speaking, the multidimensional problem to a one-dimensional one in the bifurcation region if the random forces are small. This approach is used in Sec. 3.5 to consider fluctuations in Markov systems in the case when one of the points produced upon bifurcation is stable (node or focus) and corresponds to a metastable equilibrium state, while the other is a saddle (see also [45, 46]). In

particular, the probability W of an escape from a metastable state through a saddle point to some other stable state of the system is calculated.

Since the motion near a bifurcation point can be regarded as one-dimensional, the problem is much simpler than in the general case considered in Sec. 3.3. It reduces in natural fashion to the known first passage time problem in the Brownian motion theory [47]. This simplification makes it possible to determine not only the argument of the exponential in the expression for W (this argument depends substantially on the distance to the bifurcation point), but also the pre-exponential factor. In this respect the analysis of the transition probability in the considered, generally speaking, multidimensional systems (but located in the bifurcation region) turns out to be more complete than in Sec. 3.3, where W is determined only with logarithmic accuracy.

In two-parameter systems the curves in the plane of the parameters c_1 and c_2, which describe bifurcations of the general type considered just above, can have singular points of the form of spinodes. The shape of the curve near the spinode K ($c_1 = c_{1K}$, $c_2 = c_{2K}$) is shown in Fig. 10. In the parameter region bounded by the solid curves in Fig. 10 the system has two stable states, as well as an unstable equilibrium state—a saddle point. As the point K is approached (in parameter space), these states come closer together (in phase space), the probabilities of the transitions between them increase rapidly, the rigidity of the system decreases, and as a result the fluctuations in the vicinity of

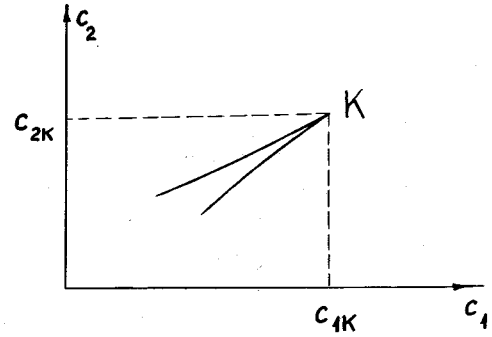

Figure 10 Schematic representation of the bifurcation curves (solid lines) in two-dimensional parameter space near the spinode point K.

the point K increase sharply. In a certain sense, the point K is analogous to the critical point on the gas–liquid phase-transition curve.

The theory of fluctuations near such a spinode is considered in Sec. 3.6. The asymptotic form of the damping of the correlation functions for long times is obtained. Far from the point K, where the stationary probability-distribution function has two sharp maxima, this asymptotic form is determined by the probabilities of the transitions between the corresponding equilibrium positions. On the other hand, in the region of very short distances to K, the damping decrement of the critical fluctuations, as a function of two parameters, is investigated numerically. In Sec. 3.7, the results of Secs. 3.5 and 3.6 are applied to an investigation of the fluctuations of a Duffing oscillator in an external resonant field near bifurcation points and a spinode point.

A special question is that of the absorption of an external field by a bistable system. This question is considered in Sec. 3.8 for a nonlinear oscillator. In the same section are analyzed the characteristic features of the spectral dependence of the absorption (or amplification) of the additional weak field, particularly those connected with bistability.

3.2 *Forced Oscillations of a Nonlinear Oscillator that Interacts with a Medium in a Resonant Field*

The Hamiltonian of an isolated Duffing oscillator in an external field is of the form

$$H_0 = \tfrac{1}{2}(p^2 + \omega_0^2 q^2) + \tfrac{1}{4}\gamma q^4 - qh\cos\omega t. \tag{3.1}$$

To describe the oscillator relaxation and fluctuations, which are due to interaction with the medium, we can use the model considered in Sec. 2, in which the medium is represented as a set of vibrations having continuous frequency spectrum, and the interaction is linear in the coordinate q of the considered oscillator. The total Hamiltonian of the system is then described by Eqs. (2.1), (2.2), and (3.1), while the equation of motion takes, after excluding the vibrations of the continuous spectrum, a form similar to Eq. (2.10):

$$\frac{d^2q}{dt^2} + \omega_0^2 q + \gamma q^3 = h\cos\omega t + f(t) + L[q]. \tag{3.2}$$

Here $f(t)$ is the random force exerted on the oscillator by the medium, and $L[q]$ is an integral operator that describes the reaction of the medium, with retardation taken into account; the explicit forms of $f(t)$ and $L[q]$ are given in Eqs. (2.11) and (2.12) respectively.

Resonant phenomena manifest themselves most distinctly in the case of weak interaction with the medium, when the characteristic damping $\Gamma \ll \omega_0$. If, in addition, the nonlinearity is weak enough and the resonance condition $|\omega - \omega_0| \ll \omega_0$ is satisfied, it is expedient to use the averaging method for the analysis of the motion of the oscillator in the time region of greatest interest $t \gg \omega_0^{-1}$. In the case considered it is convenient to introduce a slowly varying complex amplitude of the oscillator $\tilde{u}_1(t)$ with the aid of the substitution

$$q = \tilde{u}_1 e^{i\omega t} + \tilde{u}_2 e^{-i\omega t}, \qquad \frac{dq}{dt} = i\omega(\tilde{u}_1 e^{i\omega t} - \tilde{u}_2 e^{-i\omega t}), \qquad \tilde{u}_2 = \tilde{u}_1^*,$$

which is analogous to Eq. (1.5). In accord with the idea of the averaging method we can discard in the equation for $\tilde{u}_1$ (just as in the absence of a resonant field) the fast oscillating terms that are proportional to the small parameters γ and Γ (see Appendix B). Transforming the operator $L[q]$ in the region $t \gg t_c$ (the characteristic "fast" time t_c is defined in Eq. (2.6)) in the same manner as was done in the derivation of Eq. (2.14), we reduce the second-order equation (3.2) to a first-order complex equation

$$\frac{d\tilde{u}_1}{dt} = -\frac{3\gamma}{2i\omega}|\tilde{u}_1|^2\tilde{u}_1 - (\Gamma \operatorname{sgn} t + i\delta\omega)\tilde{u}_1 + \frac{h}{4i\omega} + \frac{1}{2i\omega}e^{-i\omega t}f(t), \tag{3.3}$$

$$\delta\omega = \omega - \omega_0 - P.$$

Explicit expressions for the damping Γ and for the frequency shift P in the model (2.1), (2.2) of interaction with the medium are given in Eq. (2.15) (strictly speaking, when expressions (2.15) for Γ and P are substituted in Eq. (3.3), ω_0 in them should be replaced by ω, but the renormalization of Γ and P can be neglected with accuracy to the small corrections $\sim\delta\omega(d\Gamma(\omega)/d\omega)$ and $\delta\omega(dP(\omega)/d\omega)$).

Besides the fast oscillating terms proportional to γ and Γ, we have discarded in Eq. (3.3) also the terms $\sim\delta\omega\tilde{u}_2 \cdot \exp(-2i\omega t)$ and $h\omega^{-1}\exp(-2i\omega t)$. The criterion for the applicability of Eq. (3.3) takes

therefore the form:

$$|\delta\omega|, \Gamma, |P|, |\gamma\tilde{u}_1^2|\omega^{-1} \ll t_c^{-1}, \qquad \left|\frac{d\Gamma(\omega)}{d\omega}\right|, \left|\frac{dP(\omega)}{d\omega}\right| \ll 1.$$

We shall assume hereafter that the frequency detuning $\delta\omega$ differs from zero (at $\delta\omega = 0$ the oscillator has only one stable state).

3.2.1 Transition to Dimensionless Variables It is convenient to choose the dimensionless variables (the time τ, the complex amplitude u, the field amplitude $\beta^{1/2}$, and the relative frequency detuning $\tilde{\Omega}$) in the following manner:

$$\tau = |\delta\omega|t, \qquad u = \left(\frac{3|\gamma|}{2\omega|\delta\omega|}\right)^{1/2}\tilde{u}_1$$
$$\beta = \frac{3|\gamma|h^2}{32\omega^3|\delta\omega|^3}, \qquad \tilde{\Omega} = \frac{\delta\omega}{\Gamma}. \tag{3.4}$$

In terms of these variables, Eq. (3.3) takes the form

$$\dot{u} = v + \frac{1}{\tilde{\Omega}}\tilde{f}(\tau),$$

$$v = v(u, u^*) = -\frac{u}{|\tilde{\Omega}|}\operatorname{sgn}\tau + iu(uu^*\operatorname{sgn}\gamma - \operatorname{sgn}\tilde{\Omega}) - i\beta^{1/2}\operatorname{sgn}h, \tag{3.5}$$

$$\hat{f}(\tau) = -i\left(\frac{3|\gamma|}{8\omega^3\Gamma^2|\delta\omega|}\right)^{1/2}\exp\left(-i\frac{\omega}{|\delta\omega|}\tau\right)f\left(\frac{\tau}{|\delta\omega|}\right), \qquad \dot{u} = \frac{du}{d\tau}.$$

From a comparison of Eqs. (3.3), (3.5) with Eqs. (2.1), (C1) it is clear that the random force $\tilde{f}(\tau) = \tilde{f}'(\tau) + i\tilde{f}''(\tau)$ is asymptotically δ-correlated, and its distribution is Gaussian. According to Eq. (C1)

$$\langle\tilde{f}'(\tau_1)\tilde{f}'(\tau_2)\rangle = \langle\tilde{f}''(\tau_1)\tilde{f}''(\tau_2)\rangle = 2\alpha\delta(\tau_1 - \tau_2),$$
$$\langle\tilde{f}'(\tau_1)\tilde{f}''(\tau_2)\rangle = 0, \qquad \alpha = 3|\gamma|T(8\omega^3\Gamma)^{-1}. \tag{3.6}$$

Eq. (3.5) with a random force having the statistical properties (3.6), just as Eqs. (2.4) and (C1), is valid also for a Hamiltonian H_i of the interaction with the medium of more general form than that in Eq. (2.2); all that is important is that the interaction be linear in q.

By changing from the variables q and dq/dt to u and u^* we can reduce to Eq. (3.5) also the problem of forced oscillations of a nonlinear Brownian oscillator in a field of an arbitrary (not necessarily due to thermal fluctuations) random δ-correlated force f_L:

$$\frac{d^2q}{dt^2} + 2\Gamma\frac{dq}{dt} + \omega_0^2 q + \gamma q^3 = h\cos\omega t + f_L(t),$$
$$\langle f_L(t)f_L(t')\rangle = 2\mathscr{B}\Gamma\delta(t - t'). \tag{3.7}$$

The parameter $\alpha = 3|\gamma|\mathscr{B}(16\omega^3\Gamma)^{-1}$ in Eq. (3.6) is now determined by the characteristic of the noise $\mathscr{B}$. Actually Eq. (3.5) is the Langevin equation for a nonlinear oscillator in an external field.

3.2.2 Stable States In the absence of a random force the complex nonlinear equation of motion (3.5) has stationary solutions ($\dot{u} = 0$) whose amplitudes are determined by the equation

$$\varphi(|u_{st}|^2) = 0, \qquad \varphi(x) = x(x\operatorname{sgn}(\gamma\tilde{\Omega}) - 1)^2 + \frac{x}{\tilde{\Omega}^2} - \beta \tag{3.8}$$

(the stationary points on the phase plane (u', u'') correspond to limit cycles on the phase plane $(q, dq/dt)$). Equation (3.8) has three solutions if β and $\tilde{\Omega}$ in Fig. 11 are located in the region bounded by the solid curve, and one solution if β and Ω are located outside this region. The boundaries of the region of existence of three solutions are described analytically by the following equations (equations of the bifurcation curves):

$$\beta_B^{(1,2)}(\tilde{\Omega}) = \frac{2}{27}\left[1 + g\tilde{\Omega}^{-2} \mp (1 - 3\tilde{\Omega}^{-2})^{3/2}\right], \qquad \tilde{\Omega}\operatorname{sgn}\gamma \geqslant \sqrt{3}. \tag{3.9}$$

The spinode point K corresponds to the values $\tilde{\Omega}_K\operatorname{sgn}\gamma = \sqrt{3}$ and $\beta_K = 8/27$.

The dependence of the reduced amplitude $2|u_{st}|$ of the oscillator oscillations on the reduced amplitude $\beta^{1/2}$ of the resonant field

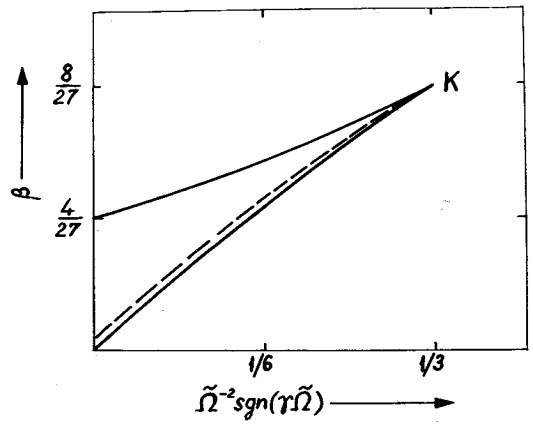

Figure 11 Region of existence of two stable states of a nonlinear oscillator. The bifurcation points lie on the solid lines, K is the spinode point. The dashed curve corresponds to the parameter values at which the probabilities of the transitions $1 \to 2$ and $2 \to 1$ are equal.

($\beta \propto h^2$) at $\tilde{\Omega} \operatorname{sgn} \gamma \geqslant \sqrt{3}$ is S-shaped (cf. Fig. 2). In the region where three solutions exist, two of them are stable with respect to small fluctuations (with the largest and smallest amplitude $2|u_{st}|$), while the third corresponds to a saddle point. On the phase plane (u', u'') $(u = u' + iu'')$ a separatrix passes through this point (see Fig. 9). If the fluctuations are not taken into account, hysteresis is present in the system: If β increases from zero, then u_{st} also increases remaining on the lower branch of the amplitude versus field characteristic (see Fig. 2). Next, when β goes through the value $\beta_B^{(2)}(\tilde{\Omega})$, where $d|u_{st}|/d\beta = \infty$, u_{st} goes over jumpwise to the upper branch. When β decreases from large values, $|u_{st}|$ varies along the upper branch of the characteristic, and at $\beta = \beta_B^{(1)}(\tilde{\Omega})$ it goes over jumpwise to the lower branch.

Even a small random force in Eq. (3.5) leads to randomization of the motion. This causes the relatively rare fluctuation transitions between stable states, which are discussed in Sec. 3.1 above, and after a time of the order $\sim W^{-1}$ the system "forgets" the initial state, hysteresis effects no longer appear, and a stationary distribution of the probabilities over the stable states is formed. As a result, the dependence of $\langle |u_{st}| \rangle$ on β becomes single-valued (in contrast to the ambiguous dependence at times $t \ll W^{-1}$ at which the hysteresis phenomena manifest themselves).

3.3 *General Expression for the Transition Probability*

We consider the probability density $w(u_b; u_a; \tau_b - \tau_a)$ of the transition of the system from a certain point $u_a = u_a' + iu_a''$ on the phase plane, in which the system was located at the instant of time τ_a, to a point u_b at the instant τ_b. With an aim at calculating the probability of a transition between stable states, we shall assume that the point u_a is located in the vicinity of one of the foci (for example f_1), and u_b is located near a separatrix. It is convenient to express the function $w(u_b; u_a; \tau_b - \tau_a)$ in the form of a path integral

$$w(u_b; u_a; \tau_b - \tau_a) = \int_{u(\tau_a)=u_a} \mathscr{D}\tilde{f}(\tau) \mathscr{P}[f(\tau)] \delta(u(\tau_b) - u_b) \times \left\{ \int \mathscr{D}\tilde{f}(\tau) \mathscr{P}[\tilde{f}(\tau)] \right\}^{-1} \tag{3.10}$$

In the calculation of the path integral with respect to the complex functions $\tilde{f}(\tau)$ $(\mathscr{D}\tilde{f}(\tau) \equiv \mathscr{D}\tilde{f}'(\tau)\mathscr{D}\tilde{f}''(\tau))$ it is assumed that at $\tau = \tau_a$ the system is located at the point u_a. As seen from Eq. (3.10), contributions to the path integral are made only by such realizations of the random force $\tilde{f}(\tau)$ which transfer the system over a time $\tau_b - \tau_a$ from the state u_a into the state u_b.

The functional $\mathscr{P}[\tilde{f}]$ determines the probability distribution of the random function $\tilde{f}(\tau)$. It is known (see [32]) that for a random function of the white-noise type with correlators (3.6) we have

$$\mathscr{P}[\tilde{f}(\tau)] = \exp\left[-\frac{1}{4\alpha} \int |\tilde{f}(\tau)|^2 d\tau \right]. \tag{3.11}$$

We shall regard hereafter in this section the parameter α, which characterizes the intensity of the random force, as small.

It is of interest to calculate the probability density (3.10) in the range of times

$$W|\delta\omega|^{-1} \gg \tau_b - \tau_a \gg 1 \qquad (t_b - t_a \gg |\delta\omega|^{-1}),$$

where W is the probability of the transition from the focus f_1 to another focus f_2 per unit time. In this case the system approaches after the dimensionless time $\tau - \tau_a \sim 1$ the focus f_1 and forgets the

initial state u_a. The "hop" into the point u_b from a certain point in the vicinity of the focus f_1 is also rapid, within a dimensionless time ~ 1 (but has low probability).

If u_b lies near a separatrix and the criteria indicated below are satisfied, then the probability density of the transition (3.10) is exponentially small. It contains a small parameter α in the denominator of the argument of the exponential. Performing the calculation with logarithmic accuracy, we shall determine only the argument of the exponential and ignore the pre-exponential factor, which depends on α weakly. In Eq. (3.10), following [32], it is convenient to change from integration with respect to the random force $\mathscr{D}\tilde{f}(\tau)$ to integration with respect to the trajectories $\mathscr{D}u(\tau)$ of the nonlinear oscillator. As seen from Eq. (3.5), the Jacobian for the transformation is independent of α and influences only the pre-exponential factor, which is of no interest to us. Within the adopted accuracy, it suffices also to separate in the path integral the main exponential factor, which corresponds to the extremal path. The probability density (3.10), with allowance for Eqs. (3.11) and (3.5), is then equal to

$$w(u_b\,;u_a\,;\tau_b-\tau_a)=\text{const}\times\max\exp\Big(-\frac{\tilde{\Omega}^2S}{4\alpha}\Big),$$

$$S=\int_{\tau_c}^{\tau_b}d\tau\ \mathscr{L}(u,\dot{u}), \tag{3.12}$$

$$\mathscr{L}(u,\dot{u})=|\dot{u}-v(u,u^*)|^2,\qquad u(\tau_c)=u_c\,,\qquad u(\tau_b)=u_b\,.$$

Here τ_c is the instant of time when the large fluctuation starts, that transfers the system into u_b, u_c is a point in the immediate vicinity of the focus ($|u_c-u_{f_1}|^2\lesssim\alpha$), and the maximum is taken with respect to the possible paths and with respect to $\tau_b-\tau_c$. In fact, it would be necessary to integrate in Eq. (3.12) with respect to u_c with an appropriate distribution function as a weight, but this integration would influence only the pre-exponential factor.

The quantity $\mathscr{L}(u,\dot{u})$ in Eq. (3.12) can be regarded as the Lagrangian of a certain auxiliary particle, and S as its action. Since $\mathscr{L}$ does not depend explicitly on the time, it follows that $\partial S/\partial\tau=-E$, where E is the particle energy (see [27]), and the condition that

S in Eq. (3.12) be extremal with respect to $\tau_b-\tau_c$ reduces to the equality

$$E=|\dot{u}|^2-|v(u,u^*)|^2=0. \tag{3.13}$$

The probability W of the transition between states in the vicinities of the foci f_1 and f_2 is determined, accurate to a factor $\sim 1/2$, by the integral of the expression (3.12) with respect to the points u_b in the vicinity of the separatrix, and in the approximation adopted it is necessary to find the extremum with respect to $\dot{u_b}$ in this vicinity. Since $\partial S/\partial u_b=\partial\mathscr{L}/\partial\dot{u}$, it follows that

$$\delta S=(\dot{u}^*-v^*)\delta u_b+(\dot{u}-v)\delta u_b^*\,.$$

When the point u_b is shifted along the separatrix we have $\delta u_b/v=\delta u_b^*/v^*$; therefore, taking Eq. (3.13) into account, $\delta S=-|\dot{u}-v|^2 v^{-1}\delta u_b$. If the point u_b shifts in the direction of the saddle, then $v^{-1}\delta u_b>0$, and consequently from among the paths that terminate on the separatrix, the extremal ones are those that pass through the saddle point u_s. At this point $v=0$, and according to Eq. (3.13) the particle moves slowly near it ($\tau_b-\tau_c\to\infty$ as $u_b\to u_s$). In the vicinity of the saddle point, an important role is assumed by small fluctuations about the extremal trajectory, which carry the system through the separatrix within a time $\Delta\tau\sim 1$ with probability $>1/2$. Therefore, accurate to the pre-exponential factor, we can write W in the form

$$W=\text{const}\times\exp\Big(-\frac{Q_a}{\alpha}\Big),\qquad Q_a=\frac{1}{4}\tilde{\Omega}^2\min\int_0^{\tau}d\tau_1\,|\dot{u}-v|^2,$$
$$u(0)\approx u_f\,,\qquad u(\tau)\approx u_s\,. \tag{3.14}$$

The criterion for the applicability of Eq. (3.14) is the inequality

$$Q_a\gg\alpha. \tag{3.15}$$

It can be seen from Eqs. (3.13) and (3.14) that calculation of the probability of a transition between stationary states of a Duffing oscillator has reduced to the solution of the variational problem of

finding a minimum of the action

$$\int_0^\tau d\tau_1 |\dot{u} - v|^2$$

for an auxiliary particle with zero energy, moving from the point u_f into u_s. It is important that this particle is two-dimensional: it has coordinates u' and u'' and corresponding momenta $\dot{u}' - v'$ and $\dot{u}'' - v''$ ($v = v' + iv''$), i.e., this particle has twice as many degrees of freedom as the considered oscillator.

The only characteristic parameter of the δ-correlated random force, its intensity, enters in expression (3.14) for the transition probability as the factor α^{-1} in the argument of the exponential. In the case of a nonlinear oscillator interacting with a medium, as can be seen from Eq. (3.6), $\alpha \propto T$, i.e., the probability of the transition depends exponentially on T^{-1} (with activation energy $Q_a T/\alpha$). This result is valid also for other subsystems that interact weakly with the medium.

3.4 *Probabilities of Transitions Between Stationary States of a Duffing Oscillator at Relatively Large Frequency Detuning*

The activation energy Q_a in expression (3.14) for the transition probability depends only on the dynamic characteristics of the subsystem. In the considered case of a nonlinear oscillator, in accordance with Eqs. (3.5) and (3.14), Q_a is a function of two dimensionless parameters, β and $\tilde{\Omega}$. For any concrete values of β and $\tilde{\Omega}$ we can calculate Q_a by solving numerically the Euler equation for the variational problem (3.14), (3.13). In the limiting cases it is possible to obtain for $Q_a(\beta, \Omega)$ explicit analytic expressions. This can be done if $|\tilde{\Omega}| \gg 1$ or if the values of β and $\tilde{\Omega}$ are close to the bifurcation values and lie in the vicinity of the solid lines of Fig. 11. These limiting cases are considered respectively in this subsection and in subsection 3.7.

Since two stable states exist only at $\gamma\tilde{\Omega} > 0$, and Q_a does not depend on the signs of γ and h, we shall assume hereafter that $\gamma > 0$, $\tilde{\Omega} > 0$ and $h > 0$. In terms of the real variables

$$u' = \mathrm{Re}\, u, \qquad u'' = \mathrm{Im}\, u$$

expressions (3.14) for Q_a and (3.13) for E take the form

$$Q_a = \frac{1}{4\epsilon^2} \min \int_0^\tau d\tau_1 \left[\left(\dot{u}' + \frac{\partial g}{\partial u''} + \epsilon u' \right)^2 + \left(\dot{u}'' - \frac{\partial g}{\partial u'} + \epsilon u'' \right)^2 \right],$$

$$E = \dot{u}'^2 + \dot{u}''^2 - \left(\frac{\partial q}{\partial u''} + \epsilon u' \right)^2 - \left(\frac{\partial g}{\partial u'} - \epsilon u'' \right)^2 = 0 \qquad (3.16)$$

$$g(u', u'') = \tfrac{1}{4}(u'^2 + u''^2 - 1)^2 - u'\sqrt{\beta}\,, \qquad \epsilon = \tilde{\Omega}^{-1},$$

and the extremal trajectories are described by the following Euler equations:

$$\frac{d^2u'}{d\tau^2} + \frac{du''}{d\tau}\Delta g - \frac{1}{2}\frac{\partial}{\partial u'}\left[\left(\frac{\partial g}{\partial u'} \right)^2 + \left(\frac{\partial g}{\partial u''} \right)^2 \right] - \epsilon^2 u' = 0$$

$$\frac{d^2u''}{d\tau^2} - \frac{du'}{d\tau}\Delta g - \frac{1}{2}\frac{\partial}{\partial u''}\left[\left(\frac{\partial g}{\partial u'} \right)^2 + \left(\frac{\partial g}{\partial u''} \right)^2 \right] - \epsilon\sqrt{\beta} - \epsilon^2 u'' = 0, \qquad (3.17)$$

$$\Delta g = \frac{\partial^2 g}{\partial u'^2} + \frac{\partial^2 g}{\partial u''^2}\,.$$

The latter can be written in the form

$$\frac{d^2u'}{d\tau^2} - \frac{du''}{d\tau}\mathscr{H}_{\text{eff}} + \frac{\partial U}{\partial u'} = 0 \qquad \frac{d^2u''}{d\tau^2} + \frac{du'}{d\tau}\mathscr{H}_{\text{eff}} + \frac{\partial U}{\partial u''} = 0, \qquad (3.18)$$

$$U(u', u'') = -\frac{1}{2}\left(\frac{\partial g}{\partial u''} + \epsilon u' \right)^2 - \frac{1}{2}\left(\frac{\partial g}{\partial u'} - \epsilon u'' \right)^2, \qquad \mathscr{H}_{\text{eff}} = -\Delta g.$$

Equations (3.18) can be regarded as the equations of motion of a charged two-dimensional particle with potential energy $U(u', u'')$ in a transverse magnetic field whose intensity is proportional to $\mathscr{H}_{\text{eff}} =$

$-\Delta g$. From the definition of the potential it can be seen that $U(u',u'') \leqslant 0$. The shape of the potential is quite complicated; at large distances from the origin $U(u',u'')$ decreases rapidly (as $-(u'^2 + u''^2)^3$), while in the region $(u'^2 + u''^2) \sim 1$ the function $U(u', U'')$ has three domes. The maximum value of $U(u',u'')$ is reached at the peaks of the domes and equals zero. The peaks of the domes are located at points corresponding to the foci $f_{1,2}$ and to the saddle point of the Duffing oscillator (this can be seen from the relation $U(u',u'') = -(1/2)|v(u' + iu'', u' - iu'')|^2$, where $v(u,u^*)$ is defined in Eq. (3.5)).

The complicated shape of the potential and the presence of a coordinate-dependent magnetic field make it practically impossible to describe analytically the motion of the auxiliary particle in the general case of arbitrary β and $\tilde{\Omega}$. The character of the motion becomes simpler in the case of a relatively large detuning of the external field frequency relative to the eigenfrequency of the oscillaror, $\tilde{\Omega} \gg 1$. In the zeroth approximation in $\epsilon = \tilde{\Omega}^{-1}$, Eqs. (3.17) have a solution that corresponds to "fast" motion along a closed trajectory. This trajectory is described by the equation $g(u',u'') = \text{const}$. Motion over a closed trajectory is in a certain sense degenerate: the periods of the oscillations are the same for u' and u''.

In Appendix D we obtain a self-similar solution to Eqs. (3.17) with the aid of an asymptotic perturbation theory (in the region $\tau \sim \epsilon^{-1}$) with account taken of the degeneracy of the "fast" motion. In this case the first-order corrections in ϵ are determined from the secular equation obtained when terms of second order are considered. As a result we obtain for the quantity Q_a, which determines according to Eq. (3.14) the transition probability, the expression (D.13) in the form of quadratures. It can be represented in analytic form at small β. According to Eqs. (D.13) and (D.15), the values of Q_{a1} for a transition from the focus f_1 (corresponding to the smaller oscillator amplitude), and of Q_{a2} for a transition from the focus f_2, are given by the formulas

$$\epsilon Q_{a1} = \tfrac{1}{2} - \zeta' \beta^{1/4}, \qquad \epsilon Q_{a2} = 2\sqrt{\beta}\,, \qquad \epsilon^2 \ll \beta \ll 1. \tag{3.19}$$

The constant ζ' here is a quadrature of an elliptic integral, $\zeta' \approx 0.98$.

The dependence of $Q_{a1,2}$ on β in the region of small β (but $\beta \gg \epsilon^2$) is very steep. If the parameter β is not small (but, as before, $\epsilon = \tilde{\Omega}^{-1}$

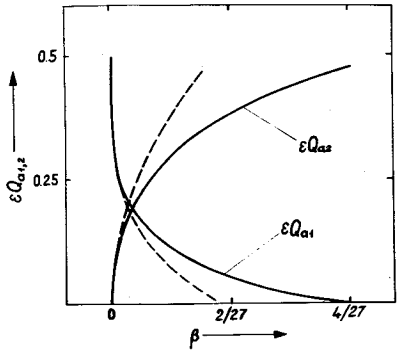

Figure 12 Dependences of the activation energies Q_{a1} and Q_{a2} on β for the transitions $1 \to 2$ and $2 \to 1$ at $\tilde{\Omega} \gg 1$.

$\ll 1$), then Q_{a1} and Q_{a2} can be calculated numerically from Eqs. (D.11) and (D.13). The results of the corresponding calculations are given in Fig. 12 (the dashed curves are plots of the expressions (3.19)). From Eqs. (3.14), (D.13), and (3.19) and from Fig. 12 we can see the dependence of the argument Q_a/α of the exponential on the oscillator parameters. Q_a/α depends on the external field amplitude only in terms of the parameter $\beta \propto h^2|\delta\omega|^{-3}$, and Q_{a1} decreases monotonically with the field, while Q_{a2} increases. The frequency detuning $\delta\omega$ enters in both β and $\tilde{\Omega}$, and at fixed β we have according to Eq. (D.13) $Q_a \propto |\delta\omega|$. The damping parameter Γ at $\Gamma \ll |\delta\omega|$ drops out completely from the expression for Q_a/α. In the considered case $\epsilon \ll 1$, the values of Q_a/α are proportional not only to $\alpha^{-1} \gg 1$ but also to $\epsilon^{-1} \gg 1$.

Special consideration must be given to the regions of β close to the bifurcation values (3.9), i.e., $\beta \approx \epsilon^2 \ll 1$ or $(4/27) - \beta \sim \epsilon^2$. In the region $\beta \approx \epsilon^2$, Eq. (3.19) for Q_{a2} is not applicable, since the focus f_2 is too close to the saddle point. To calculate Q_{a2} we can use formula (3.44) obtained in Sec. 3.7 for the transition probability near a bifurcation point (this formula describes also Q_{a1} at $\beta \approx 4/27$). According to Eq. (3.44), Q_{a2} depends here on the relation between the two small parameters β and ϵ.

The expressions obtained for the transition probabilities W enable us to determine the probabilities $w_{1,2}$ of finding the system near the foci $f_{1,2}$. The ratio of the probabilities w_1 and w_2, taking Eq. (3.14)

into account is equal to

$$\frac{w_1}{w_2} = \frac{W_{21}}{W_{12}} = \text{const} \times \exp \frac{Q_{a1} - Q_{a2}}{\alpha}. \tag{3.20}$$

With logarithmic accuracy, as seen from Fig. 12, at $\beta_0 - \beta \gg \alpha\epsilon$ ($\beta_0 \approx 0.013$) we have $w_2 \ll w_1 \approx 1$, and $w_1 \ll w_2 \approx 1$ at $\beta - \beta_0 \gg \alpha\epsilon$. In the region $|\beta - \beta_0| \approx \alpha\epsilon$ the probabilities of both states are of the same order of magnitude.

3.5 *Probability of Escape of a System from a Metastable Equilibrium State Located Near a Saddle Point*

If a dynamic system is near a bifurcation point, the stochastic motion of the system acquires a universal character and depends only on the type of the bifurcation, but not on the details of the system. This leads, in particular, to a simplification of the expressions for the probabilities of the transitions between different states.

We consider a multidimensional dynamic system acted upon by small random δ-correlated (in time) forces $f_i(t)$ which have a Gaussian distributions. The equations of motion for the dimensionless dynamic variables x_i of the system are

$$\frac{dx_i}{dt} = P_i(\mathbf{c}, \mathbf{x}) + f_i(\mathbf{x}, t)$$
$$\langle f_i(\mathbf{x}, t) f_j(\mathbf{x}', t') \rangle_m = 2d_{ij}(\mathbf{x}, \mathbf{x}')\delta(t - t'). \tag{3.21}$$

Here $\mathbf{c} = \{c_i\}$ denotes the set of parameters of the system. Near the bifurcation points $\mathbf{c} = \mathbf{c}_B$ which correspond to emergence of a stable stationary state and a saddle point (marginal points), we can transform the variables in Eqs. (3.21) in such a way that the functions $P_i(\mathbf{c}, \mathbf{x})$ acquire the form

$$P_1(\mathbf{c}, \mathbf{x}) = \epsilon_1(\mathbf{c}) - \sum_{i,j} B_{ij} x_i x_j + \cdots,$$
$$P_i(c, x) = -\mathscr{A}_i x_i + \sum_{j,l} B_{ijl} x_j x_l + \cdots \qquad (i \geqslant 2), \tag{3.22}$$
$$\mathscr{A}_i' \geqslant 0, \qquad |\epsilon_1| \ll \mathscr{A}_i', \qquad |\alpha_{ij}| \ll \mathscr{A}_i', \qquad \mathscr{A}_i' \equiv \mathrm{Re}\,\mathscr{A}_i$$

(cf. [44], where analogous expressions in the absence of random forces are given). The parameter ϵ_1 vanishes here at the bifurcation points, and the variable x_1 describes slow motion of the system in the bifurcation region ($|x_i|$ are assumed to be small). In Secs. 3.5 and 3.6 we shall choose the time scale such that $\min \mathscr{A}_i' \sim 1$.

In the absence of random forces in the region of small $|x_i|$ the motion of the system has the following character: over a time $\sim t_0$ ($t_0 = \min \mathscr{A}_i' \sim 1$) the variables x_i with $i \geqslant 2$ relax to the values $x_i = X_i(\mathbf{c}, x_1)$, which at a given x_1 are stationary. Motion along x_1 is substantially slower. Therefore in the region $t \gg t_0$ it can be described in the adiabatic approximation, assuming that $x_i = X_i(c, x_1)$ ($i \gtrsim 2$). Thus, the problem becomes one-dimensional.

It is shown in Appendix E that in the time range $t \gg t_0$ a multidimensional Markov random process $\mathbf{x}(t)$ near a bifurcation point is also described by a one-dimensional equation (cf. Eq. (E.9)). The adiabatic approximation used in the derivation of Eq. (E.9) is valid in the most favorable case $\epsilon_1 = 0$ accurate to corrections $\sim \alpha^{1/3}$. The EFP equation (E.9) corresponds to the Langevin equation

$$\frac{dz}{d\tau'} = -\frac{dU}{dz} + \tilde{f}_1(\tau'), \qquad U(z) = z\left(\frac{1}{3} b_2 z^2 - b_0\right), \tag{3.23}$$

where

$$z = \alpha_1^{-1/3} x_1, \qquad \tau' = \alpha_1^{1/3} t, \qquad b_0 = \alpha_1^{-2/3} \epsilon_1,$$
$$b_2 = B_{11}, \qquad \alpha_1 \equiv \alpha_{11}(0, 0), \tag{3.24}$$
$$\tilde{f}_1(\tau') = \alpha_1^{-2/3} f_1(0, t), \qquad \langle \tilde{f}_1(\tau_1') \tilde{f}_1(\tau_2') \rangle = 2\delta(\tau_1' - \tau_2').$$

In the absence of a random force, Eq. (3.23) has at $b_0 > 0$ a stable stationary point $z = z_0$ and a saddle point $z = z_s$:

$$z_0 = \left(\frac{b_0}{b_2}\right)^{1/2}, \qquad z_s = -\left(\frac{b_0}{b_2}\right)^{1/2}, \tag{3.25}$$

which correspond to a local minimum and maximum of the potential $U(z)$ (sec Fig. 13; it is assumed for the same of concreteness here and

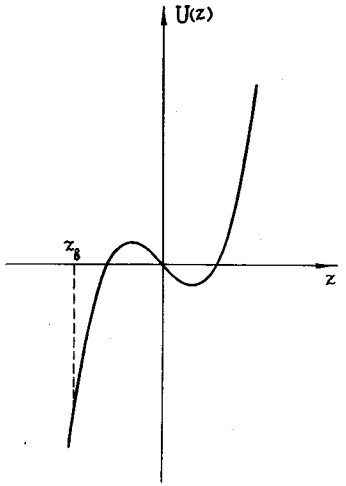

Figure 13 The potential $U(z)$ near the metastable equilibrium state.

below that $b_2 > 0$). Obviously, the state z_0 is metastable at sufficiently small b.

The one-dimensional Markov process (3.23) can be investigated by standard methods. In particular, it is of interest to determine the probability $W_{\text{m.s.}}$ of the escape of the system from the metastable equilibrium state at $b_0 \gg b_2^{1/3}$. In this case $W_{\text{m.s.}}$ is small compared with the reciprocal characteristic time $|b_2 z_0|$ of the motion of the system near the equilibrium position:

$$W_{\text{m.s.}} \ll |b_2 z_0| = (b_0 b_2)^{1/2}.$$

As a result, the duration of the transition process itself is small compared with $W_{\text{m.s.}}^{-1}$. The probability $W_{\text{m.s.}}$ is equal to the reciprocal average time $(\overline{\tau'})^{-1}$ during which the system stays near the equilibrium position. The quantity $\overline{\tau'}$ can be characterized by the average time $\tau'(z, z_b)$ required for the system (initially located at some point z in the attraction region of the metastable equilibrium position) to reach the boundary point z_b chosen in a certain conventional way (see Fig. 13). Starting from physical reasons, it is necessary to choose the position of the boundary point z_b such that the probability of returning to the attraction basin of z_0 from the region $z \leqslant z_b$ be negligibly

small. Therefore z_b should be far enough to the left of the saddle point z_s corresponding to the maximum of $U(z)$. Since far enough to the left of z_s the system "rolls off" rapidly, the time $\overline{\tau'}(z, z_b)$ depends weakly (non-exponentially) on z_b.

The function $\overline{\tau'}(z, z_b)$ satisfies a known equation [47] for the average time of the first passage through the boundary in the theory of Markov processes (see also [46])

$$\frac{\partial^2 \overline{\tau'}(z, z_b)}{\partial z^2} - \frac{\partial U}{\partial z} \frac{\partial \overline{\tau'}(z, z_b)}{\partial z} + 1 = 0. \tag{3.26}$$

The boundary conditions for the function $\overline{\tau'}(z, z_b)$ follow from physical grounds. First, from the definition of the function $\overline{\tau'}(z, z_b)$ as the average time of first reaching the point z_b (starting from z), it is clear that

$$\overline{\tau'}(z_b, z_b) = 0.$$

In the region of large positive z, the function $\overline{\tau'}(z, z_b)$ tends to a constant, and consequently

$$\partial \overline{\tau'}(z, z_b)/\partial z \to 0 \qquad \text{at} \quad z \to \infty.$$

The last relation follows from the fact that in the region of large $b_2 z$ (and large $|\partial U/\partial z|$) the fluctuations have practically no effect on the motion of the system. Therefore the time of motion from the point z to z_1 ($z > z_1 \gg z_0$) is, according to Eq. (3.23), approximately equal to $b_2^{-1}(z_1^{-1} - z^{-1})$ and remains finite as $z \to \infty$. Taking these boundary conditions into account, we can write the solution of Eq. (3.26) in the form

$$\overline{\tau'}(z, z_b) = \int_{z_b}^{z} dx\, e^{U(x)} \int_{x}^{\infty} dx'\, e^{-U(x')} \tag{3.27}$$

The integral (3.27) for the potential (3.23) under consideration can be calculated asymptotically at $b_0^{3/2} \gg b_2^{1/2}$ (i.e., at $\alpha_1 \ll \epsilon_1^{3/2} B_{11}^{-1/2}$) by the saddle-point method. The principal exponential term is of the

form

$$\overline{\tau'}(z,z_b) = W_{\text{m.s.}}^{-1} = \frac{\pi}{\sqrt{b_0 b_2}} e^{Q_a'},$$
$$Q_a' = \frac{4}{3}\frac{b_0^{3/2}}{b_2^{1/2}} = \frac{4}{3}\frac{\epsilon_1^{3/2}}{\alpha_1 B_{11}^{1/2}} \qquad (Q_a' \gg 1). \tag{3.28}$$

In this approximation $\overline{\tau'}(z,z_b)$ is independent of z and of z_b.

The argument of the exponential Q_a' in the expression for the probability $W_{\text{m.s.}}$ of leaving the metastable state varies with the distance to the bifurcation point (along the axis ϵ_1) like $\epsilon_1^{3/2}$, and with the fluctuation intensity like $1/\alpha_1$ (the argument of the exponential in Eq. (3.14) is also inversely proportional to the intensity of the random-force fluctuations).

The parameters ϵ_1, B_{11} in Eqs. (3.22) are expressed in simple fashion in terms of the set of parameters $\mathbf{c} = \{c_i\}$, which enter in the function $P_i(\mathbf{c},\mathbf{x})$ in the initial equations (3.21) (before the latter are reduced to standard form (3.22)). The bifurcation values $c_i = c_{iB}$ are determined from the condition $\det\|P_{ij}\| = 0$, where $P_{ij} = (\partial P_i/\partial x_j)_{\mathbf{x}=\mathbf{x}_0}$ and $\mathbf{x}_0$ are the coordinates of the stationary point. The quantities $-\mathscr{A}_i$ in Eq. (3.22) are the eigenvalues of the matrix $\|P_{ij}\|$, with $\mathscr{A}_1(\mathbf{c}_B) = 0$ (some $\mathscr{A}_i$ can be complex). We denote by $\hat{\Lambda}$ the matrix that diagonalizes $\hat{P} = \|P_{ij}\|$ at $\mathbf{c} = \mathbf{c}_B$:

$$(\hat{\Lambda}\hat{P}\hat{\Lambda}^{-1})_{ij} = -\mathscr{A}_i \delta_{i,j}.$$

Expanding the functions P_i in Eq. (3.21) in powers of $\mathbf{c} - \mathbf{c}_B$ and $\mathbf{x} - \mathbf{x}_{0B}$ ($\mathbf{x}_{0B}$ is the value of $\mathbf{x}_0$ at $\mathbf{c} = \mathbf{c}_B$) and transforming from the variables $\mathbf{x}$ to new variables $\hat{\Lambda}(\mathbf{x} - \mathbf{x}_{0B})$ (which coincide, accurate to ϵ_1, with the variables in Eq. (3.22)), we obtain

$$\epsilon_1 = \sum_j \Lambda_{1j} \frac{\partial P_j(\mathbf{c},\mathbf{x})}{\partial \mathbf{c}} (\mathbf{c} - \mathbf{c}_B),$$
$$B_{11} = -\frac{1}{2}\sum_{m,n,j} \Lambda_{1j}\Lambda_{n1}^{-1}\Lambda_{m1}^{-1}\frac{\partial^2 P_j}{\partial x_n \partial x_m}, \tag{3.29}$$
$$\alpha_1 \equiv \alpha_{11} = \sum_{i,j} \Lambda_{1i}\alpha_{ij}(\mathbf{x}_{0B},\mathbf{x}_{0B})\Lambda_{1j},$$

where the derivatives are calculated at $\mathbf{x} = \mathbf{x}_{0B}$ and $\mathbf{c} = \mathbf{c}_B$.

Equations (3.28) and (3.29) yield a simple expression for the probability of escape from the metastable state in the general case of a multidimensional and a multiparameter system (situated, however, near the bifurcation curve in the parameter space). An example of the application of these formulas to a nonlinear Duffing oscillator will be presented in Sec. 3.7 below.

3.6 *Fluctuations Near the Coalescence Point of Two Stable States of a Dynamic System*

Near the spinode point on the bifurcation curve, at which the stable states coalesce (the point K in Fig. 10), the coefficient B_{11} in Eq. (3.22) tends to zero. Accordingly, the right-hand sides of the equations of motion (3.21) at $\mathbf{c} = \mathbf{c}_K$ (the values of the components of $\mathbf{c}_K$ correspond to the point K) and at stationary values of the variables $\mathbf{x}$ (which are assumed equal to zero at the point K) satisfy, after the variables are transformed in a way similar to the one used above, the relations

$$(P_i)_K = 0, \qquad \left(\frac{\partial P_i}{\partial x_j}\right)_K = -\mathscr{A}_i \delta_{ij}, \qquad \mathscr{A}_1 = 0,$$
$$\operatorname{Re}\mathscr{A}_{i>1} > 0, \qquad \left(\frac{\partial^2 P_1}{\partial x_1^2}\right)_K = 0. \tag{3.30}$$

Near the spinode point K, just as in the case considered in the preceding section, the motion of the system can be divided into fast, with a characteristic time $\sim\min(\mathscr{A}_i'^{-1})$, and slow. In the region of long times the slow motion is one-dimensional and, as shown in Appendix E, is described by a one-dimensional EFP equation. According to Eq. (E.10), this equation is of the form

$$\frac{\partial w}{\partial \tau'} = \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial z} w\right) + \frac{\partial^2 w}{\partial z^2}, \quad U(z) = z\left(\frac{1}{4}d_3 z^3 - \frac{1}{2}d_1 z - d_0\right), \tag{3.31}$$
$$z = \alpha_1^{-1/4} x_1, \qquad \tau' = \alpha_1^{1/2} t, \qquad d_3 > 0$$

(expressions that relate d_0, d_1, and d_3 to the values of the derivatives of the functions $P_i(\mathbf{c}, \mathbf{x})$ at the point K and to the distance to the point K in the space of the parameters $\mathbf{c}$ are given in Eq. (E.11)). The error due to the use of the adiabatic approximation at the point K is of the order of $\alpha_1^{1/4}$.

At sufficiently long times, the probability distribution tends to a stationary solution $w_{st}(z)$ of Eq. (3.31):

$$w_{st}(z) = Z^{-1}\exp(-U(z)), \qquad Z = \int_{-\infty}^{\infty} dz \exp(-U(z)). \quad (3.32)$$

Depending on the relation between the parameters d_0, d_1, and d_3 the potential $U(z)$ in Eq. (3.32) can have one or two minima corresponding to one or two stable equilibrium positions of the system in the absence of a random force. The conditions for the existence of two minima of $U(z)$ (separated by a region in which $U(z)$ has a local maximum corresponding to a saddle point) are the inequalities

$$d_1 > 0, \qquad |\Delta| \leqslant 1, \qquad \Delta = \frac{3(3d_3)^{1/2}d_0}{2d_1^{3/2}}. \quad (3.33)$$

The region of values of $\mathbf{c}$ where the inequalities (3.33) are satisfied is bounded by the curves in Fig. 11. In this region, $w_{st}(z)$ has two maxima corresponding to the minima of $U(z)$, and outside this region it has one maximum.

The character of the fluctuations in the system depends essentially on the distance to the point K, which determines the depths of the minima of the potential $U(z)$. At $d_1 \gg d_3^{1/2}$ the minima of $U(z)$ are deep, and the maxima of $w_{st}(z)$ are exponentially sharp respectively. In this case, if $|\Delta| < 1$, the system stays relatively long near one of the stationary states, experiencing small fluctuations, and only rarely goes over to another state. The ratio of the populations of the states is in this case, with logarithmic accuracy,

$$\frac{w_1}{w_2} = \exp \delta U, \qquad \delta U = U_2 - U_1,$$
$$U_{1,2} \equiv U(z_{1,2}), \qquad \left(\frac{\partial U}{\partial z}\right)_{z=z_{1,2}} = 0. \quad (3.34)$$

It is convenient, taking Eq. (3.34) into account, to represent the

potential difference δU in the form

$$\delta U = \delta\tilde{U} d_1^2 d_3^{-1},$$

where $\delta\tilde{U} = \delta\tilde{U}(\Delta)$ is a function of a single parameter Δ, and

$$\delta\tilde{U} = \frac{4}{3\sqrt{3}}\Delta \qquad \text{at} \quad |\Delta| \ll 1,$$
$$\delta\tilde{U} = \frac{3}{4}\frac{\Delta}{|\Delta|}\left[1 - \frac{8}{9}(1 - |\Delta|)\right] \qquad \text{at} \quad 1 - |\Delta| \ll 1. \quad (3.35)$$

We have assumed here $z_1 > z_2$. It is seen from Eq. (E.11) that Δ does not depend on α_1, while $\delta U \propto \alpha_1^{-1}$.

The probability W_{12} of the transition from the state 1 into 2 can be determined with logarithmic accuracy in the same way as in Sec. 3.5:

$$W_{12} = \text{const} \times \exp(-Q'_{a1})$$
$$Q'_{a1} = U_s - U_1 = \frac{d_1^2}{d_3}\delta\tilde{U}_{1s}(\Delta), \qquad U_s \equiv U(z_s). \quad (3.36)$$

The position of the saddle point z_s is defined here as the middle root of the equation $\partial U/\partial z = 0$. The function $\delta U_{1s}(\Delta)$, just as $\delta\tilde{U}(\Delta)$, is a function of one parameter Δ and takes in the limiting cases the form

$$\delta\tilde{U}_{1s}(\Delta) = \frac{1}{4} + \frac{2}{3\sqrt{3}}\Delta \qquad \text{at} \quad |\Delta| \ll 1,$$
$$\delta\tilde{U}_{1s}(\Delta) = \frac{3}{4}\left[1 - \frac{8}{9}(1 - \Delta)\right] \qquad \text{at} \quad 1 - \Delta \ll 1, \quad (3.37)$$
$$\delta\tilde{U}_{1s}(\Delta) = \frac{16}{27\sqrt{6}}(1 + \Delta)^{3/2} \qquad \text{at} \quad 1 + \Delta \ll 1.$$

The argument of the exponential in the expression for the transition probability W_{21} can also be represented in the form (3.36), and it is clearly from symmetry arguments that $\delta\tilde{U}_{2s}(\Delta) = \delta U_{1s}(-\Delta)$. At $\Delta = 0$ the probabilities W_{12} and W_{21} are equal. It is obvious that $W_{12}/W_{21} = w_2/w_1$. Since the problem has been reduced to one-dimensional and the motion is potential, expression (3.36) for W_{12} (just as (3.28)) is

of the same style as the expression for the probability of the escape of a Brownian particle from a potential well, given in the well known paper of Kramers [36].

In the region of the values of the parameters $d_1 \lesssim d_3^{1/2}$ the maxima of the function $w_{st}(z)$ cease to be sharp, and the concept of the probability of a transition between equilibrium positions becomes meaningless. In this case the dynamics of the system in the region of long times $\tau' \gg \mathscr{A}_i'^{-1}$ is characterized by the time correlation functions of the type $\langle z(\tau')z(0)\rangle$ for the slow variable $z = \alpha_1^{-1/4}x_1$. It is convenient to determine such correlators by starting from Eq. (3.31). Carrying out in Eq. (3.31) the standard separation of the variables (see, e.g., [41, 48, 49])

$$w(z,\tau') = \sum_n C_n e^{-\lambda_n \tau'} \exp\left[-\frac{1}{2}U(z)\right]\psi_n(z), \tag{3.38}$$

we obtain an eigenvalue equation for the functions $\psi_n(z)$:

$$\frac{d^2\psi_n}{dz^2} + \left[\frac{1}{2}\frac{d^2U}{dz^2} - \frac{1}{4}\left(\frac{dU}{dz}\right)^2\right]\psi_n + \lambda_n\psi_n = 0. \tag{3.39}$$

This is a Schrödinger equation with a potential that presents, according to Eq. (3.31), a polynomial of sixth degree in z (cf. [49]). The lowest eigenvalue in Eq (3.39) is $\lambda_0 = 0$. It corresponds to the eigenfunction $\psi_0(z) \propto \exp[-(1/2)U(z)]$, which has no nodes (the stationary distribution $w_{st}(z) \propto \psi_0^2(z)$).

The behavior of the time correlation functions at large τ' is determined by the lowest nonzero eigenvalue λ_1:

$$\langle z(\tau')z(0)\rangle \propto \exp(-\lambda_1\tau') = \exp(-\lambda_1\alpha_1^{1/2}t), \qquad \tau' \gg 1.$$

The dependence of $\lambda_1 d_3^{-1/2}$ on the parameters $d_1 d_3^{-1/2}$ and $d_0 d_3^{-1/4}$ was determined by numerically solving Eq. (3.39). The results are shown in Fig. 14 (the sign of the quantity plotted as abscissa in the analogous figure 3 of [46] must be reversed). In particular, at the spinode point itself, i.e. at $d_0 = d_1 = 0$, we have

$$\lambda_1 \approx 1.37 d_3^{1/2}, \qquad \langle z(t)z(0)\rangle \propto \exp\left[-1.37(d_3\alpha_1)^{1/2}t\right], \quad (d_3\alpha_1)^{1/2}t \gg 1. \tag{3.40}$$

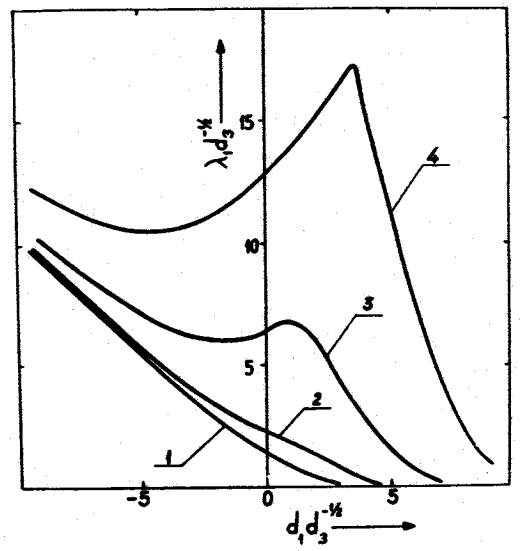

Figure 14 Dependence of the decrement λ_1 on the parameter $d_1 d_3^{-1/2}$. Curves 1–4 correspond to the values of the parameter $d_0 d_3^{-1/4} = 0$, 2, 5, and 10.

We note that at large $|d_0|d_3^{-1/4}$ the dependence of $\lambda_1 d_3^{-1/2}$ on $d_1 d_3^{-1/2}$ has a rather sharp maximum and a relatively smooth minimum. The maximum is located at positive d_1 in the region where the dynamic system has one stationary state. Analytic expressions for λ_1 can be easily obtained at large $-d_1 d_3^{-1/2}$ or at large $|d_0|d_3^{-1/4}$. In both cases, the potential in the Schrödinger equation (3.39) has one deep minimum and is described near the minimum by a parabola. As a result of a simple calculation we obtain for the energy of the first excited level

$$\lambda_1 = -d_1 + 3d_1^{-1}d_3 + 3d_0^2 d_1^{-2}d_3 \quad \text{at} \quad (-d_1 d_3^{-1/2})^{3/2} \gg |d_0|d_3^{-1/4}, \quad -d_1 d_3^{-1/2} \gg 1,$$

$$\lambda_1 = 3d_0^{2/3}d_3^{1/3} + d_1 \quad \text{at} \quad |d_0|d_3^{-1/4} \gg |d_1 d_3^{-1/2}|^{3/2}, \quad |d_0|d_3^{-1/4} \gg 1.$$

It is seen from these expressions that in the region of negative $d_1 d_3^{-1/2}$ with large absolute values the curves in Fig. 14 take the form of the straight line $\lambda_1 d_3^{-1/2} = -d_1 d_3^{-1/2}$, and at large $|d_0|d_3^{-1/4}$ and small $|d_1 d_3^{-1/2}|$ the dependence of λ_1 on d_1 has a positive slope.

3.7 *Fluctuations Near the Bifurcation Points of a Nonlinear Duffing Oscillator in a Resonant Field*

3.7.1 Marginal Points The bifurcation of a Duffing oscillator in a resonant field, as is clear from the results of Sec. 3.2, pertain to the types considered in Secs. 3.5 and 3.6. When the lines $\beta_B(\tilde{\Omega})$ described by Eq. (3.9) are intersected on the $(\beta, \tilde{\Omega})$ plane (see Fig. 11), a new stable and unstable equilibrium state appear (or vanish) (bifurcations of codemension 1, see [44]). At the point K, the three equilibrium positions coalesce (bifurcation of codimension 2).

Since bifurcations arise only at $\gamma\Omega > 0$, this inequality is assumed hereafter to be satisfied and we put for convenience $\tilde{\Omega} > 0$ and $h > 0$. The equations of motion (3.5) of the considered Duffing oscillator, now expressed in terms of real variables

$$\frac{du'}{d\tau} = \frac{u'}{\tilde{\Omega}} - u''(|u|^2 - 1) + \frac{1}{\tilde{\Omega}}\tilde{f}'(\tau),$$

$$\frac{du''}{d\tau} = -\frac{u''}{\tilde{\Omega}} + u'(|u|^2 - 1) - \beta^{1/2} + \frac{1}{\tilde{\Omega}}\tilde{f}''(\tau), \qquad (3.41)$$

$$\tau = |\delta\omega|t, \qquad \tau > 0,$$

are similar in form to Eqs. (3.21). The stationary values of u' and u'' at the bifurcation values of the parameters β and $\tilde{\Omega}$ are

$$u'_{0B} = \beta_B^{-1/2} p(p-1), \qquad u''_{0B} = -\beta_B^{-1/2}\tilde{\Omega}^{-1}p,$$

$$p \equiv (u'_{0B})^2 + (u''_{0B})^2 = (1 - 3\tilde{\Omega}^{-2})^{-1}\left(1 + \tilde{\Omega}^{-2} - \frac{9}{2}\beta_B\right), \qquad (3.42)$$

$$\beta_B \equiv \beta_B^{(1,2)}(\tilde{\Omega})$$

(the parameters β_B and $\tilde{\Omega}$ are connected by relation (3.9) and lie on the solid curves of Fig. 11).

It is easy to verify that near the bifurcation points the variable u'' in Eq. (3.41) describes the slow motion: the terms linear in $(u' - u'_{0B})$ and $(u'' - u''_{0B})$ drop out of the equation for $\dot{u}''$ (this is the consequence of the choice of the phase of the external field in Eq. (3.1)). It is therefore easy to reduce Eqs. (3.41) far from the point K to the

standard form (3.22) with coefficients

$$\epsilon_1 = \epsilon_1(\beta, \tilde{\Omega}) = \frac{\beta - \beta_B(\tilde{\Omega})}{2[\beta_B(\tilde{\Omega})]^{1/2}},$$

$$B_{11} = p[\beta_B(\tilde{\Omega})]^{-1/2}[5p - 3 + 3(2p-1)^2(p-1)\tilde{\Omega}^2]. \qquad (3.43)$$

Substituting expression (3.43) in Eq. (3.28) and recognizing that in the considered case $\alpha_1 = \alpha\tilde{\Omega}^{-2}$ in accordance with Eqs. (3.6), (3.21), and (3.24), we find that the argument Q'_a of the exponential in the expression for the probability of the escape of the nonlinear oscillator from the metastable equilibrium state is of the form

$$Q'_a = \frac{G_a(\tilde{\Omega})}{\alpha}|\beta - \beta_B(\tilde{\Omega})|^{3/2},$$

$$G_a(\tilde{\Omega}) = \frac{\sqrt{2}}{3}\frac{\tilde{\Omega}^2}{|B_{11}(\tilde{\Omega})|^{1/2}[\beta_B(\tilde{\Omega})]^{3/4}}. \qquad (3.44)$$

The dependence of the coefficient $G_a(\tilde{\Omega})$ on the dimensionless frequency detuning is shown in Fig. 15, where curves 1 and 2 correspond

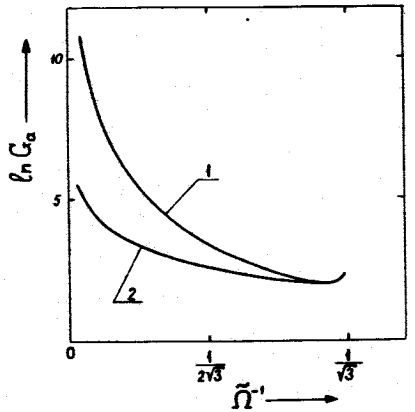

Figure 15 Dependence of the coefficient $G_a(\Omega)$ in the expression for the Duffing-oscillator transition probability (from the metastable state) near the bifurcation point on the dimensionless frequency detuning. Curves 1 and 2 correspond to the lower and upper curves of Fig. 11, respectively.

respectively to the function $\beta_B^{(1)}(\tilde{\Omega})$ and $\beta_B^{(2)}(\tilde{\Omega})$ (3.9). At small $\tilde{\Omega}^{-1}$ we have asymptotically $G_a^{(1)}(\tilde{\Omega}) \approx (2/3)\tilde{\Omega}^3$ and $G_a^{(2)}(\tilde{\Omega}) \approx 4.5\tilde{\Omega}$ (expression (3.44) coincides at $\tilde{\Omega} \gg 1$ with the results obtained in [29] by an essentially different method). With increasing $\tilde{\Omega}^{-1}$, the functions $G_a^{(1)}(\tilde{\Omega})$ and $G_a^{(2)}(\tilde{\Omega})$ first decrease sharply and approach each other. In the immediate vicinity of the spinode point K ($\tilde{\Omega}^{-1} = 1/\sqrt{3}$) they increase, owing to the vanishing of the coefficient $B_{11}(\tilde{\Omega})$ at $\tilde{\Omega}^{-1} = 1/\sqrt{3}$.

3.7.2 Spinode Point To consider the motion of a nonlinear oscillator at values of β and $\tilde{\Omega}$ in the vicinity of the point K on Fig. 11 ($\beta_K = 8/27$, $\tilde{\Omega}_K = \sqrt{3}$) it is convenient to transform in Eq. (3.41) from the variables u' and u'' to the new variables x_1 and x_2 introduced above in Eqs. (3.22) and (3.30) for the description of the slow and fast motion, respectively, in the vicinity of the spinode point

$$x_1 = u'' - u_K'', \qquad x_2 = u' - u_K' + \frac{1}{\sqrt{3}}(u'' - u_K''),$$
$$u_K' = -\frac{1}{\sqrt{6}}, \qquad u_K'' = -\frac{1}{\sqrt{2}}. \tag{3.45}$$

Neglecting in the equation obtained in this manner for x_1 at $\tau \gg 1$ the small terms $\sim x_2^2$ and $x_1^2 x_2$ ($|x_2| \ll |x_1|$), we can rewrite this equation in the form

$$\frac{dx_1}{d\tau} = -d_3 x_1^3 + \tilde{d}_1 x_1 + \tilde{d}_0 + \frac{1}{\sqrt{3}}\tilde{f}''(\tau),$$
$$\tilde{d}_0 = -\frac{9\sqrt{3}(\beta - \beta_K) + 4(\tilde{\Omega} - \tilde{\Omega}_K)}{12\sqrt{2}}, \tag{3.46}$$
$$\tilde{d}_1 = \frac{\tilde{\Omega} - \tilde{\Omega}_K}{3}, \qquad d_3 = \frac{4}{3\sqrt{3}}.$$

The EFP equation corresponding to (3.46) is of the same form as Eq.

(3.31) in which, taking (3.6) into account

$$d_0 = \left(\frac{3}{\alpha}\right)^{3/4}\tilde{d}_0 = -\frac{9\sqrt{3}(\beta - \beta_K) + 4(\tilde{\Omega} - \tilde{\Omega}_K)}{2^{5/2}3^{1/4}\alpha^{3/4}},$$
$$d_1 = \sqrt{\frac{3}{\alpha}}\,\tilde{d}_1 = \frac{\tilde{\Omega} - \tilde{\Omega}_K}{(3\alpha)^{1/2}}. \tag{3.47}$$

Thus, the fluctuations of the oscillator in the vicinity of the point K of Fig. 11 are described by Eqs. (3.34)–(3.37) (see also Fig. 14), which were obtained for the general case of systems situated near a spinode point. The dependences of the parameters Δ, d_1, and d_3 in these equations on the characteristics of the dynamic subsystem, i.e., on the frequency detuning and the amplitude of the resonant field in the concrete case of a Duffing oscillator, are determined by Eqs. (3.33), (3.46), and (3.47). Expressions (3.36), (3.37), and (3.47) for the probability of the transitions between the stable states of the Duffing oscillator could have been derived also using the independent method [29], which is described in Secs. 3.3 and 3.4 (and is not connected with the proximity to the bifurcation point), and solving explicitly Eqs. (3.17) for the vicinity of the point K.

As the point K is approached, the characteristic activation energy of the transitions $1 \to 2$ or $2 \to 1$ between the oscillator states, as can be seen from Eqs. (3.36) and (3.47), decreases like $(\tilde{\Omega} - \tilde{\Omega}_K)^2$. The line $\beta_0(\tilde{\Omega})$ (in the space of the parameters β and $\tilde{\Omega}$), where the populations of the stationary states w_1 and w_2 are equal, is determined in the vicinity of the point K from the condition $\Delta = 0$, i.e., $d_0 = 0$ (see Eq. (3.33)), whence

$$\beta_0(\tilde{\Omega}) \approx \beta_K\left[1 - \frac{\sqrt{3}}{2}(\tilde{\Omega} - \tilde{\Omega}_K)\right].$$

Using this result for the segment of the curve $\beta_0(\tilde{\Omega}^{-2})$ adjacent to $\tilde{\Omega} = \tilde{\Omega}_K$ and the value of β_0 obtained in Sec. 3.4 at $\tilde{\Omega}^{-1} \to 0$, and taking into account the monotonic character of the dependence of β_0 on $\tilde{\Omega}^{-2}$ (which is obvious from qualitative arguments), we can obtain the interpolation curve for $\beta_0(\tilde{\Omega}^{-2})$ shown dashed in Fig. 11.

3.8 *Features of Resonant-Field Absorption*

A nonlinear Duffing oscillator in a resonant field was recently investigated in connection with nonlinear-optics problems. For example, using the Duffing-oscillator model to describe the medium, it is possible to analyze within a unified approach optical bistability, phase conjugation in reflection from a nonlinear medium, and a number of other effects [50]. In the context of optics problems, it is of interest to calculate the absorption of intense radiation by a nonlinear medium, and also to calculate the response of a nonlinear nonequilibrium medium to an additional weak signal. Analysis of the absorption of a strong and an additional weak field can be applied also to other systems simulated by a Duffing oscillator.

Without account of the fluctuations, when the dependence of the oscillator amplitude on the field becomes S-shaped (see Fig. 2), the field dependence of the absorption coefficients exhibits hysteresis. When account is taken of the fluctuations, the hysteresis vanishes (in the quasistationary regime, i.e., at a sufficiently long observation time compared with the characteristic time $W_{12}^{-1} + W_{21}^{-1}$ within which the distribution over the equilibrium states f_1 and f_2 is established). Although the dynamic characteristics of the oscillator become single-valued, they have peculiar features when the relation between the field h and the frequency detuning $\delta\omega$ is such that $\beta \approx \beta_0(\tilde{\Omega})$, i.e., when the probabilities w_1 and w_2 of the occupation of the stable states are comparable.

3.8.1 Strong-Field Absorption We consider first the absorption of a strong resonant field by a Duffing oscillator. The power I drawn from the field, upon averaging over the period $2\pi\omega^{-1}$ and over the realizations of the random force, is given by

$$I = \left\langle \frac{dq}{dt} h \cos \omega t \right\rangle = -\omega h \sum_{i=1}^{2} w_i \operatorname{Im} \tilde{u}_{1f_i} \tag{3.48}$$

Equation (3.48) is valid in the case of sufficiently weak random forces, when the fluctuational variations of $\tilde{u}_1$ about the stable stationary values $\tilde{u}_{1f_i}$ are in the main small (compared with $|\tilde{u}_{1f_i}|$). A small fluctuational straggling of the amplitude near the foci $f_{1,2}$ necessitates corrections $\sim\alpha$ to Eq. (3.48). These corrections are of a type similar

to those considered in Sec. 2 for the equilibrium oscillator (there, however, α was not assumed small and the corrections were substantial, whereas here they can be disregarded). The populations of the states w_i have a non-analytic dependence on α (like $\exp(Q_{ai}/\alpha)$, so that perturbation theory in α cannot be used to calculate w_i and the special methods considered in Secs. 3.3, 3.4, and 3.7 must be used.

Equating $v(u, u^*)$ in Eq. (3.5) to zero we can, taking Eq. (3.4) into account, express $\tilde{u}_{1f_i}$ in terms of $|u_{f_i}|^2$, where $|u_{f_i}|^2$ is the smallest (at $i = 1$) and largest ($i = 2$) root of the cubic equation (3.8) (the intermediate root corresponds to a saddle point). After this we obtain for the quantity $\mu = 2I/h^2$, which determines the absorption coefficent,

$$\mu = \frac{\Gamma}{2\beta|\delta\omega|^2}\left(w_1|u_{f_1}|^2 + w_2|u_{f_2}|^2\right). \tag{3.49}$$

At small h and accordingly at small β we have $w_1 \gg w_2$ (cf. Eqs. (3.19) and (3.20)) and

$$\mu = \frac{\Gamma}{2\beta|\delta\omega|^2}|u_{f_1}|^2, \qquad \beta < \beta_0$$

(at $\beta < \beta_B^{(1)}(\tilde{\Omega})$ there exists on the whole only one equilibrium state). Using Eq. (3.8), we easily verify that $|u_{f_1}|^2\beta^{-1}$ increases monotonically with increasing β. Therefore μ also increases monotonically and smoothly at $\beta < \beta_0$. In the region $\beta \approx \beta_0$, at a relatively small change (proportional to the small parameter α) of the parameters h or $\delta\omega$, the expression in the parentheses in Eq. (3.49) changes abruptly (at an exponential rate in accordance with Eq. (3.20)) from $|u_{f_1}|^2$ to $|u_{f_2}|^2$ or vice versa (when h is increased and $\delta\omega$ is fixed, the transition is precisely from $|u_{f_1}|^2$ to $|u_{f_2}|^2$). A somewhat blurred kink appears therefore on the plot of $\mu(h)$ in the vicinity of $\beta \approx \beta_0$ (see Fig. 16). To the right of the kink

$$\mu = \frac{\Gamma}{2\beta|\delta\omega|^2}|u_{f_2}|^2, \qquad \beta > \beta_0.$$

At $\beta_0 > \tilde{\Omega}^{-2}$ the value of μ decreases with increasing β to the right of the kink because of the decrease of $|u_{f_2}|^2\beta^{-1}$.

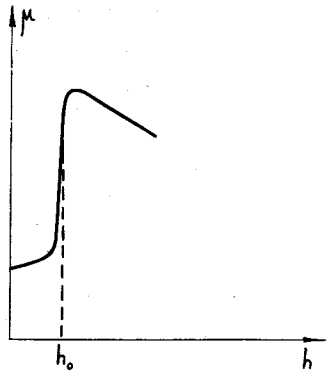

Figure 16 Schematic representation of the dependence of the strong-field absorption coefficient μ on the field amplitude h. At $h = h_0$ the populations of the stationary states of the Duffing oscillator are equal.

3.8.2 Absorption of Additional Weak Field Besides the investigation of the absorption of a strong field at frequency ω, it may be of interest to study the absorption of an additional weak field h' at another frequency ω'. To determine the corresponding quantity $\mu' = 2I'/h'^2$ in the resonance region $|\omega' - \omega| \sim |\omega - \omega_0|$ we can include in Eq. (3.2) a term $h'\cos\omega' t$ and solve Eq. (3.3) (in the absence of the random force) with account taken of the term $\propto h'$, by linearizing this equation with respect to $\delta\tilde{u}_1 \propto h'$. Then

$$\mu' = \sum_{i=1}^{2} w_i \mu_i', \qquad \mu_i' = \frac{1}{2\Gamma} \frac{1 + \left(\tilde{\Omega} - \tilde{\Omega}' - 2\tilde{\Omega}|u_{f_i}|^2\right)^2 - \tilde{\Omega}^2|u_{f_i}|^4}{\left(\nu_i^2 - \tilde{\Omega}'^2\right)^2 + 4\tilde{\Omega}'^2}, \tag{3.50}$$

$$\tilde{\Omega}' = \frac{\omega' - \omega}{\Gamma}, \qquad \nu_i^2 = 1 + \tilde{\Omega}^2\left(3|u_{f_i}|^4 - 4|u_{f_i}|^2 + 1\right) \qquad (\tilde{\Omega} > 0).$$

According to expressions (3.50), the spectrum $\mu'(\tilde{\Omega}')$ consists of two partial spectra μ_i', whose intensities are comparable at $\beta = \beta_0$. The structure of the spectra $\mu_i'(\tilde{\Omega}')$ is determined by the values of $|u_{f_i}|^2$. At

$$\tilde{\Omega}|u_{f_i}|^2 > 1$$

the spectrum $\mu_i'(\tilde{\Omega}')$ has two maxima (which are generally speaking broad), between which there exists a region in which $\mu_i' < 0$, i.e., the weak field is enhanced by the strong one (but the absorption integrated over the frequency ω' is positive and is equal to $\pi/2$).

Thus, at a strong-field amplitude h corresponding to $\beta \approx \beta_0$ we can, by slightly changing h, not only alter greatly the absorption coefficient μ' (by going from the region $\beta < \beta_0$ to $\beta > \beta_0$, i.e., from $\mu' \approx \mu_1'$ to $\mu' \approx \mu_2'$), but even reverse the sign of μ', i.e., transform from absorption to enhancement of the weak field. If $|u_{f_i}|^2\tilde{\Omega} \gg 1$ (in this case $\nu_i^2 \gg 1$), the function μ_i' has relatively narrow peaks at the frequencies $\tilde{\Omega}' \approx \pm\nu_i$. The widths of these peaks are ~ 1 (of the order of Γ in frequency units), and the distance between them is considerably larger ($\sim\nu_i$). One of the peaks corresponds to absorption and the other to enhancement of the additional weak field.

3.8.3 Narrow Absorption Peak of Additional Weak Field In the region $|\omega' - \omega| \sim W_{21} + W_{12}$ there is produced, besides the absorption (or enhancement) μ', which is described by Eq. (3.50) and is due to the small-amplitude oscillations about the equilibrium positions $f_{1,2}$, induced by weak field h', also an additional absorption $\tilde{\mu}'$ due to the field-induced redistribution over the states $f_{1,2}$. This redistribution can be easily calculated in the case

$$|\omega' - \omega| \ll \Gamma, \qquad |\delta\omega|. \tag{3.51}$$

In the absence of an additional weak field, the populations of the stable states w_1 and w_2 satisfy the balance equation

$$\frac{dw_1}{dt} = -W_{12}w_1 + W_{21}w_2, \qquad w_1 + w_2 = 1. \tag{3.52}$$

The transition probabilities W depend on the strong-field amplitude h as a parameter (obviously, W is independent of the rapidly changing phase of the strong field). If the condition (3.51) is satisfied, the presence of the additional weak field can be interpreted as slow time modulation of the strong field: h is replaced in Eq. (3.3) by

$$\tilde{h}(t) = h + h'\exp\left[i(\omega' - \omega)t\right].$$

Since the duration of the transition process is $\sim|\delta\omega|^{-1}$ or $\sim\Gamma^{-1}$ (~ 1 in the scale of the dimensionless time τ), the field $\tilde{h}(t)$ can be regarded as constant during the time of the transition itself if (3.51) is satisfied. However, the time dependence of $\tilde{h}(t)$ leads to a parametric dependence of W on the time. Recognizing that W is the function of only $|h|$ (i.e., $\beta^{\frac{1}{2}}$), we obtain in first order in h'

$$W_{ij}(t) = W_{ij} + 2\frac{\partial W_{ij}}{\partial\beta}\sqrt{\beta\beta'}\cos(\omega' - \omega)t,$$
$$W_{ij} \equiv W_{ij}(\beta), \qquad \beta' = \frac{3|\gamma|h'^2}{32\omega^3|\delta\omega|^3}. \tag{3.53}$$

The dependence of the transition probability on the time leads to time-dependent addition to the populations w_1 and w_2. In first order in h' it follows from Eqs. (3.52) and (3.53) that

$$w_1 = w_1^{(0)} + w_1', \qquad w_1^{(0)} = \frac{W_{21}}{W_{12} + W_{21}}, \tag{3.54}$$

$$w_1' = -2\sqrt{\beta\beta'}\,\frac{W_{21}^2}{W_{12} + W_{21}}\,\frac{\partial}{\partial\beta}\left(\frac{W_{12}}{W_{21}}\right)\mathrm{Re}\,\frac{\exp[i(\omega' - \omega)t]}{W_{12} + W_{21} + i(\omega' - \omega)}.$$

Expression (3.54) can be transformed by taking into account the relation (3.14) $W_{ij} \propto \exp(-Q_{ai}/\alpha)$ (Q_{ai} is the activation energy of the transition from the state i). Since $Q_{ai} \gg \alpha$, we have

$$\frac{\partial W_{ij}}{\partial\beta} \approx -\frac{1}{\alpha}\frac{\partial Q_{ai}}{\partial\beta}W_{ij}$$

and

$$w_1' = 2\frac{\sqrt{\beta\beta'}}{\alpha}\,\frac{\partial}{\partial\beta}(Q_{a1} - Q_{a2})\frac{W_{12}W_{21}}{W_{12} + W_{21}}\,\mathrm{Re}\,\frac{\exp[i(\omega' - \omega)t]}{W_{12} + W_{21} + i(\omega' - \omega)}. \tag{3.55}$$

The expression for the power $\tilde{I}'$ drawn from the weak field on account of the redistribution over the oscillator states, which is induced by this field, may be easily shown to be

$$\tilde{I}' = -h'\omega\,\mathrm{Im}\sum_{i=1}^{2}(\tilde{u}_1)_{fi}[w_1'(t)]_{\omega'}\exp[-i(\omega' - \omega)t] \tag{3.56}$$

(the brackets $[\ldots]_{\omega'}$ indicate that it is necessary to retain in the corresponding quantity only the term $\propto \exp[i(\omega' - \omega)t]$). Taking Eqs. (3.54) and (3.55) into account, we find, that in addition $\tilde{\mu}'$ to the absorption coefficient is

$$\tilde{\mu}' = -\frac{\sqrt{\beta}}{2\alpha|\delta\omega|}\,\frac{W_{12}W_{21}}{W_{12} + W_{21}}\,\frac{\partial}{\partial\beta}(Q_{a1} - Q_{a2})\mathrm{Im}\,\frac{u_{f_1} - u_{f_2}}{W_{12} + W_{21} + i(\omega' - \omega)} \tag{3.57}$$

The absorption $\tilde{\mu}'(\omega')$ has a very sharp maximum (with a width $\sim(W_{12} + W_{21})$, that is much smaller than the characteristic width Γ) in the region $\tilde{\Omega}' = 0$ ($|\omega' - \omega| \ll \Gamma$). In the vicinity of the point $\beta \approx \beta_0$, where $W_{12} \approx W_{21}$, the height of the peak of $\tilde{\mu}'$ is independent of W and is proportional to the large quantity Q_a/α, i.e., it exceeds the absorption (3.50). Outside the vicinity of the point $\beta = \beta_0$, the height of the peak of $\tilde{\mu}'$ is proportional to the ratio of the smaller transition probability W_{ij} to the larger, i.e., it is exponentially small. The presence of such a specific absorption peak in the vicinity of a "first-order kinetic phase transition" ($\beta \approx \beta_0$) makes it possible to observe the transition itself. The peak is of interest also from the point of view of applications: it makes it possible to compare with high accuracy the frequencies of the strong and weak fields. The frequency error is $\sim W$, and thus can be smaller by many orders of magnitude not only than the frequency ω of the strong field itself, but also than the small damping decrement Γ. If the oscillator Hamiltonian contains a term $\propto q^3$, the average static dipole moments of the stable states are different and an analogous relaxation-type peak occurs at low frequencies

$$\omega' \lesssim W_{12} + W_{21}.$$

§4 Quantum Theory of Nonlinear Oscillator Interacting with a Medium

The classical theory of relaxation of a nonlinear oscillator interacting with a medium, developed in the preceding sections, is obviously applicable only at sufficiently high temperatures. At low temperatures there appear specific quantum effects connected with the discrete character of the energy spectrum. For example, since the levels of the nonlinear oscillator are not equidistant (see Fig. 1), a fine structure appears in the spectral distribution $Q(\omega)$ of the correlator of the coordinates in the case of small damping. The individual lines of the structure correspond to transitions between various neighboring levels.

The standard method of describing relaxation in quantum statistical physics is the method of the quantum kinetic equation (QKE). It is usually employed for systems with a small number of levels, since the number of relaxation parameters increases rapidly with increasing number of levels. An oscillator is in this sense a singular system, since it has an infinite number of levels, while the damping is described by a small number of parameters. Frequently only one parameter is used at all: in the classical limit this parameter corresponds to the coefficient of proportionality of the friction force to the velocity (see Eq. (1.1)).

The relaxation described by linear friction is due to decay processes (see below), in which the oscillator goes over from one level to a neighboring one. Since the energy spectrum of a harmonic oscillator is equidistant the energy change in such transitions is the same for all levels. It is this which allows us to introduce one damping parameter. Since the spectrum is equidistant, the matrix elements $\rho(m,n;t)$ of the density matrix with identical $m-n$ oscillate with the same period (like $\exp[-i\omega_0(m-n)t]$). They are therefore coherent and become mixed with one another in the course of relaxation. This complicates significantly the problem of oscillator compared with the case of a two-level system or a system with an essentially nonequidistant spectrum. At the same time, the total degeneracy of the Bohr frequencies, which is a characteristic of the harmonic oscillator, makes it possible to obtain a simple solution of the problem in the case of linear friction [4, 11, 12].

The anharmonicity of the oscillator makes the energy spectrum nonequidistant. This suppresses partially the mixing of the off-diagonal matrix elements $\rho(m,n;t)$. Obviously, the degree of suppression depends on the relation between the non-equidistance of the energy levels (see Fig. 1) and the reciprocal relaxation time Γ. If $|V| \ll \Gamma$, the matrix elements $\rho(m,m+n)$ (with different m) manage to die down within the time necessary for their phases to become mismatched as a result of anharmonicity, i.e., the anharmonicity has practically no effect and the solution obtained for the harmonic oscillator can be used. If, on the contrary, $|V| \gg \Gamma$, the variation of the phase difference is fast compared with the relaxation (at the same time the frequency ω_0 can exceed $|V|$ considerably, i.e., the nonequidistance of the levels is small), and the damping has the same character as for a system with a strongly non-equidistant spectrum.

At $|V| \sim \Gamma$ the problem differs qualitatively from the completely degenerate problem of the harmonic oscillator and, at the same time, the multilevel character of the system manifests itself strongly. This intermediate case is the most complicated and most interesting. In the general case of arbitrary relation between V and Γ it appears to be possible [21] to obtain for the problem with linear friction, in quantum as well as in classical theory, an explicit expression for the time correlation function $Q(t)$. Its form at $|V| \gtrsim \Gamma$ differs substantially from the simple exponential function that describes the relaxation of a harmonic oscillator.

At $|V| \gg \Gamma$ the spectral distribution $Q(\omega)$ of the time correlation function has a clearly pronounced fine structure. The individual lines (partial spectra) correspond to transitions between different neighboring levels of the oscillator (see Fig. 1), and are spaced in frequency $\approx |V|$ apart. With decreasing $|V|$ the lines overlap. However, a formal representation of the spectral distribution of $Q(\omega)$ in the form of a sum of partial spectra corresponding to the individual transitions turns out to be possible also at $|V| \lesssim \Gamma$, but here the partial spectra are distorted by the interference between the different transitions that make up the spectrum $Q(\omega)$ (corresponding to the interference of the off-diagonal matrix elements of the density matrix).

The character of the oscillator relaxation depends not only on the nonequidistance of the energy levels, but also on the form of the effective friction force determined by the interaction with the medium. If the friction force has a nonlinear dependence on the oscillator velocity, the damping of the correlation function $Q(t)$ becomes nonexponential, and the peak of $Q(\omega)$ has a non-Lorentz shape even if the energy levels are equidistant ($V = 0$). The reason is the depen-

dence of the effective damping decrement on the vibration amplitude in the case of nonlinear friction, as a result of which the decrement changes with changing amplitude, and the damping on the whole is nonexponential. In contrast to the internal nonlinearity of an isolated oscillator, the nonlinearity of the friction does not lead to the onset of a fine structure of the spectrum.

In Sec. 4.1 and in Appendix F below we shall obtain and solve, by the generating functions method, a generalized quantum kinetic equation for a nonlinear oscillator with linear friction. These results are used in Sec. 4.2 to determine its time correlation function, and an analysis of the spectral distribution $Q(\omega)$ is given. The results of Sec. 4.1 are used also in Sec. 4.3 to investigate the effects of interference of transitions, and the paradox of the harmonic oscillator is analyzed from this point of view. In Sec. 4.4 the results of Secs. 4.1–4.3 are generalized to include the case of an oscillator with nonlinear friction; in particular, the role of nonlinear friction is discussed in the classical limit.

4.1 *General Expressions for the Time Correlation Functions of a Quantum Oscillator Interacting with a Medium*

4.1.1 System of Differential-Difference Equations Describing the Relaxation of an Oscillator The Hamiltonian of a Duffing oscillator interacting with a medium, expressed in terms of the creation and annihilation operators $\hat{a}^+$ and $\hat{a}$ for the oscillator oscillations and $\hat{a}_k^+$ and $\hat{a}_k$ for the continuous-spectrum vibrations, takes the form

$$H = \mathscr{H}_0 + H_i, \quad \mathscr{H}_0 = H_0 + H_m, \quad H_0 = \omega_0 \hat{n} + \tfrac{1}{2} V \hat{n}^2, \quad \hat{n} = \hat{a} + \hat{a}, \tag{4.1}$$

$$H_m = \sum_k \omega_k \hat{a}_k^+ \hat{a}_k, \quad H_i = \sum_k \epsilon_k (4\omega_0 \omega_k)^{-1/2} (\hat{a} + \hat{a}^+)(\hat{a}_k + \hat{a}_k^+), \quad \hbar = 1.$$

Here, as in the preceding sections, the medium is simulated by a set of vibrations of a quasi-continuous spectrum. The interaction H_i in Eq. (4.1) is chosen linear both in the coordinates of the Duffing oscillator and in the coordinates of the oscillators of the media (a more general case is considered in Appendix F and in Sec. 4.4).

In quantum theory, the time correlation function of the operators $\hat{L}$ and $\hat{M}$, which depend on the dynamic variables of the singled-out oscillator, is defined by the formula

$$\langle \hat{L}(t)\hat{M}(0)\rangle \equiv Z^{-1}\mathrm{Tr}(e^{iHt}\hat{L}e^{-iHt}\hat{M}e^{-H/T}), \quad Z = \mathrm{Tr}\exp(-H/T). \tag{4.2}$$

The functions of such type, on the one hand, describe the relaxation, and on the other they determine the linear response to external perturbations of various types and can therefore be directly measured in a number of cases. Greatest interest attaches to calculation of the time correlation functions in the region of relatively long times

$$t \gg t_c = \max(\omega_0^{-1}, \omega_e^{-1})$$

(ω_e is the characteristic frequency at which the density of states of the medium, weighted with allowance for the interaction, has a maximum.) The behavior of the correlators in the region $t \gg t_c$ in the considered case $\Gamma t_c \ll 1$ determines the shapes of the peaks of their spectral distributions.

To calculate the correlators, it is convenient to transform to the interaction representation and to introduce the operator

$$\hat{U}(t) = e^{i\mathscr{H}_0 t}e^{-iHt} = T_{t_1}\exp\left[-i\int_0^t dt_1 \tilde{H}_i(t_1)\right], \qquad \tilde{H}_i(t) = e^{i\mathscr{H}_0 t}H_i e^{-i\mathscr{H}_0 t} \tag{4.3}$$

where T_{t_1} is the chronological-ordering operator. Then, recognizing that $\hat{L}$ and $\hat{M}$ act only on the wave functions of the singled-out oscillator, we can rewrite Eq. (4.2) in the form

$$\langle \hat{L}(t)\hat{M}(0)\rangle = \mathrm{Tr}_0\left[e^{iH_0 t}\hat{L}e^{-iH_0 t}F_M(t)\right], \qquad F_M(t) = Z^{-1}\mathrm{Tr}_m\left[\hat{U}(t)\hat{M}\exp(-H/T)\hat{U}^t(t)\right] \tag{4.4}$$

Here Tr_0 and Tr_m are respectively the traces over the wave functions of the isolated oscillator and of the medium.

It is seen from Eq. (4.4) that $F_M(t)$ is an operator in the oscillator wave-function space. This operator describes the oscillator relaxation due to the interaction with the medium, and is a generalization of the density matrix. The elimination of the vibrations of the continuous spectrum as dynamic variables, carried out by introducing the operator $F_M(t)$, is analogous in a certain sense to elimination of the oscillators of the medium in the classical theory (see Sec. 2.1). It is shown in Appendix F that $F_m(t)$ satisfies the quantum kinetic equation (F.14) written in operator form.

For concrete calculations it is convenient to transform from the operator $F_M(t)$ to its matrix elements

$$F_M(n,m;t) = (n|F_M(t)|m), \tag{4.5}$$

taken on the wave functions $|n)$, which are the eigenfunctions of $\hat{n} = \hat{a}^+\hat{a}$ (i.e., the eigenfunctions of the harmonic oscillator). From the operator equation (F14), with allowance for the relation

$$\begin{gathered}(n|\tilde{a}(t)|m) = m^{1/2}\delta_{n+1,m}\exp\left[-i\omega_0 t - iV(m-\tfrac{1}{2})t\right] \\ (\tilde{a}(t) = \exp(i\mathscr{H}_0 t)\hat{a}\exp(-i\mathscr{H}_0 t)),\end{gathered} \tag{4.6}$$

follows a system of differential-difference equations for $F_M(n,m;t)$:

$$\frac{\partial F_M(n,m;t)}{\partial t} = \sum_j d_j(n,m)\exp\left[-iV_j(n-m)t\right]F_M(n+j,m+j;t), \tag{4.7}$$

where

$$\left.\begin{aligned} d_0(n,m) &= -\Gamma(n) - \Gamma(m) - i(n-m)P, \\ d_{\pm 1}(n,m) &= 2(\bar{n} + \tfrac{1}{2} \pm \tfrac{1}{2})\left[(n+\tfrac{1}{2}\pm\tfrac{1}{2})(m+\tfrac{1}{2}\pm\tfrac{1}{2})\right]^{1/2}1 \end{aligned}\right\} \tag{4.8}$$

$$d_j = 0 \qquad \text{at} \quad |j| \geqslant 2 \tag{4.9}$$

$$\Gamma(n) = \left[n(2\bar{n}+1) + \bar{n}\right]\Gamma, \quad \bar{n} = \bar{n}(\omega_0), \quad \bar{n}(\omega) = \left[\exp(\omega/T) - 1\right]^{-1}. \tag{4.10}$$

The expressions for the damping parameter Γ and for the frequency shift P in Eqs. (4.8) and (4.10) coincide with Eq. (2.15) in the considered case of bilinear interaction (4.1), and are determined by formula (F.15) in the more general case of the interaction (F.11). For Eq. (4.7) to be valid it is necessary that the interaction be weak and that the relaxation parameters have a smooth dependence on the oscillator frequency, as well as that the oscillator nonlinearity be small enough

$$\begin{gathered}\Gamma, |P| \ll \omega_0, \omega_e, T, \\ \left|\frac{d\Gamma}{d\omega_0}\right|, \left|\frac{dP}{d\omega_0}\right| \ll 1, \qquad \frac{|V|}{\omega_0}(2\bar{n}+1) \ll 1.\end{gathered} \tag{4.11}$$

The initial condition for Eq. (4.7) takes according to Eq. (F10) the form

$$F_M(n,m;0) = Z_0^{-1}M_{nm}\exp(-\omega_0 m/T), \qquad M_{nm} = (n|\hat{M}|m), \tag{4.12}$$

$$Z_0 = \bar{n} + 1.$$

In addition to the corrections $\sim\Gamma/T$ omitted here we have discarded in the argument of the exponential the term $(1/2)Vm^2/T$ as compared with $\omega_0 m/T$; this is valid for actual $m \lesssim \bar{n}$ by virtue of the condition (4.11).

Equation (4.7) demonstrates the connection, which is typical for the oscillator problem, between the matrix elements $F_m(n,m;t)$ and $F_M(n+j,m+j;t)$. In contrast to the case of the harmonic oscillator, however, the coefficients in Eq. (4.7) depend explicitly on the time.

4.1.2 The Method of Generating Functions Obviously, the function $F_M(n,m;t)$ is linearly related to the matrix elements of the operator $\hat{M}$. It is convenient to single out this relation explicitly by introducing the Green's function $G(n,m;m',n';t)$ for Eq. (4.7) with the aid of the formula

$$F_M(n,m;t) = \sum_{m',n'} G(n,m;m',n';t)Z_0^{-1}M_{n'm'}\exp(-\omega_0 m'/T) \tag{4.13}$$

Obviously the Green's function $G(n,m;m',n';t)$ satisfies Eq. (4.7); all that changes is the form of the initial condition. Taking Eqs. (4.7), (4.12), and (4.13) into account we have

$$\frac{\partial G(n,m;m',n';t)}{\partial t} = \sum_j d_j(n,m)\exp\left[-iV_j(n-m)t\right] \times G(n+j,m+j;m',n';t),$$
$$G(n,m;m',n';0) = \delta_{n,n'}\delta_{m,m'}. \tag{4.14}$$

The system (4.14) of differential-difference equations for the Green's function $G(n,m;m',n';t)$ can be solved for arbitrary $|V|/\Gamma$ by the method of generating functions (a similar method was used in [11] to solve the kinetic equation in the case of a harmonic oscillator). We introduce the function

$$G_l(x,t;m',n) = \sum_{\eta_1=0}^{\infty}\left(\frac{\eta_2!}{\eta_1!}\right)^{1/2} G(n,m;m',n';t)e^{-x\eta_1} \times \exp\left[-iPlt + \tfrac{1}{2}i(m^2-n^2)Vt\right],$$
$$\eta_1 = \min(m,n), \qquad \eta_2 = \max(m,n), \qquad l = m-n. \tag{4.15}$$

From Eqs. (4.14) and (4.8)–(4.10) we obtain a partial differential equation for the function $G_l(x,t;m',n')$:

$$\frac{\partial G_l(x,t)}{\partial t} + C_1\frac{\partial G_l(x,t)}{\partial x} = C_2 G_l(x,t) \qquad (G_l(x,t) \equiv G_l(x,t;m',n')),$$
$$C_1 = 2(e^x-1)\left[(1-e^{-x})\bar{n}+1\right]\Gamma + ilV, \tag{4.16}$$
$$C_2 = \left[2\bar{n}(|l|+1)(e^{-x}-1) - |l|\right]\Gamma + \tfrac{1}{2}il|l|V.$$

The initial condition for this equation takes according to Eq. (4.14)

the form

$$G_l(x,0;m',n') \equiv G_l^{(0)}(x;m',n'),$$
$$G_l^{(0)}(x;m',n') = \left(\frac{\eta_2'!}{\eta_1'!}\right)^{1/2}\delta_{l,m'-n'}e^{-x\eta_1'}, \tag{4.17}$$
$$\eta_1' = \min(m',n'), \qquad \eta_2' = \max(m',n').$$

The first-order partial differential equation (4.15) is solved by the method of characteristics. The result is

$$G_l(x,t;m',n') = \left[\operatorname{ch} a_l t + \frac{2(2\bar{n}+1)\Gamma - 4\bar{n}e^{-x}\Gamma - ilV}{2a_l}\operatorname{sh} a_l t\right]^{-|l|-1} \times \exp\left[(\Gamma - \tfrac{1}{2}ilV)t\right]G_l^{(0)}(x^{(0)};m',n'),$$
$$x^{(0)} \equiv x^{(0)}(x,t),$$
$$e^{-x^{(0)}} = \frac{e^{-x}\left[y_{l1} - y_{l2}\exp(2a_l t)\right] - y_{l1}y_{l2}\left[1-\exp(2a_l t)\right]}{e^{-x}\left[1-\exp(2a_l t)\right] - \left[y_{l2} - y_{l1}\exp(2a_l t)\right]}, \tag{4.18}$$

where

$$a_l^2 = \Gamma^2 - il(2\bar{n}+1)\Gamma V - \tfrac{1}{4}l^2V^2 \qquad (\operatorname{Re} a_l > 0),$$
$$y_{l1,2} = \frac{2(2\bar{n}+1)\Gamma - ilV \pm 2a_l}{4\Gamma\bar{n}}. \tag{4.19}$$

Expressions (4.17)–(4.19) describe completely the relaxation of the nonlinear oscillator in the case when the interaction with a medium is linear in its coordinates, and reduces in classical terms to a friction force proportional to the velocity. Together with Eqs. (4.13), (4.4), and (4.5) these expressions make it possible, neglecting the small corrections (4.11), to obtain the time correlation function for arbitrary operators. In the classical region ($\bar{n} \gg 1$; $m', n' \gg 1$; $x \ll 1$) the function $G_l(x,t;m',n)$ goes over into the generating function for the solution of the EFP equation (see Appendix C).

4.2 *Analysis of the Time Correlation Function for Annihilation and Creation Operators and of Its Spectral Distribution*

When developing a quantum theory it is convenient to express the time correlation function of the oscillator coordinates in terms of the correlator $Q'(t)$ of the annihilation and creation operators $\hat{a}$ and $\hat{a}^+$:

$$Q(t) = \langle q(t)q(0)\rangle = (2\omega_0)^{-1}\left[Q'(t) + \frac{\bar{n}}{\bar{n}+1}\, Q'^*(t)\right], \qquad |t| \gg t_c,$$

$$Q'(t) = \langle \bar{a}(t)\hat{a}^+(0)\rangle \qquad \left(\langle \hat{a}^+(t)\hat{a}(0)\rangle \approx \frac{\bar{n}}{\bar{n}+1}\, Q'^*(t)\right). \tag{4.20}$$

Taking into account the smallness of the interaction with the medium and of the nonlinearity, we have discarded here the correlators $\langle \hat{a}(t)\hat{a}(0)\rangle$ and $\langle \hat{a}^+(t)\hat{a}^+(0)\rangle$, and used the relation

$$q = (2\omega_0)^{-1/2}(\hat{a} + \hat{a}^+).$$

Putting $\hat{L} = \hat{a}$ and $\hat{M} = \hat{a}^+$ in Eqs. (4.4) and (4.7) and calculating Tr_0 in Eq. (4.4) for the wave functions $|n)$, we can, taking Eqs. (4.6) and (4.13) into account, express $Q'(t)$ in the form

$$Q'(t) = \sum_{n,n'} Z_0^{-1}\exp\left(-\frac{\omega_0 n'}{T}\right)(n+1)^{1/2}(n'+1)^{1/2}$$
$$\times G(n+1, n; n', n'+1; t)\exp\left[-i\omega_0 t - i\left(n+\tfrac{1}{2}\right)Vt\right] \tag{4.21}$$

Thus, the quantity $Q'(t)$ is determined by the value of the generating function $G_l(x, t; n', n'+1)$ (4.15) at $l = -1$ and $x = 0$. From Eqs. (4.21) and (4.17)–(4.19) we obtain

$$Q'(t) = (\bar{n}+1)\exp\left[-i\left(\omega_0 + P - \tfrac{1}{2}V\right)t\right]\tilde{Q}^*(t), \quad \tilde{Q}(t) = e^{\Gamma t}\psi^{-2}(t),$$

$$\psi(t) = \mathrm{ch}\, at + \left[1 - i\frac{V}{2\Gamma}(2\bar{n}-1)\right]\frac{\Gamma}{a}\,\mathrm{sh}\, at, \tag{4.22}$$

$$a \equiv a_1 = \left[\Gamma^2 - i(2\bar{n}+1)\Gamma V - \frac{1}{4}V^2\right]^{1/2}.$$

(For convenience in comparing the classical and quantum expressions for the time correlation functions, the quantities $\psi(t)$ and a in Eq. (4.22) are defined to be the complex conjugates of the corresponding quantities $\psi_{\kappa\kappa}(t)$ and $a_{\kappa\kappa}$ in [21] and of $\psi(t)$ and a in [51]. The quantities $G_l(x, t; m', n')$ and a_l correspond to the quantities $f_l^*(e^{-x}, t; m', n')$ and a_l^* in [51].)

It is easy to verify that Eqs. (4.20) and (4.22) go over into the formulas (2.46), (2.47), and (2.40) of the classical theory for the time correlation function of the coordinates in the limit as $\hbar \to 0$. To this end, we reveal the constant $\hbar$ explicitly in the formulas (we have chosen above in this section units in which $\hbar = 1$) and recognize that according to Eq. (1.14)

$$V = \frac{3\hbar\gamma}{4\omega_0^2}$$

i.e., the nonlinearity parameter V is indeed proportional to $\hbar$ and consequently $|V| \ll \Gamma$ as $\hbar \to 0$. We can therefore regard the conditions

$$T \gg \omega_0, \qquad |V| \ll \Gamma,$$

to be satisfied in the classical limit, and thus replace $\bar{n}$ by T/ω_0 and discard the last term in Eq. (4.22) for a. The expressions (4.20) and (4.22) then coincide with Eqs. (2.46), (2.47), and (2.40) of the classical theory.

It is seen from Eqs. (4.20) and (4.22) that in quantum theory, just as in the classical theory, the time correlation function $Q(t)$ is expressed in terms of elementary functions, i.e., it is determined in explicit form for an arbitrary ratio of V and Γ. Its spectral representation $Q(\omega)$ is expressed according to Eqs. (1.8), (4.20), and (4.22) in quadratures, and for any concrete set of parameters it can be easily determined by numerical integration, just as in the classical theory. However, while the equations of the quantum theory for the envelope $\tilde{Q}(t)$ of the correlator $Q(t)$ are formally of the same form as in the classical theory, they are in fact much more productive and can yield, in particular, spectral distributions with fine structure.

The integral that determines $Q(\omega)$ can be calculated analytically in the limiting cases of large and small ratios of the parameters V and Γ.

In the case of strong nonlinearity, when the condition

$$|V| \gg (2\bar{n}+1)\Gamma \tag{4.23}$$

is satisfied, the nonequidistance of the levels of the singled-out oscillator is substantial, and the spectral distribution has a pronounced fine structure. Expanding a in Eq. (4.22) in powers of $\Gamma/|V|$ and retaining the quadratic terms of the expansion, and also expanding the last factor in expression (4.22) for $\tilde{Q}(t)$ in powers of $\exp(-at)$ and taking Eqs. (1.8), (4.20) and (4.22) into account, we can represent the spectral distribution

$$Q(\omega) = \frac{1}{2\omega_0} Q'(\omega) = \frac{1}{2\omega_0}(\bar{n}+1)\tilde{Q}(\Omega),$$
$$\tilde{Q}(\Omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty dt\, e^{i\Omega t}\tilde{Q}^*(t), \qquad \Omega = \omega - \omega_0 - P + \frac{1}{2}V \tag{4.24}$$

as a set of relatively narrow almost equidistant lines

$$\Omega_n = \omega - \omega_0 - P - \left(n - \frac{1}{2}\right)V - 8n\bar{n}(\bar{n}+1)\frac{\Gamma^2}{V},$$
$$\Gamma_n = \Gamma(n) + \Gamma(n-1) = [2n(2\bar{n}+1) - 1]\Gamma, \tag{4.26}$$
$$\alpha_n = 4\frac{\Gamma}{V}(n - 1 - 2\bar{n}).$$

At small $\Gamma/|V|$ and not very large n we can put $\cos\alpha_n \approx 1$ and $\sin\alpha_n \approx 0$. The individual lines are then Lorentzians having widths $2\Gamma_n$. It is easy to verify that the Γ_n are equal to the sums of the reciprocal lifetimes $\Gamma(n-1)$ and $\Gamma(n)$ of the levels $n-1$ and n, between which the corresponding transition takes place. This agrees with the Weisskopf and Wigner general theory of spectral line widths [26]. The widths of the individual lines increase linearly with the line number n and increase with rising temperature.

At sufficiently large n, the widths Γ_n become comparable with the distance $|V|$ between the lines, and the lines should overlap. At very

low temperatures ($T \ll \omega_0$) the spectrum contains only one line (corresponding to $n = 1$ in Eq. (4.25)). At somewhat higher temperatures there appear first very weak lines corresponding to $n = 2, 3, \ldots$. With further rise in temperature, the number of lines of noticeable intensity and their widths increase, and the fine structure begins to smear out. At sufficiently high temperatures it vanishes completely. Thus, the fine structure can manifest itself only in a limited interval of intermediate temperatures.

To illustrate the temperature-dependent smearing of the fine structure, Fig. 17 shows the spectral distribution $\tilde{Q}(\Omega)$ at different temperatures. It can be seen how the contribution of the maxima of higher order increases with rising temperature, and the intensity oscillations decrease.

In the opposite limiting case of relatively strong interaction with the medium, when

$$|V| \ll \Gamma,$$

there is no fine structure in the spectrum (the spectral distribution is smooth). Expanding a and $\psi(t)$ in Eq. (4.22) in powers of V/Γ up to terms of second order inclusive, and substituting the result of the expansion in Eq. (4.24), we can obtain in this case an analytic

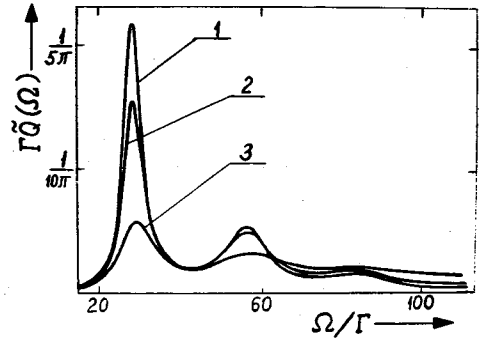

Figure 17 Fine structure of the spectral distribution $\tilde{Q}(\Omega)$ for $V = 28\Gamma$ at different temperatures. Curves 1–3 correspond to $\bar{n} = \frac{3}{8}$, $\frac{1}{2}$, and 1 (T/ω_0—0.77, 0.91, and 1.44). Curve 3 passes higher and curve 1 lower than curve 2 near the third peak ($\Omega/\Gamma \approx 80$).

expression for the spectral distribution:

$$\tilde{Q}(\Omega) = \frac{1}{\pi}\left\{\frac{\Gamma}{\Gamma^2+\Omega_T^2} + \frac{V^2}{2\Gamma^2}\bar{n}(\bar{n}+1)\right.$$
$$\left.\times\left[\frac{\Gamma}{\Gamma^2+\Omega_T^2} - \frac{3\Gamma}{9\Gamma^2+\Omega_T^2} - \frac{2\Gamma(\Gamma^2-\Omega_T^2)}{(\Gamma^2+\Omega_T^2)^2}\right]\right\}, \tag{4.27}$$
$$\Omega_T = \Omega - (2\bar{n}+1)V.$$

This expression agrees with an equation derived for the case of small $|V|/\Gamma$ by the Green's-function method [22].

4.2.1 Case of Nonequilibrium Oscillator The results presented above pertained to the case when the nonlinear oscillator was in thermodynamic equilibrium with the medium. The method of quantum kinetic equation and the explicit solution of the quantum kinetic equation (4.18) makes it also possible to analyze a nonequilibrium nonlinear oscillator that interacts with the medium. Such an analysis was carried out in [51] for the case when the thermodynamic equilibrium is upset by an intense pulsed external force (the pulse duration is $\lesssim t_c$, and the interval between pulses is large) that acts upon the oscillator. The response of a nonequilibrium nonlinear oscillator to an additional weak periodic field has a number of peculiarities compared with the response of an equilibrium oscillator (which is described by the function $Q(\omega)$). Thus, under nonequilibrium conditions the fine structure of the spectral distribution usually manifests itself much more strongly, since the broadening of the lines (4.26) is relatively small at low temperatures of the medium, and at the same time the occupation of the excited levels of the oscillator is determined by an external non-thermal action and can be large. At sufficiently large $|V|/\Gamma$, when the fine structure is pronounced, a nonequilibrium oscillator amplifies, in a certain frequency range, the weak resonant field (to obtain amplification in the stationary regime, in the case when the external force constitutes shot noise, it is necessary that the average frequency of the pulses exceeds 2Γ [51]). We note that the response of a nonequilibrium oscillator depends on its initial state only in the presence of nonlinearity.

4.3 *Representation of the Spectral Distribution in the Form of a Sum of Partial Spectra, and the Harmonic-Oscillator Paradox*

Besides the generating-function method used in Sec. 1 to calculate the correlators $\langle \hat{L}(t)\hat{M}(0)\rangle$ and their spectral distributions, the quantum kinetic equation (4.7) can be investigated by an alternate method based on an analysis of the equations for the partial spectra. This method makes it possible, in particular, to consider a problem of an oscillator with a more general (nonlinear in q) interaction with the medium, as well as certain problems with a more complicated energy spectrum, the solution of which by the method of generating functions does not lead to simple results.

The spectral distribution $Q'(\omega)$ of the time correlation function $Q'(t)$ of the operators $\hat{a}(t)$ and $\hat{a}^+(0)$ can be represented, with allowance for expressions (4.4)–(4.6), in the form [52]

$$Q'(\omega) = \frac{1}{\pi}\operatorname{Re}\sum_{n=1}^{\infty}\varphi(n,\omega), \tag{4.28}$$

$$\varphi(n,\omega) = \int_0^\infty dt\, n^{1/2}F_{a^+}(n,n-1;t)\exp\left[i\left(\omega-\omega_0-\left(n-\frac{1}{2}\right)V\right)t\right]$$

(according to Eq. (4.20) the operator F_M in Eq. (4.4) is taken for $\hat{M}=\hat{a}^+$).

The individual terms $\pi^{-1}\operatorname{Re}\varphi(n,\omega)$ in expression (4.8) for $Q'(\omega)$ can be interpreted as the partial spectra that correspond to a transition from the level $n-1$ to the level n (see Fig. 1). The system of equations for the functions $\varphi(n,\omega)$ is obtained by taking the Fourier transform of the differential-difference equation (4.7):

$$(i\tilde{\Omega}_n - \Gamma_n)\varphi(n,\omega) + \sum_{j\neq 0} d_j(n,n-1)\left(\frac{n}{n+j}\right)^{1/2}\varphi(n+j,\omega)$$
$$= -(\bar{n}+1)^{-1}n\exp\left[-(n-1)\omega_0/T\right], \tag{4.29}$$
$$\tilde{\Omega}_n = \omega - \omega_0 - P - (n-\tfrac{1}{2})V$$

(the quantity Γ_n is defined in Eq. (4.26)).

that the boundary condition for Eqs. (4.29) is of the form

$$\varphi(0,\omega) = \varphi(\infty,\omega) = 0. \tag{4.30}$$

It can be seen from Eq. (4.29) that in the oscillator problem the functions $\varphi(n,\omega)$ with different n, which describe the partial spectra, are interrelated. Since these functions are complex and the phase relations between them are important, it can be stated that Eq. (4.29) describes interference of the transitions between the oscillator levels. The interference causes the shape of the partial spectrum $\mathrm{Re}\,\varphi(n,\omega)$ to be, generally speaking, non-Lorentzian.

Interference of transitions between almost equidistant levels is a common phenomenon inherent in multilevel systems. The corresponding interrelated partial spectra are described by a system of linear algebraic equations of the type (4.29) (but with other, generally speaking much more complicated, coefficients). Analytic solutions of these systems are possible only in some limiting cases. However, since the inhomogeneous terms in systems of equations of this type decrease exponentially rapidly with increasing number of the level (since the level population decreases with the energy), they are very convenient for numerical computer solutions. Examples of a numerical analysis of the problem of a Duffing oscillator with nonlinear friction and of the problem of a triply degenerate nonlinear oscillator by the method of partial spectra are given in Secs. 4.4 and 6.2.2.

The system of equations (4.29) can be solved analytically in several limiting cases, without specifying the expressions for Γ_n and d_j (but recognizing that $d_j \sim \Gamma_n$). Thus, if the condition (4.23) is satisfied, the peaks of the partial spectra for several lowest transitions (with small n) are separated in frequency by a distance $\sim|V|$ greatly exceeding their widths $2\Gamma_n$. To describe the shape of an individual peak in the zeroth approximation, we can discard in Eqs. (4.29) the terms with $j \neq 0$. As a result we obtain

$$\mathrm{Re}\,\varphi(n,\omega) = \Phi_n(\tilde{\Omega}_n), \quad \Phi_n(\tilde{\Omega}_n) = \frac{n}{\bar{n}+1}\exp\left[-\frac{(n-1)\omega_0}{T}\right]\frac{\Gamma_n}{\Gamma_n^2+\tilde{\Omega}_n^2}, \tag{4.31}$$

$$|\tilde{\Omega}_n| \lesssim \Gamma_n \ll |V|,$$

i.e., the standard equation known from the Weisskopf and Wigner theory (the parameter Γ_n is equal to the sum of the reciprocal lifetimes of the oscillator in the states $|n)$ and $|n-1)$).

In first-order perturbation theory in the parameter $\Gamma_n/|V|$ we have in the region $|\Omega_n| \ll |V|$

$$\mathrm{Re}\,\varphi(n,\omega) = \Phi_n(\Omega_n)\left(1 - \frac{\alpha_n'\Omega_n}{\Gamma_n}\right),$$

$$\mathrm{Re}\,\varphi(n+j,\omega) = \left(\frac{n+j}{n}\right)^{1/2}\frac{d_{-j}(n+j,n+j-1)}{jV}\,\frac{\Omega_n}{\Gamma_n}\,\Phi_n(\Omega_n)$$

$$(j \neq 0) \tag{4.32}$$

$$\Omega_n = \omega - \omega_0 - P - \left(n - \frac{1}{2}\right)V$$

$$-\frac{1}{V}\sum_{j\neq 0} j^{-1} d_j(n,n-1)d_{-j}(n+j,n+j-1),$$

$$\alpha_n' = \frac{1}{V}\sum_{j\neq 0} n^{1/2}(n+j)^{1/2} j^{-1} d_j(n,n-1)\exp\left(-\frac{\omega_0}{T}j\right).$$

It can be seen from Eq. (4.32) that, owing to the interference of the transitions, in the frequency region $|\Omega_n| \sim \Gamma_n$, which corresponds to the transition $n-1 \to n$, a contribution to the peak of the spectral distribution is made not only by the partial spectrum $\varphi(n,\omega)$, but also by the partial spectra $\varphi(n+j,\omega)$ for the neighboring transitions. Their role increases with increasing $d_j/|V|$. The "interference interaction" of the transitions with nearby frequencies, as can be seen from Eq. (4.32), leads to distortion of individual partial spectra and to the onset of their shift and asymmetry. In the particular case of linear friction, the partial spectra (4.32) taken together lead to the distribution (4.25) (multiplied by $\pi(\bar{n}+1)$).

The system (4.29) can be solved analytically also in the case of low temperatures, when only several partial spectra with small n are of importance (see [52]). The summary spectrum consists then either of several peaks that differ noticeably in intensity (at large $|V|$), or constitutes a single but asymmetric peak (at small $|V|$).

A special case in which the system (4.29) has an exact solution is that of a linear oscillator ($V = 0$) that interacts linearly with a medium. In this case, as seen from Eq. (4.22), the correlator of the coordinates decays exponentially in time, and its spectral distribution $Q'(\omega)$ has a Lorentz shape. Equation (4.29) takes in this case, according to Eqs. (4.8)–(4.10), the form

$$\begin{aligned}&\left[i\Omega - (4n\bar{n} + 2n - 1)\Gamma\right]\varphi(n,\omega) + 2\Gamma n(\bar{n}+1)\varphi(n+1,\omega)\\ &+2\Gamma n\bar{n}\varphi(n-1,\omega) = -(\bar{n}+1)^{-1}n\exp\left[-\frac{(n-1)\omega_0}{T}\right],\\ &\Omega = \omega - \omega_0 - P.\end{aligned} \tag{4.33}$$

It is easy to verify directly that the system (4.33) has the following solution:

$$\varphi(n,\omega) = (\bar{n}+1)^{-1}n\exp\left[-\frac{(n-1)\omega_0}{T}\right]\frac{\Gamma + i\Omega}{\Gamma^2 + \Omega^2}, \tag{4.34}$$

i.e., all the partial spectra have the same shape and a common maximum, and differ only in intensity.

Each of the partial spectra (into which the resultant spectrum is subdivided conventionally), and consequently also the entire distribution as a whole, has the same width 2Γ. This width does not depend on the number n and does not coincide with the value $2\Gamma_n$ obtained from the formulas of the Weisskopf–Wigner theory for nondegenerate systems [26]; this just constitutes the known "harmonic-oscillator paradox" (see, e.g., [11]). In fact, however, the results of the Weisskopf–Wigner theory are directly applicable only when the frequency of the considered transition differs noticeably from the frequencies of the other quantum transitions in the system. In the problem of the singled-out oscillator, this situation arises for an essentially nonlinear oscillator with noticeable nonequidistance of the levels. The degeneracy is absent here, and, as noted above, the results agree fully with those of the Weisskopf–Wigner theory. If the nonlinearity decreases, the problem becomes almost degenerate (in the case of a linear singled-out oscillator it is completely degenerate in the transition frequencies) and it is necessary to take into account the interference

between the transitions. This leads naturally to a substantial change in the spectral distribution, and in the limit as $V \to 0$ to the formation of the simple distribution (4.34).

Thus, by using Eq. (4.29) we can investigate the distortion of the shapes of the individual partial spectra, which arises due to the interference of the transitions when the peaks come closer with decreasing nonlinearity, and the onset of a single Lorentz distribution as $V \to 0$.

4.4 *Effect of Nonlinear Interaction with the Medium (Nonlinear Friction) on the Spectral Distribution of the Singled-Out Oscillator*

4.4.1 Kinetic Equation for an Oscillator with Nonlinear Friction The previously considered interaction (4.1) or (F.11) with the medium was linear in the coordinates of the singled-out oscillator and corresponded, in terms of classical mechanics, to linear friction (the friction force is proportional to the velocity, with a proportionality coefficient independent of the amplitude). New effects arise if the friction is nonlinear, i.e., if H_i contains terms proportional to q^2 or to higher powers of q. Since in this case the friction coefficient depends on the amplitude and consequently varies with time, the relaxation process becomes more complicated, and the decay of the correlator $Q(t)$ becomes nonexponential and the spectral distribution $Q(\omega)$ becomes non-Lorentzian even for a linear singled-out oscillator (neglecting the interaction).

The simplest interaction that leads to nonlinear friction is described by the Hamiltonian

$$\begin{aligned}H_i^{(2)} &= \sum_k \epsilon_k^{(2)} q^2 q_k\\ &\times \left(q = (2\omega_0)^{-1/2}(\hat{a} + \hat{a}^+), q_k = (2\omega_k)^{-1/2}(\hat{a}_k + \hat{a}_k^+)\right),\end{aligned} \tag{4.35}$$

which is added to the Hamiltonian (4.1). Substituting the total Hamiltonian of the interaction in the operator kinetic equation (F.9) and integrating with respect to t_1, just as in the case of an interaction linear in the q, we obtain the system of differential-difference equations (4.7) for the matrix elements $F_M(n,m;t)$. Now, however, the

right-hand side contains terms with $j = \pm 2$, corresponding to a change of the quantum number of the oscillator by 2. The coefficients of these terms are equal to

$$d_{\pm 2}(n,m) = 2(\bar{n}_2 + \tfrac{1}{2} \pm \tfrac{1}{2}) \times \left[(n \pm 1)(n+1 \pm 1)(m \pm 1)(m+1 \pm 1)\right]^{1/2} \Gamma^{(2)}, \tag{4.36}$$

$$\bar{n}_2 \equiv \bar{n}(2\omega_0) = \frac{\bar{n}^2}{2\bar{n}+1}, \qquad \Gamma^{(2)} = \frac{\pi}{8\omega_0^2} \sum_k (\epsilon_k^{(2)})^2 \omega_k^{-1} \delta(2\omega_0 - \omega_k).$$

The parameter $\Gamma^{(2)}$, as seen from Eq. (4.36), determines the probability of the decay process, at which a transition between oscillator levels, whose numbers differ by 2 and energies differ accordingly by $\approx 2\omega_0$, is accompanied by creation or annihilation of a vibration of the medium.

The interaction (4.35) causes also adding to $d_0(n,m)$ in Eq. (4.8) the term

$$-\frac{1}{2} i(n^2 - m^2) V^{(2)} - i(n-m) P^{(2)},$$

$$V^{(2)} = \text{v.p.} \frac{1}{2\omega_0^2} \sum_k (\epsilon_k^{(2)})^2 \omega_k^{-1} \left(\frac{\omega_k}{4\omega_0^2 - \omega_k^2} - \frac{2}{\omega_k} \right), \tag{4.37}$$

$$P^{(2)} = -\text{v.p.} \frac{1}{8\omega_0^2} \sum_k (\epsilon_k^{(2)})^2 \omega_k^{-1} \left[\frac{4\bar{n}(\omega_k) + 1}{2\omega_0 - \omega_k} + \frac{4\bar{n}(\omega_k) + 3}{2\omega_0 + \omega_k} + \frac{4}{\omega_k} \right]$$

and the renormalization of the expressions for the level half-widths $\Gamma(n)$ and $\Gamma(m)$ contained in $d_0(n,m)$:

$$\Gamma(n) = \left[n(2\bar{n}+1) + \bar{n}\right]\Gamma + \left[2\bar{n}_2(n^2 + n + 1) + n(n-1)\right]\Gamma^{(2)}. \tag{4.38}$$

It is seen directly from Eqs. (4.8) and (4.37) that the term $\propto P^{(2)}$ reduces to a renormalization of the frequency shift P; we assume hereafter this renormalization to be performed. It is easy to verify also that the term $\propto V^{(2)}$ in $d_0(n,m)$ leads to a renormalization of the nonlinearity parameter V,

$$V \to V + V^{(2)}. \tag{4.39}$$

The system of differential-difference equations (4.7) for the problem with nonlinear interaction (4.35) differs from the analogous system for the problem with interaction (4.1) because of the appearance of terms with $j = \pm 2$ and of terms $\propto \Gamma^{(2)}$ (quadratic in n and m) in $d_0(n,m)$. When these terms are taken into account, it is inexpedient to solve Eqs. (4.7) and (4.14) by the method of generating functions, since for $G_I(x,t;m',n')$ a second-order partial differential equation is obtained here. To analyze the problem with nonlinear interaction we can transform from the differential-difference equations (4.7) to difference equations for the Fourier transforms of the functions $F_M(n, m; t)$. Such equations were discussed in Sec. 4.3. They are convenient for numerical calculations and for the analysis of limiting cases.

In particular, in the case when the oscillator levels are relatively strongly nonequidistant, i.e. when $|V| \gg \Gamma_n$, the spectral distribution $Q'(\omega)$, just as in the problem with linear friction, presents a set of the weakly overlapping peaks (4.31) and (4.32), but the parameters in Eqs. (4.31) and (4.32) are now determined by expressions (4.8), (4.36), and (4.38). At low temperatures, when the terms $\propto \exp(-\omega_0/T)$ can be neglected, the nonlinear friction, as seen from Eqs. (4.29), (4.36), and (4.38), does not influence $Q'(\omega)$, since $Q'(\omega)$ is formed as a result of only a transition from the ground state to the first excited level, while the considered nonlinear interaction, which causes decays in which the oscillator energy changes by $\approx 2\omega_0$, cannot lead at $\exp(-\omega_0/T) \ll 1$ to decay of the first excited state.

4.4.2 Classical Limit When account is taken of the nonlinear friction, Eqs. (4.7) and (4.29) contain three dimensionless parameters: V/Γ, $\Gamma^{(2)}/\Gamma$, and T/ω_0. These parameters determine the entire manifold of the spectra of the nonlinear oscillator, including the spectra with fine structure.

The dynamics of an oscillator with nonlinear friction and its spectral distribution turn out to be simpler in the region of high temperatures, when there is no fine structure of the spectrum and the

classical approximation can be used. An analog of the quantum kinetic equation in the classical case is the Einstein–Fokker–Planck (EFP) equation. In this case the Green's function $G(n,m;m',n';t)$ introduced in Eq. (4.13) corresponds to the classical probability density $w(r,\varphi;r(0),\varphi(0);t)$ of the transition from a state $\{r(0),\varphi(0)\}$ at the instant $t=0$ into a state $\{r,\varphi\}$ at the instant t (φ denotes the phase of the oscillator and r is proportional to the harmonic part of the energy). The EFP equation can be obtained by going to the classical limit in the quantum equation (4.14). It can be obtained also directly in classical theory by starting from the Langevin equation for a nonlinear oscillator interacting with a medium. To derive the Langevin equation in the case of an interaction described by formulas (4.1) and (4.35), it is convenient to use the method developed in Sec. 2. The stochastic differential equation obtained in this manner for the complex amplitude $u_1(t)$ introduced in Eq. (1.5) and varying slowly over times $\sim t_c$, is a generalization of Eq. (2.14):

$$\dot{u}_1 = -\left[(\Gamma + 4\Gamma^{(2)}\omega_0|u_1|^2)\operatorname{sgn} t - iP\right]u_1 - \frac{3\tilde{\gamma}}{2i\omega_0}|u_1|^2u_1 + f_1(t) + u_1^*f_1^{(2)}(t), \qquad \tilde{\gamma} = \gamma + \tfrac{4}{3}\omega_0^2V^{(2)}. \tag{4.40}$$

Here P includes the term $P^{(2)}$; $f_1(t)$ and $f_1^{(2)}(t)$ are independent Gaussian δ-correlated (in a scale of times that are long compared with t_c) processes

$$\langle f_1(t)f_1(t')\rangle = \langle f_1(t)f_1^{(2)}(t')\rangle = \langle f_1(t)f_1^{(2)*}(t')\rangle = \langle f_1^{(2)}(t)f_1^{(2)}(t')\rangle = 0, \tag{4.41}$$

$$\langle f_1(t)f_1^*(t')\rangle = \frac{\Gamma T}{\omega_0^2}\delta(t-t'), \qquad \langle f_1^{(2)}(t)f_1^{(2)*}(t')\rangle = \frac{4\Gamma^{(2)}\Gamma}{\omega_0}\delta(t-t').$$

Corresponding to the Langevin equation (4.40) for a nonlinear oscillator with nonlinear interaction with the medium, with account taken of Eq. (4.41), is the EFP equation

$$\frac{\partial w}{\partial t} = \Gamma\left[\frac{\partial}{\partial r}\left(\frac{T}{\omega_0^2}r\frac{\partial w}{\partial z} + 2rw\right) + \frac{T}{4\omega_0^2 r}\frac{\partial^2 w}{\partial\varphi^2}\right] + 4\Gamma^{(2)}\omega_0\left[\frac{\partial}{\partial r}\left(\frac{T}{\omega_0^2}r^2\frac{\partial w}{\partial r} + 2r^2w\right) + \frac{T}{4\omega_0^2}\frac{\partial^2 w}{\partial\varphi^2}\right] - \left(P + \frac{3\tilde{\gamma}r}{2\omega_0}\right)\frac{\partial w}{\partial\varphi}, \qquad w = w(r,\varphi;r(0),\varphi(0);t). \tag{4.42}$$

The variables in this equation are

$$r = |u_1|^2 = \tfrac{1}{4}(q^2 + \omega_0^{-2}\dot{q}^2), \qquad \varphi = \arg u_1. \tag{4.43}$$

In the absence of nonlinear friction, Eq. (4.42) coincides with the EFP equation (C.5). Just as in Appendix C, the initial condition and the normalization condition are of the form

$$w(r,\varphi;r(0),\varphi(0);0) = 2\delta(r-r(0))\delta(\varphi-\varphi(0)), \qquad \frac{1}{2}\int dr\,d\varphi\, w(r,\varphi;r(0),\varphi(0);t) = 1. \tag{4.44}$$

It can be seen from Eq. (4.40) that decay processes with transfer of an energy $\approx 2\omega_0$ to the medium, which lead to a damping $\propto\Gamma^{(2)}$, correspond in terms of classical theory to a regular friction force that has a nonlinear dependence on u_1. Also connected with such processes is the random force $u_1^*f_1^{(2)}(t)$ in Eq. (4.40), which is proportional to the oscillation amplitude. In the phenomenological description, the nonlinear friction force in Eq. (4.40) results from a friction force $\propto q^2\dot{q}$ in the equation of motion (1.10) of a Van der Pol oscillator. The fluctuations of a nonequilibrium Van der Pol oscillator were investigated in [10, 41, 13] by using a model in which the random force in Eq. (1.10) was assumed independent of q and $\dot{q}$, i.e., the term $u_1^*f_1^{(2)}(t)$ was disregarded in Eq. (4.40). We note that this term is necessary for the description of the fluctuations of an oscillator that is in thermodynamic equilibrium with the medium: without it

the term $\propto \Gamma^{(2)}$ in Eq. (4.42) would take the form

$$8\Gamma^{(2)}\omega_0 \frac{\partial}{\partial r}(r^2 w)$$

(cf. [10, 13]), and therefore the Gibbs distribution

$$w_{st}(r,\varphi) = \frac{2\omega_0^2}{\pi T}\exp\left(-\frac{2\omega_0^2 r}{T}\right) \tag{4.45}$$

would not be a stationary solution of the EFP equation (even if the intensity of the random force $f_1(t)$ were to differ from $\Gamma T/\omega_0^2$).

Thus, analysis of the microscopic model of a nonlinear oscillator whose interaction with the medium is quadratic in q enables us to investigate equilibrium fluctuations of both the Duffing oscillator and of the Van der Pol oscillator. These fluctuations, as seen from Eqs. (4.42) and (4.45), are described in the classical limit by two parameters, which can be chosen to be

$$\Gamma_T^{(2)} = \frac{2\Gamma^{(2)}T}{\omega_0}, \qquad \tilde{\alpha} = \frac{3}{8}\frac{\tilde{\gamma}T}{\omega_0^3\Gamma} \tag{4.46}$$

4.4.3 Spectral Distribution of Time Correlation Function of the Coordinates for an Oscillator with Nonlinear Friction In the presence of nonlinear friction, the attempts to obtain an analytic solution of the EFP equation (4.42) failed. The task of calculating the peak of the spectral distribution $Q(\omega)$ of the time correlation function of the coordinates with the aid of a series of transformations of the EFP equation can be reduced to the solution of a second-order ordinary differential equation. This equation can be obtained also directly from the system of equations (4.29) for the functions $\varphi(n,\omega)$ that describe the partial spectra in quantum theory. In the classical limit the main contribution to Eq. (4.28), which connects $Q'(\omega)$ (or $Q(\omega)$ in accordance with Eq. (4.24)) with $\varphi(n,\omega)$, is made by the terms with large n. This makes it possible to introduce in place of the discrete variable n the continuous variable $R = \omega_0 n/T$, and to obtain for the function $W(R,\omega) = (\omega_0/T)\varphi(n,\omega)$ of this variable, in place of the

difference equation (4.29), the differential equation

$$2R(\Gamma + \Gamma_T^{(2)}R)\frac{d^2W}{dR^2} + 2R\left[\Gamma + \Gamma_T^{(2)}(1+R)\right]\frac{dW}{dR}$$
$$+\left[i(\Omega - 2\tilde{\alpha}\Gamma R) + \Gamma - \Gamma_T^{(2)}(1-3R)\right]W = -\mathrm{Re}^{-R}, \tag{4.47}$$
$$W(0,\omega) = W(\infty,w) = 0.$$

In this approximation, when account is taken of Eqs. (4.24) and (4.28), we have

$$Q(\omega) = \frac{1}{2\omega_0}Q'(\omega) = \frac{T}{2\omega_0^2}\tilde{Q}(\Omega), \qquad \tilde{Q}(\Omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty dR\, W(R,\omega). \tag{4.48}$$

Equation (4.47) was solved numerically [52] for different sets of parameters $\tilde{\alpha}\Gamma/\Gamma_T^{(2)}$ and $\Gamma_T^{(2)}/\Gamma$ and at different $\Omega/\Gamma_T^{(2)}$, and it was this which made it possible to investigate the spectral distribution $Q(\omega)$ in the classical limit. For the case of linear friction ($\Gamma_T^{(2)} = 0$) the results agreed with those discussed in Sec. 2.3.2. In the opposite case of purely nonlinear friction ($\Gamma = 0$, $\Gamma_T^{(2)} \neq 0$), the numerically obtained spectral distributions for different $\tilde{\alpha}\Gamma/\Gamma_T^{(2)}$ are shown in Fig. 18. It can be seen from this figure that even if the levels are equidistant ($\tilde{\alpha} = 0$), in the case of nonlinear friction the spectral distribution, being symmetrical, differs substantially from a Lorentz distribution (which corresponds to linear friction). Although curve 1, just as the Lorentz curve, decreases at large $|\Omega|$ in accordance with the law

$$\tilde{Q}(\Omega) \approx \frac{\Gamma_T^{(2)}}{\pi\Omega^2} \qquad (|\Omega| \gg \Gamma_T^{(2)}, \Gamma_T^{(2)} \gg \Gamma), \tag{4.48a}$$

in the central part of the spectrum it is much narrower and higher than a Lorentz curve. The height of the maximum of curve 1 is $1.67/\pi$ (as against $1/\pi$ for a Lorentzian), and its width at half maximum is half as large as for a Lorentz curve. The narrowing of the central part of the spectrum in the case of nonlinear friction is the

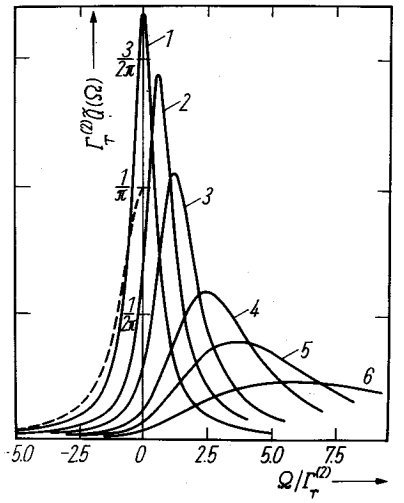

Figure 18 Spectral distribution $\tilde{Q}(\Omega)$ in the classical limit in the case of purely nonlinear friction. Curves 1–6 correspond to the parameter values $\tilde{\alpha}\Gamma/\Gamma_T^{(2)} = 0$, 0.25, 0.5, 1, 1.5, and 2.5. The dashed line corresponds to a Lorentz curve having the same area and the same asymptote at large $|\Omega|$ as curves 1–6.

consequence of the fact that, as can be seen from Eq. (4.40) small-amplitude oscillations decay slowly (the friction force tends to zero as $|u_1|^2 \to 0$), and therefore the contribution of such oscillations to $\tilde{Q}(\Omega)$ is described by a narrow peak. With increasing parameter $\tilde{\alpha}\Gamma$ (or V), which determines the nonequidistance of the levels, the spectral distribution broadens and its maximum shifts. Expression (4.48a) remains valid in this case on the wings.

For the general case, when both linear and nonlinear friction are present, the parameters that characterize the width, asymmetry, and deviation from Lorentzian of the distributions $Q(\omega)$ as functions of the ratio $\Gamma_T^{(2)}/\Gamma$ (at different $\tilde{\alpha}\Gamma/\Gamma_T^{(2)}$) were also calculated in [52]. It is interesting that these parameters reach the asymptotic values corresponding to the limit $\Gamma_T^{(2)}/\Gamma \to \infty$ only at very large $\Gamma_T^{(2)}/\Gamma \sim 10^2$.

The foregoing results are generalized in [52] also to include the case of several singled-out oscillators. The role of nonlinear friction corresponding to decay processes with participation of different oscillators can be quite substantial in this case, and if the oscillator frequencies

differ strongly the correlators can be calculated analytically (see also Sec. 6.2.2b below). From the formal point of view, nonlinear friction includes also friction due to quasielastic scattering of the vibrations of the medium by the oscillator (see Sec. 6.2.1).

4.4.4 Time Correlation Function of the Occupation Numbers and Its Spectral Distribution Besides the investigation of the influence of nonlinear friction on the coordinate correlators, it is of interest to analyze its influence on the time correlation function $\mathcal{N}(t)$ of the oscillator occupation numbers and its spectral distribution $\mathcal{N}(\omega)$:

$$\mathcal{N}(t) = \langle \hat{n}(t)(\hat{n}(0) - \bar{n}) \rangle, \qquad \mathcal{N}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{i\omega t} \mathcal{N}(t). \quad (4.49)$$

In the case of weak nonlinearity, the function $\mathcal{N}(t)$, as seen from Eq. (4.1) determines also the oscillator-energy correlator.

Putting $\hat{L} = \hat{n}$ and $\hat{M} = \hat{n} - \bar{n}$ in Eqs. (4.4) and (4.5), we can represent $\mathcal{N}(t)$ in the form

$$\mathcal{N}(t) = \sum_{n=0}^{\infty} n\chi(n,t), \qquad \chi(n,t) = F_{\hat{n}-\bar{n}}(n,n;t), \quad (4.50)$$

where the function $\chi(n,t)$ is a solution of Eq. (4.7) (cf. Eqs. (4.7) and (4.12)):

$$\frac{\partial \chi(n,t)}{\partial t} = \sum_j d_j(n,n)\chi(n+j,t),$$
$$\chi(n,0) = (\bar{n}+1)^{-1}(n-\bar{n})\exp(-\omega_0 n/T). \quad (4.51)$$

Using the explicit form (4.8) and (4.36)–(4.38) of the coefficients $d_j(n,n)$ in Eq. (4.51), we can easily show that

$$\sum_{n=0}^{\infty} \chi(n,t) = 0. \quad (4.52)$$

In the absence of nonlinear friction ($\Gamma^{(2)} = 0$) we can calculate $\mathcal{N}(t)$ by using the explicit expression, obtained in Eq. (4.18), for the

generating function $G_l(x,t;m',n')$

$$\mathcal{N}(t) = (\bar{n}+1)^{-1}\sum_{n=0}^{\infty}(n-\bar{n})\exp\left(-\frac{\omega_0}{T}n\right)\left(-\frac{\partial}{\partial x}G_0(x,t;n,n)\right)_{x=0}$$

$$= \bar{n}(\bar{n}+1)\exp(-2\Gamma t), \qquad t \geqslant 0 \qquad (\Gamma^{(2)}=0) \tag{4.53}$$

(the connection between $\mathcal{N}(t)$ and $G_l(x,t;m',n')$ was obtained from expressions (4.13), (4.15), and (4.50)). Thus, weak internal anharmonicity does not influence the damping of the correlator $\mathcal{N}(t)$. (Expression (4.53) for the occupation-number correlator for an oscillator interacting with a medium can also be obtained by the Green's-function method [53].)

In the presence of nonlinear friction we did not succeed in obtaining a general solution of Eq. (4.51). In the case of small $\Gamma^{(2)}/\Gamma$ the correlator $\mathcal{N}(t)$ can be calculated by using the method of moments, transforming from Eq. (4.51) to the system of equations for the functions

$$\chi_k(t) = \sum_{n=0}^{\infty} n^k \chi(n,t).$$

At $\Gamma^{(2)}=0$ the function $\chi_k(t)$ turns out to be connected only with functions $\chi_{k'}(t)$ having $k'<k$, while the presence of the terms $\propto \Gamma^{(2)}$ leads to coupling to the succeeding moments. Solving the equation for the moments $\chi_k(t)$ by iteration with respect to $\Gamma^{(2)}/\Gamma$, we obtain

$$\mathcal{N}(t) = \bar{n}(\bar{n}+1)e^{-2\Gamma t} - 16\bar{n}_2(\bar{n}+1)^2\Gamma^{(2)}te^{-2\Gamma t}, \tag{4.54}$$

$$\Gamma^{(2)} \ll \Gamma, \qquad \frac{1}{\Gamma^{(2)}} > t \gg \omega_0^{-1}.$$

According to Eq. (4.54), when nonlinear friction is taken into account $\mathcal{N}(t)$ decreases with time non-exponentially (faster than $\exp(-2\Gamma t)$).

The non-exponential decay of $\mathcal{N}(t)$ and the non-Lorentz character of $\mathcal{N}(\omega)$ manifest themselves particularly noticeably in the case of appreciable nonlinear friction. At sufficiently high temperatures the function $\mathcal{N}(\omega)$ can be calculated in the classical-limit approximation, when the difference equations (4.51) reduce to a differential equation

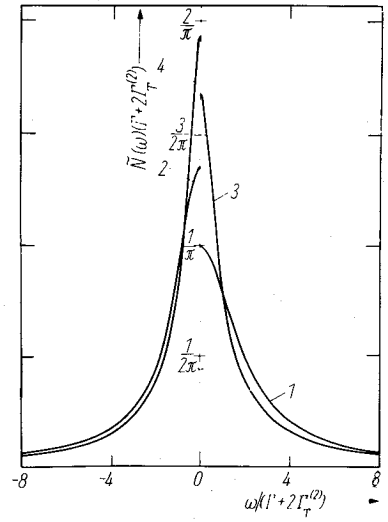

Figure 19 Spectral distribution $\mathcal{N}(\omega)$ in the classical limit. The ordinates are the values of the function $\bar{N}(\omega) = 2(\omega_0/T)^2\mathcal{N}(\omega)$. Curves 1–4 correspond to the values $\Gamma_T^{(2)}/\Gamma = 0$, 2, 10, and 100. The figure shows only the left or the right parts of the symmetrical curves.

of the type (4.47), and the function $\mathcal{N}(\omega)$ itself contains only one parameter $\Gamma_T^{(2)}/\Gamma$ (see Ref. 52). The results of the numerical calculations are shown in Fig. 19. It can be seen from this figure, in particular, that for large nonlinear friction ($\Gamma_T^{(2)} \gg \Gamma$) the $\mathcal{N}(\omega)$ peak is much narrower than a Lorentz peak with the same asymptotic form in the region of large $|\omega|$ which corresponds to $\Gamma_T^{(2)}=0$.

§5 Heating and Cooling, by an External Periodic Field, of an Oscillator Nonlinearly Interacting with a Medium

In the analysis of the relaxation of eigenoscillations of the oscillator, as well as in the analysis of its motion in a resonant field (of frequency $\omega \approx \omega_0$), we used in the preceding sections the fact that the motion of a weakly nonlinear oscillator could be divided into fast and slow. It is important that the fast motion of an isolated oscillator was characterized by one frequency, namely the eigenfrequency ω_0 of the

oscillations. As a result, the damping could be described with the aid of two parameters, the coefficients Γ and $\Gamma^{(2)}$ of linear and nonlinear friction. These coefficients are determined (see Eqs. (2.15) and (4.36)) by the probabilities of the decay processes whereby the oscillator transfers to the medium an energy ω_0 and $2\omega_0$, respectively ($\hbar = 1$).

A different situation arises in the analysis of the dynamics of an oscillator in a periodic field whose frequency ω differs substantially from ω_0 (but is also "fast"):

$$\omega, |\omega - \omega_0| \gg \Gamma, \Gamma^{(2)}, |V|(2\bar{n} + 1). \tag{5.1}$$

In this case the "fast" motion of the oscillator is characterized by two frequencies, ω_0 and ω, and in addition to the decay processes indicated above it is possible to have decay or scattering processes in which energies ω, 2ω, as well as $\omega + \omega_0$ and $|\omega - \omega_0|$ are tranferred to the medium. The processes of the last two types, just as nonlinear friction, are due to nonlinearity of the interaction with the medium with respect to the oscillator coordinates, and describe in fact the eigenoscillation (of frequency ω_0) relaxation due to the forced oscillations at the frequency ω. It is important that the corresponding damping can be "negative" and describe the buildup of eigenoscillations.

The possibility of such an instability can be easily understood in the case when the medium constitutes an assembly of vibrations with a quasicontinuous spectrum of frequencies ω_k. The field can then induce either processes of the type shown in Figs. 20a and 20b, which correspond to decay of the oscillator (to a transition to a lower level) accompanied by creation or annihilation of a "photon" (of energy ω) and excitation of a vibration of the medium, or processes in which the energy balance is described by the relation $\omega = \omega_k + \omega_0$ (see Fig. 20c). Processes of the last type correspond to decay of a "photon" with production of a vibration of the medium and with transition of the oscillator to a higher energy level, i.e., to excitation of the oscillator ("negative" damping). Since the transition probability, which is determined by the oscillator-coordinate matrix element, increases with the number of the level, the excitation probability is higher the higher the oscillator energy is, and if the mechanism indicated in Fig. 20c dominates, "runaway" sets in—the energy of the eigenoscillations of the oscillator increases with time.

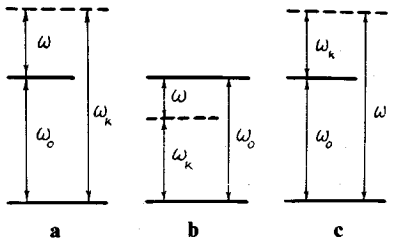

Figure 20 Scheme of decay processes induced by a field of frequency ω. The solid lines show neighboring energy levels of the oscillator (ω_0 is its eigenfrequency and ω_k are the vibration frequencies of the continuous spectrum). The energy balance for the processes a–c is described by the equations

$$\omega_0 + \omega = \omega_k\,; \qquad \omega_0 - \omega = \omega_k\,; \qquad \omega = \omega_0 + \omega_k\,.$$

We obtain below (see also [30]) a quantum kinetic equation for an oscillator in an external field (Sec. A) and analyze its solution (Secs. B and C). In Sec. C it is shown, in particular, that no "runaway" occurs if the nonlinear friction is taken into account. In Sec. D we discuss the response of the oscillator to an additional weak field.

5.1 *Change of the Oscillator Relaxation Parameters in an External Field*

5.1.1 *General Description of Oscillator Motion in a Nonresonant Field*

The Hamiltonian of a Duffing quantum oscillator nonlinearly interacting with a medium and acted upon by an external force $h(t)$ takes, in the case when the medium constitutes a set of vibrations of a continuous spectrum, the form

$$H(t) = \mathscr{H}_0(t) + \mathscr{H}_i\,, \quad \mathscr{H}_0(t) = H_0(t) + H_m\,, \quad \mathscr{H}_i = H_i + H_i^{(2)},$$

$$H_0(t) = \omega_0\hat{n} + \tfrac{1}{2}V\hat{n}^2 - qh(t), \qquad \hat{n} = \hat{a}^+\hat{a}. \tag{5.2}$$

Here H_i and $H_i^{(2)}$ are determined by Eqs. (4.1) and (4.35). It is assumed hereafter that the force $h(t)$ is quasimonochromatic and

nonresonant (see (5.1)):

$$h(t) = h_1(t)\exp(-i\omega t) + \text{c.c.},$$
$$\left|\frac{dh_1}{dt}\right| \ll \frac{|h_1|}{t_c}, \qquad t_c = \min(\omega, \omega_0, |\omega - \omega_0|, \omega_e) \tag{5.3}$$

(ω_e denotes the characteristic reciprocal damping time of the correlations in the medium).

When conditions (5.1) and (5.3) are satisfied, the motion of an oscillator isolated from the medium (it is described by the Hamiltonian $H_0(t)$) constitutes a superposition of eigenoscillations at the frequency $\approx\omega_0$, and of forced oscillations at the frequency ω, accurate to small corrections of the order of

$$\left|\frac{V}{\omega_0 \pm \omega}\right|, \left|\frac{V}{\omega - 3\omega_0}\right|, |Vh_1^2|t_c^3, \frac{|Vh_1^3|t_c^3}{|3\omega - \omega_0|}, \tag{5.4}$$

and also neglecting terms $\sim\dot{h}_1(\omega_0 h_1)^{-1}$. Indeed, the equation of motion for the annihilation operator $\tilde{a}$ of the isolated oscillator

$$\frac{d}{dt}\tilde{a} = -i[\tilde{a}(t), H_0(t)] \tag{5.5}$$

has, neglecting the corrections (5.4), a solution

$$\tilde{a}(t) = \hat{a}_0(t)e^{-i\omega_0 t} + v_1(t),$$
$$v_1(t) = (2\omega_0)^{-1/2}\left[\frac{h_1(t)e^{-i\omega t}}{\omega_0 - \omega} + \frac{h_1^*(t)e^{i\omega t}}{\omega_0 + \omega}\right], \tag{5.6}$$

where the operator $\hat{a}_0(t)$ describes the eigenoscillations and satisfies the equation

$$\frac{d\hat{a}_0(t)}{dt} = -iV\left[\frac{1}{2} + \hat{n}_0 + \frac{2(\omega^2 + \omega_0^2)|h_1(t)|^2}{\omega(\omega_0^2 - \omega^2)^2}\right]\hat{a}_0(t), \qquad \hat{n}_0 = \hat{a}_0^+\hat{a}_0. \tag{5.7}$$

It is obvious that $\hat{a}_0(t)$ varies slowly over a time $\sim t_c$.

In the case of an oscillator interacting with the medium, Eqs. (5.5)–(5.7) correspond to the interaction representation. The transformation to the interaction representation is effected with the aid of the operator

$$\hat{S}(t, t_0) = T_{t_1}\exp\left[-i\int_{t_0}^{t} dt_1 \mathscr{H}_0(t_1)\right]. \tag{5.8}$$

Here T_{t_1} is the chronological-ordering symbol. The initial instant t_0 is chosen generally speaking arbitrarily (in particular, it can be chosen to precede the instant when the force is turned on, i.e., $h_1(t)$ may be supposed to vanish at $t \leqslant t_0$). Obviously, in the interaction representation (5.8) the initial condition for Eq. (5.7) is of the form

$$\hat{a}_0(t_0) = [\hat{a} - v_1(t_0)]\exp(i\omega_0 t_0) \qquad (\tilde{a}(t_0) \equiv \hat{a}). \tag{5.9}$$

The Hamiltonian of the interaction of the oscillator with the medium, in the interaction representation (5.8) with allowance for Eqs. (5.6), (4.1), and (4.35), is equal to

$$\tilde{\mathscr{H}}_i(t) = \hat{S}^+(t, t_0)\mathscr{H}_i\hat{S}(t, t_0), \qquad \tilde{\mathscr{H}}_i(t) = \mathscr{H}_i' + \mathscr{H}_i'' + H_i''',$$
$$\mathscr{H}_i' = \sum_k \tilde{\epsilon}_k(t)q_0(t)\tilde{q}_k(t), \qquad \mathscr{H}_i'' = \sum_k \epsilon_k^{(2)}q_0^2(t)\tilde{q}_k(t),$$
$$\mathscr{H}_i''' = (2\omega_0)^{-1/2}\operatorname{Re} v_1(t)\sum_k [\tilde{\epsilon}_k(t) + \epsilon_k]\tilde{q}_k(t), \tag{5.10}$$
$$\tilde{\epsilon}_k(t) = \epsilon_k + \frac{2\sqrt{2}}{\sqrt{\omega_0}}\epsilon_k^{(2)}\operatorname{Re} v_1(t) = \epsilon_k + 2\epsilon_k^{(2)}(\omega_0^2 - \omega^2)^{-1}h(t),$$

where

$$q_0(t) = (2\omega_0)^{-1/2}[\hat{a}_0(t)\exp(-i\omega_0 t) + \text{H.c.}],$$
$$\tilde{q}_k(t) = (2\omega_k)^{-1/2}[\hat{a}_k\exp(-i\omega_k(t - t_0)) + \text{H.c.}] \tag{5.11}$$

The term $\mathscr{H}_i'$ in Eq. (5.10) determines the interaction with the medium, linear in the coordinate q_0 (q_0 describes the oscillator oscillations at the eigenfrequency). It can be seen from Eqs. (5.10) and (5.6)

that in an external nonresonant field the parameters of the linear interaction $\tilde{\epsilon}_k(t)$ depend periodically on the time, owing to the nonlinearity of the total Hamiltonian of the interaction $\mathscr{H}_i$ with respect to q. It is this which makes possible the decay processes shown in Fig. 20. The term $\mathscr{H}_i''$ in Eq. (5.10), just as the Hamiltonian $H_i^{(2)}$ (4.35) in the absence of a field, describes decay processes with participation of two isolated oscillator quanta. The term $\mathscr{H}_i'''$ in Eq. (5.10) describes the time-dependent shift of the equilibrium positions of the oscillators of the medium.

5.1.2. Quantum Kinetic Equation for a Singled-Out Oscillator The dynamics of a nonequilibrium oscillator interacting with a medium is described by its density matrix $\rho(t)$. It is most convenient to calculate $\rho(t)$ in the interaction representation:

$$\rho(t) = \hat{S}(t,t_0)\tilde{\rho}(t)\hat{S}^{\dagger}(t,t_0), \qquad \tilde{\rho}(t) = \mathrm{Tr}_m\left[\hat{U}(t,t_0)\rho^{(s)}(t_0)\hat{U}^{\dagger}(t,t_0)\right], \tag{5.12}$$

where $\rho^{(s)}$ is the density matrix of the "oscillator + medium" system, and

$$\hat{U}(t,t_0) = T_{t_1}\exp\left[-i\int_{t_0}^{t}dt_1\tilde{\mathscr{H}}_i(t_1)\right]. \tag{5.13}$$

The operator $\hat{U}(t,t_0)$ is a generalization of the operator $\hat{U}(t)$ introduced in Eq. (4.3), to include the case of an oscillator with a Hamiltonian that depends explicitly on the time. As regards $\rho^{(s)}(t_0)$, we shall assume either that at $t \leqslant t_0$ the force $h(t) = 0$ and at the instant of time t_0 the system is in a thermodynamic-equilibrium state

$$\rho^{(s)}(t_0) = Z^{-1}\exp(-H/T), \qquad Z = \mathrm{Tr}\exp(-H/T),$$

or that at $t = t_0$ the operator $\rho^{(s)}$ represents the product of the density matrix of the equilibrium medium by the density matrix of the singled-out oscillator (actually, in the case of weak interaction, the first model is a particular case of the second, accurate to small corrections $\sim\Gamma/T$). The equation for $\tilde{\rho}(t)$ in the region $t - t_0 \gg t_c$ then takes the same form as Eq. (F.9) for $F_M(t)$.

It is convenient to calculate the matrix elements $\tilde{\rho}(t)$ using the

eigenfunctions $|n)$ of the occupation-number operator $\hat{n}_0(t_0)$ of the eigenoscillations of the singled-out oscillator (it is seen from Eq. (5.7) that when the corrections (5.4) are neglected $\hat{n}_0$ is independent of time). According to Eq. (5.7), the matrix elements of the annihilation operator $\hat{a}_0(t)$ on the functions $(n|, |m)$ are equal to:

$$(n|\hat{a}_0(t)|m) = \delta_{m,n+1}m^{1/2}\exp\left\{-iV\left[\left(m-\frac{1}{2}\right)(t-t_0) + \frac{2(\omega_0^2+\omega^2)}{\omega_0(\omega_0^2-\omega^2)^2}\int_{t_0}^{t}dt_1|h_1(t_1)|^2\right]\right\}. \tag{5.14}$$

We note that the functions $|n)$ at $h(t_0) \neq 0$ are not eigenfunctions of the Hamiltonian $H_0(t_0)$.

When account is taken of Eq. (5.14), the kinetic equation of type (F.9) for $\tilde{\rho}(t)$, after integration with respect to t_1 using expressions (5.7), (5.10), (5.11), and (B.3), reduces to a system of differential-difference equations for $\tilde{\rho}(n,m;t)$, similar to the system (4.7):

$$\frac{\partial\tilde{\rho}(n,m;t)}{\partial t} = \sum_j d_j^{(h)}(n,m)\exp\left[-iV_j(n-m)(t-t_0)\right] \times \tilde{\rho}(n+j,m+j;t),$$
$$\tilde{\rho}(n,m;t) = (n|\tilde{\rho}(t)|m). \tag{5.15}$$

For the considered Hamiltonian of the interaction quadratic in the coordinates q of the oscillator, only the coefficients $d_j^{(h)}$ with $|j| \leqslant 2$ differ from 0 in the sum (5.15) over j. Since the nonresonant field does not influence the probability of decay processes in which the oscillator energy is changed by $2\omega_0$ (see expression (5.10) for $\mathscr{H}_i''$), it follows that $d_{\pm 2}^{(h)}(n,m)$ coincides with $d_{\pm 2}(n,m)$, where $d_{\pm 2}(n,m)$ is defined by Eq. (4.36). The coefficients $d_j^{(h)}(n,m)$ with $|j| \leqslant 1$ depend explicitly on $h(t)$ because the field influences the probabilities of the decay processes in which the energy of the eigenoscillations is changed by ω_0 (this is seen from Eq. (5.10) for $\mathscr{H}_i'$). The expressions

for $d_0^{(h)}(n,m)$ and $d_{\pm 1}^{(h)}(n,m)$ are obtained from equations (4.8) and (4.38) for $d_0(n,m)$ and $d_{\pm 1}(n,m)$ by making in the latter the substitutions

$$\Gamma \to \Gamma_h, \qquad \bar{n} \to \bar{n}_h, \qquad P \to P_h, \tag{5.16}$$

where

$$\Gamma_h = \Gamma + \Gamma_+ + \Gamma_- - \Gamma_e,$$

$$\Gamma_\pm = \frac{\pi |h_1(t)|^2}{\omega_0 (\omega_0^2 - \omega^2)^2} \sum_k (\epsilon_k^{(2)})^2 \omega_k^{-1} \delta(\omega_0 \pm \omega - \omega_k),$$

$$\Gamma_e = \frac{\pi |h_1(t)|^2}{\omega_0 (\omega_0^2 - \omega^2)^2} \sum_k (\epsilon_k^{(2)})^2 \omega_k^{-1} \delta(\omega - \omega_0 - \omega_k), \tag{5.17}$$

$$\bar{n}_h = \big[\bar{n}(\omega_0)\Gamma + \bar{n}(\omega_0 + \omega)\Gamma_+ + \bar{n}(\omega_0 - \omega)\Gamma_- + (\bar{n}(\omega - \omega_0) + 1)\Gamma_e \big] \Gamma_h^{-1}.$$

$$P_h = P + P^{(2)} + P_+ + P_-,$$

$$P_\pm = \text{v.p.} \frac{2|h_1(t)|^2}{\omega_0 (\omega_0^2 - \omega^2)^2} \sum_k (\epsilon_k^{(2)})^2 \big[(\omega_0 \pm \omega)^2 - \omega_k^2 \big]^{-1} \tag{5.18}$$

(Γ, P, and $P^{(2)}$ are defined in Eqs. (2.15) and (4.37); the definitions of the parameters $\Gamma_\pm$ and Γ_e differ by a coefficient $\propto |h_1(t)|^2$ from those used in [30]).

Thus, the influence of an external nonresonant field $h(t)$ on the oscillations of the singled-out oscillator at the eigenfrequency reduces to a renormalization of the linear-friction coefficient, on the Planck number for the oscillator, and also of the shift of its eigenfrequency. Expressions (5.15)–(5.18) are valid at arbitrary $|\Gamma_h - \Gamma|/\Gamma$, $|\bar{n}_h - \bar{n}|/\bar{n}$, and $|(P_h - P)/P|$, i.e., the renormalization of the relaxation parameters can be strong; it is necessary only that the conditions for the applicability of a kinetic equation of the type (4.11) be satisfied for Γ_h, P_h, and $\bar{n}_h$.

5.2 *Stationary Distribution of Oscillator in the Absence of Nonlinear Friction. The "Runaway" Effect*

5.2.1 Effective Temperature of Eigenoscillations It can be seen from Eq. (5.17) that the linear-friction coefficient Γ_h, which describes the damping of the eigenoscillations of the oscillator in an external nonresonant field, is the sum of the friction coefficient in the absence of a field Γ and of the coefficients $\Gamma_\pm$ and $-\Gamma_e$, which are proportional to the field intensity $|h_1(t)|^2$. The coefficients Γ and $\Gamma_\pm$ determine the probabilities of the decay of the oscillator with transfer of an energy ω_0 and $\omega_0 \pm \omega$ to the medium, respectively, while Γ_e determines the probability of the field-induced oscillator excitation with transfer of an energy $\omega - \omega_0$ to the medium (see Fig. 20); the quantity Γ_e is therefore subtracted from $\Gamma + \Gamma_+$ in the expression for Γ_h.

The total probabilities of the transitions between the oscillator levels depend not only on the coefficients Γ, $\Gamma_\pm$, and Γ_e but also on the occupation numbers of the vibrations of the medium at the corresponding frequencies. Since the transition probabilities are additive, it is easy to verify that the ratio of the total probabilities of the oscillator transitions "up" and "down" is equal to $\bar{n}_h(\bar{n}_h + 1)^{-1}$, where $\bar{n}_h$ is the effective Planck number of the oscillator and is defined in Eq. (5.17).

The field-induced decay processes can, according to Eq. (5.17) increase as well as decrease the characteristic relaxation time. The role of such processes is easiest to trace in the simplest case, when the nonlinear friction is negligible, e.g., because the density of states of the medium at the frequency $2\omega_0$ is small ($\Gamma^{(2)} \ll |\Gamma_h|$), and the force $h(t)$ is strictly periodic ($h_1(t) = \text{const}$). In this case the relaxation parameters Γ_h, $\bar{n}_h$, and P_h, and consequently also the coefficients $d_j^{(h)}(n,m)$ in the quantum kinetic equation (5.15) are independent of time. We consider first the case $\Gamma_h > 0$, i.e., $\bar{n}_h > 0$. In this case a stationary distribution of the oscillator is established within a time $\Delta t \sim \Gamma_h^{-1}$. According to Eq. (5.15) it takes the form of a Gibbs distribution with an effective temperature T_h:

$$\tilde{\rho}_{\text{st}}(n,m) = \delta_{n,m}(\bar{n}_h + 1)^{-1} \exp(-\omega_0 n / T_h), \qquad T_h = \omega_0 \left(\ln \frac{\bar{n}_h + 1}{\bar{n}_h} \right)^{-1}. \tag{5.19}$$

It is seen from Eq. (5.19) that at $\Gamma_h > 0$ the nonresonant external field does not cause a buildup of regular (coherent) eigenoscillations of the oscillator. According to Eqs. (5.6), (5.12), (5.19), and (5.14) we have

$$\langle \hat{a}(t) \rangle = \mathrm{Tr}_0\big[\tilde{a}(t)\tilde{\rho}(t)\big] = v_1(t),$$

i.e., only regular oscillations at the field frequency are present.

Oscillations at the eigenfrequency, just as in the absence of a field, are stochastic. Their effective temperature T_h, however, can differ substantially from the temperature of the medium. Thus, if the main contribution to the damping Γ_h is made by one of the terms $\Gamma_\pm$, then

$$T_h = T\frac{\omega_0}{\omega_0 \pm \omega}, \qquad \Gamma_\pm \gg \Gamma. \tag{5.20}$$

Equations (5.20) and (5.19) are easy to understand: the oscillator distribution in the energy of the eigenoscillations coincides with the energy distribution of those continuous-spectrum vibrations that cause the relaxation. Thus, with the aid of an external field it is possible to heat strongly the singled-out oscillator (at $\Gamma_- \gg \Gamma, \Gamma_+$ and $\omega_0 \gg \omega_0 - \omega$), as well as to cool it strongly at ($\Gamma_+ \gg \Gamma, \Gamma_-$ and $\omega_0 \ll \omega_0 + \omega$). The processes described by the term Γ_e in Eq. (5.17) also lead to heating of the oscillator at $\Gamma_h > 0$.

The possibility of decreasing the population of the excited level of a two-level system with the aid of an external high-frequency field was discussed in [54, 55]. In the case of an oscillator, according to Eq. (5.19), the distribution in the occupation numbers is exponential, i.e., the introduction of the effective temperature turns out to be not a mere formality. In an external field, the distributions of several other multilevel systems with equidistant energy spectrum have a similar form (e.g., those of spin systems with $s > 1/2$ [56]). It is shown in [30] that the effective temperature T_h can differ substantially from T even in relatively weak fields $h(t)$.

5.2.2 Oscillator Instability in the Case of a Negative Friction Coefficient It is seen from Eq. (5.17) that in a sufficiently strong external field that leads to simultaneous excitation of the oscillator and of a continuous spectrum vibration ($\omega = \omega_0 + \omega_k$), the linear-friction coefficient Γ_h can become negative (if $\Gamma_e > \Gamma + \Gamma_+$). A system with negative friction is unstable. For such a system, $\bar{n}_h < -1$ according to Eq. (5.17), i.e., the probability of a transition to a higher energy ($\propto \Gamma_h \bar{n}_h$) is larger than of a down transition ($\propto \Gamma_h(\bar{n}_h + 1)$). The result is the "runaway" effect.

These qualitative considerations are confirmed by direct calculation of the change of the average energy of the eigenoscillations $\omega_0\langle \hat{n}_0 \rangle$ with time. From Eqs. (5.15), (5.16) and (4.8) at $\Gamma^{(2)} = 0$ we obtain for $\langle \hat{n}_0 \rangle$ the equation

$$\frac{d\langle \hat{n}_0 \rangle}{dt} = -2\Gamma_h(\langle \hat{n}_0 \rangle - \bar{n}_h), \qquad \langle \hat{n}_0 \rangle = \mathrm{Tr}_0(\hat{n}_0(t)\tilde{\rho}(t)) \equiv \sum_n n\tilde{\rho}(n,n;t), \tag{5.21}$$

from which it is seen that at $\Gamma_h < 0$ the average energy $\omega_0\langle n_0 \rangle$ increases like $\exp(2|\Gamma_h|t)$.

The "runaway" of the oscillator consists of a transfer of the excitation to increasingly higher levels of the eigenoscillation energy in the course of time. This transfer can be easily tracked in the simplest case when Γ and $\Gamma_\pm$ are equal to zero, and

$$\Gamma_h = -\Gamma_e, \qquad \bar{n}(\omega - \omega_0) = 0. \tag{5.22}$$

In this case, according to Eq. (5.17), $\bar{n}_h = -1$. If only the lowest level is filled at the initial instant

$$\tilde{\rho}(n,n;t_0) = \delta_{n,0},$$

then the solution of (5.15) takes the form

$$\tilde{\rho}(n,n;t) = \exp\big[-2\Gamma_e(t-t_0)\big]\big\{1 - \exp\big[-2\Gamma_e(t-t_0)\big]\big\}^n. \tag{5.23}$$

It can be seen from Eq. (5.23) that $\tilde{\rho}(n,n;t) \to 0$ as $t \to \infty$ for any finite n, but $\tilde{\rho}(n,n;t)$ increases with time in a certain time interval that depends on the number n. The maximum of the distribution (5.23),

which is located at

$$n_{\max} \approx \exp\left[2\Gamma_e(t - t_0)\right] - 1,$$

shifts towards larger n with increasing $t - t_0$. As a result, no stationary distribution is established (this is clear also from Eq. (5.19): since $\bar{n}_h \leqslant -1$ at $\Gamma_h < 0$, it follows that $T_h < 0$ and the distribution (5.19) is not normalizable, i.e., the formal stationary solution (5.19) of Eq. (5.15) has no physical meaning).

In the region

$$(2\langle \hat{n}_0 \rangle + 1)\left| V \frac{d\Gamma_h}{d\omega_0} \right| \ll |\Gamma_h|$$

the nonequidistance of the oscillator levels does not influence practically the average energy of the eigenoscillations and, what is more, the diagonal elements of $\tilde{\rho}$ at all, see Eq. (5.15). At the same time the nonequidistance alters qualitatively the character of the time dependence of the amplitude of the regular oscillations of the oscillator at the eigenfrequency ω_0. It was shown above that at $\Gamma_h > 0$, when the stationary distribution is of the form (5.19), there are no stationary regular oscillations at the eigenfrequency. The case $\Gamma_h < 0$ calls for a special analysis, since Eq. (5.19) cannot be used here.

The amplitude of the regular oscillations at the eigenfrequency is, according to Eqs. (5.6), (5.12), and (5.14),

$$A_{\text{reg}}(t) = \frac{\sqrt{2}}{\sqrt{\omega_0}} \left| \mathrm{Tr}_0\left[\hat{a}_0(t)\tilde{\rho}(t) \right] \right|$$

$$= \frac{\sqrt{2}}{\sqrt{\omega_0}} \left| \sum_n (n|\hat{a}_0(t)|n+1)\tilde{\rho}(n+1, n; t) \right|. \qquad (5.24)$$

In the calculation of $A_{\text{reg}}(t)$ it is convenient to express $\tilde{\rho}(n+1, n; t)$ in terms of the Green's function of Eq. (5.15), and use for the latter the explicit solution (4.18) obtained for Eq. (4.14), which is similar to Eq.

(5.15). In the region of long times the result takes the form

$$A_{\text{reg}}(t) \propto A_{\text{reg}}(t_0)\left| 1 - \frac{2\Gamma_h - i(V + V^{(2)})}{2a_h} \right|^{-2}$$

$$\times \exp\left[-(|\Gamma_h| + 2|a_h'|)(t - t_0) \right],$$

$$\Gamma_h < 0, \qquad |\Gamma_h|(t - t_0) \gg 1,$$

$$|(V + V^{(2)})\bar{n}_h \Gamma_h^{-1}|^{-1} \ll \exp\left[2|\Gamma_h|(t - t_0)\right], \qquad (5.25)$$

$$a_h = \left[\Gamma_h^2 - i\Gamma_h(V + V^{(2)})(2\bar{n}_h + 1) - \tfrac{1}{4}(V + V^{(2)})^2 \right]^{1/2},$$

$$\Gamma_h < 0, \qquad a_h' \equiv \mathrm{Re}\, a_h < 0.$$

According to Eq. (5.25), the regular (coherent) eigenoscillations of a nonlinear oscillator decay exponentially at long times even under "runaway" conditions.

A limiting transition to the case of the harmonic oscillator ($V + V^{(2)} = 0$) is impossible in Eq. (5.25); in particular, as $V + V^{(2)} \to 0$ the pre-exponential factor diverges. A direct calculation of $A_{\text{reg}}(t)$ in the case of a harmonic oscillator leads to a qualitatively different result, the exact opposite of (5.25):

$$A_{\text{reg}}(t) \propto A_{\text{reg}}(t_0)\exp\left[|\Gamma_h|(t - t_0) \right],$$

$$|\Gamma_h|(t - t_0) \gg 1, \qquad V + V^{(2)} = 0, \qquad \Gamma_h < 0, \qquad (5.26)$$

According to Eq. (5.26) the amplitude of the regular oscillations at the frequency ω_0 increases exponentially with time.

Such a difference between the harmonic and weakly-nonlinear oscillator at $\Gamma_h < 0$ is due to the fact that the initial eigenoscillation excitation, which corresponds to excitation of transitions between some low-lying neighboring levels, is transferred in the case of the harmonic oscillator resonantly and coherently (on account of the interference of the transitions) to higher-lying levels and the ampli-

tude of the oscillations increases. In the case of a nonlinear oscillator, since the levels are not equidistant, the energy interval between sufficiently high levels differs strongly from the interval between low-lying levels even at small $|(V + V^{(2)})\Gamma_h^{-1}|$. This suppresses the interference of the transitions. As a result, transitions between high-lying levels are not excited practically, and the amplitude of the oscillations decays (for details see [30]).

Energy runaway, in contrast to runaway in the amplitude of the coherent eigenoscillations, is due to incoherent transitions from level to level, and therefore the small nonequidistance of the Duffing-oscillator levels does not stop the energy runaway in the case of linear friction. Only the strong nonlinearity, which always takes place in real vibrating systems at high energies, stops the runaway.

5.3 *Suppression of Energy "Runaway" in Systems with Nonlinear Friction*

5.3.1 Gaussian Stationary Distribution in the Case of Weak Nonlinear Friction The role of nonlinear friction increases with increasing oscillator energy; e.g., according to Eq. (4.38) the contribution of the nonlinear friction to the expression for the level half-width $\Gamma(n)$ at large n increases like n^2, while the contribution of the linear friction increases like n. Therefore, if $\Gamma^{(2)} \neq 0$, the nonlinear friction exceeds the linear at sufficiently high energies of the eigenoscillations, and in the case when the linear friction is "negative" and energy "runaway" sets in, the nonlinear friction stabilizes the system. As a result, a stationary distribution of the oscillator develops in a monochromatic field. It was analyzed in [30] by the generating-function method and it was shown there that at sufficiently low $\Gamma^{(2)}/\Gamma_h$ population inversion sets in in a definite energy region.

Since the expression obtained in [30] for the generating function in terms of a confluent hypergeometric function is not convenient for analysis, we consider below the stationary distribution of the oscillator in the limiting cases of weak nonlinear friction ($\Gamma^{(2)} \ll |\Gamma_h|$) or high temperatures ($T, |T_h| \gg \omega_0$).

At $\Gamma^{(2)} \ll |\Gamma_h|$ and $\Gamma_h < 0$, the oscillator "runs away" from the low-energy region, where the nonlinear friction is negligible (see Eq. (5.23)), and in the stationary regime it occupies mainly the states for

which the negative linear friction and the positive nonlinear friction cancel each other. The numbers n of the corresponding levels are large; as seen from Eqs. (5.16) and (4.38),

$$n \sim |\Gamma_h|/\Gamma^{(2)} \gg 1.$$

The distribution of the oscillator over the levels $\tilde{\rho}_{st}(n,n)$ depends smoothly on the level number n near the maximum. We can then expand $\tilde{\rho}_{st}(n+j,n+j)$ ($|j| \leqslant 2$) in Eq. (5.15) in powers of j and, just as the derivation of Eq. (4.47), it suffices to retain the terms quadratic in j. As a result, Eq. (5.15) is reduced in the stationary case to the form

$$\frac{d}{dn}\left\{ n\left[n(2\bar{n}_2 + 1) + \delta^{-1}|2\bar{n}_h + 1| \right] \frac{d\tilde{\rho}_n}{dn} + n(n - 2\delta^{-1})\tilde{\rho}_n \right\} = 0, \tag{5.27}$$

$$\tilde{\rho}_n \equiv \tilde{\rho}_{st}(n,n), \qquad \delta = \frac{4\Gamma^{(2)}}{|\Gamma_h|} \equiv -\frac{4\Gamma^{(2)}}{\Gamma_h}, \qquad \delta \ll 1, \qquad n \gg 1$$

In Eq. (5.27), besides the higher-order derivatives such as $d^3\tilde{\rho}_n/dn^3$ etc., we have discarded terms proportional to lower powers of n and δ^{-1}.

Using the condition that $\tilde{\rho}_{st}(n,n)$ vanish as $n \to \infty$, we can find from Eq. (5.27) that $\tilde{\rho}_{st}(n,n)$ is described by a curve with a maximum located at

$$n = n_m = 2\delta^{-1}, \tag{5.28}$$

and the distribution near the maximum is Gaussian:

$$\tilde{\rho}_{st}(n,n) = \text{const} \times \exp\left[-\frac{(n - n_m)^2}{n_m \mathscr{H}(T)} \right], \tag{5.29}$$

$$\mathscr{H}(T) = 2(2\bar{n}_2 + 1) + |2\bar{n}_h + 1|, \qquad |n - n_m| \ll n_m \mathscr{H}(T)$$

Thus, for all the oscillator levels with $n < n_m$ the population $\tilde{\rho}_{st}(n,n)$ increases with the increasing number of the level, i.e., population

inversion is present. In the region of relatively low temperatures, where $\kappa(T)\sim 1$, the width of the maximum of the distribution is $\sim n_m^{1/2}$; it is small compared with n_m, i.e., the peak of $\tilde{\rho}_{st}(n,n)$ is narrow at $\Gamma^{(2)} \ll |\Gamma_h|$. With increasing temperature, the width increases quite rapidly (like $(T+|T_h|)^{1/2}$ at $T, |T_h| \gg \omega_0$).

At low temperatures, the expression (5.29) and Eq. (5.27) are not valid on the distribution wings, since $\tilde{\rho}_{st}(n,n)$ changes relatively strongly there when n is changed by unity, and it is incorrect to change over from the difference equation (5.15) to the differential equation (5.27). A more complete investigation of the dynamics of an oscillator in a nonresonant field is possible in the region of relatively high temperatures,

$$T \gg 2\omega_0, \qquad |T_h| \gg \omega_0, \tag{5.30}$$

when the classical description of the oscillator motion is valid. We note that, as can be seen from the definition (5.19), (5.17) of the effective temperature T_h, the second inequality in (5.30) can hold even at small $T/(\omega-\omega_0)$, if $|\Gamma_h|$ is small compared with Γ_e, i.e., the field $h(t)$ is not too strong (but at the same time $\Gamma_e > \Gamma + \Gamma_+$, i.e., $\Gamma_h < 0$).

5.3.2 Classical Description of the Dynamics of an Oscillator in a Nonresonant Field. Comparison with the Van Der Pol Oscillator At high temperatures we can analyze the dynamics of an oscillator by using a classical approach. Just as in the absence of a nonresonant field, it is convenient to describe the oscillations at the eigenfrequency with the aid of a complex amplitude $u_0(t)$ that varies slowly over a time $\sim\omega_0^{-1}$ and $|\omega-\omega_0|^{-1}$. This amplitude can be introduced in our case, in analogy with Eq. (1.5), in the following manner

$$\begin{aligned} q(t) &= \left[u_0(t)e^{i\omega_0 t} + \text{c.c}\right] + \left(\omega_0^2-\omega^2\right)^{-1}\left[h_1 e^{-i\omega t} + \text{c.c.}\right], \\ \dot{q}(t) &= i\omega_0\left[u_0(t)e^{i\omega_0 t} - \text{c.c.}\right] - i\omega\left(\omega_0^2-\omega^2\right)^{-1}\left[h_1 e^{-i\omega t} - \text{c.c.}\right] \end{aligned} \tag{5.31}$$

Eliminating from the classical equation of motion for the oscillator the continuous-spectrum vibrations (see Sec. 2.1) and neglecting the

corrections (5.4), we can obtain for $u_0(t)$ a stochastic differential equation (cf. Eqs. (2.14), (4.40)) that takes into account the spontaneous decay processes as well as those induced by an external field:

$$\begin{aligned} \dot{u}_0 = &-\left[\left(\Gamma_h + 4\Gamma^{(2)}\omega_0|u_0|^2\right) - iP_h\right]u_0 - \frac{3\tilde{\gamma}}{2i\omega_0}|u_0|^2u_0 \\ &+ f_h(t) + u_0^* f_1^{(2)}(t), \qquad u_0 \equiv u_0(t). \end{aligned} \tag{5.32}$$

Here the damping Γ_h and the frequency shift P_h are defined by Eqs. (5.17) and (5.18), while the random forces $f_h(t)$ and $f_1^{(2)}(t)$ constitute independent Gaussian δ-correlated processes (in the "slow" time scale), with

$$\begin{aligned} \langle f_h(t)f_h^*(t')\rangle &= \frac{\Gamma_h T_h}{\omega_0^2}\delta(t-t'), \\ \langle f_1^{(2)}(t)f_1^{(2)*}(t')\rangle &= \frac{4\Gamma^{(2)}T}{\omega_0}\delta(t-t'). \end{aligned} \tag{5.33}$$

Strictly speaking, the right-hand side of Eq. (5.32) contains also a term

$$\sim i\gamma|h_1|^2\omega_0^{-1}\left(\omega_0^2-\omega^2\right)^{-2}u_0,$$

which describes the Stark shift of the frequency of the nonlinear oscillator, cf. Eq. (5.7); it is regarded below as included in P_h.

Equation (5.32) differs from the analogous equation (4.40) in that the parameter Γ_h that describes the linear friction in (5.32) can be negative, and the effective temperature T_h that determines the intensity of the random force $f_h(t)$ is not equal to the temperature T of the medium (and is likewise negative if $\Gamma_h < 0$; at the same time, as seen from Eqs. (5.19) and (5.17), the random-force intensity $\propto \Gamma_h T_h > 0$).

Neglecting the fluctuations, the equation for the squared amplitude of the eigenoscillations $r_0/4$ takes according to Eq. (5.32) the form

$$\dot{r}_0 = -2\left(\Gamma_h + 4\Gamma^{(2)}\omega_0 r_0\right)r_0, \qquad r_0 = |u_0|^2. \tag{5.34}$$

At $\Gamma_h < 0$ the stable stationary solution of Eq. (5.34) is

$$r_0 = \mathrm{r}_0, \qquad \mathrm{r}_0 = \frac{|\Gamma_h|}{4\Gamma^{(2)}\omega_0} \qquad (\Gamma_h < 0). \tag{5.35}$$

Thus, in the classical case, just as in the quantum case, the nonresonant field that induces decay processes with excitation of the oscillator leads to buildup of its eigenoscillations. The mean squared amplitude, according to Eq. (5.35), is proportional to absolute value of the negative linear friction coefficient Γ_h and is inversely proportional to the nonlinear friction coefficient $\Gamma^{(2)}$.

Equation (5.34) for r_0 coincides with the equation for the squared amplitude of the Van der Pol oscillator (see Eq. (1.10)).

It is convenient to analyze the fluctuations of a classical oscillator in an exciting nonresonant field by using the EFP equation that corresponds to the Langevin equation (5.32). From a comparison of Eqs. (5.32), (5.33) with Eqs. (4.40), (4.41) it is clear that the EFP equation for the case investigated here differs from the EFP equation (4.42) for an oscillator in the absence of the external fields only in that Γ, ΓT, and P in Eq. (4.42) must be replaced by Γ_h, $\Gamma_h T_h$, and P_h respectively. The stationary solution of the EFP equation, which describes the distribution function of the oscillator in the steady state, is of the form

$$w_{\mathrm{st}}(r_0, \varphi_0) = Z_0^{-1}\left(z_0 + \mathrm{r}_0\frac{|T_h|}{T}\right)^{\zeta}\exp\left(-\frac{2\omega_0^2 r_0}{T}\right), \tag{5.36}$$

$$\zeta = \frac{2\omega_0^2 \mathrm{r}_0}{T}\left(1 + \frac{|T_h|}{T}\right), \qquad Z_0 = \pi\int dr_0\left(r_0 + \mathrm{r}_0\frac{|T_h|}{T}\right)^{\zeta}\exp\left(-\frac{2\omega_0^2 r_0}{T}\right).$$

The distribution (5.36) has a maximum at $r_0 = \mathrm{r}_0$ and is Gaussian near the maximum (cf. Eq. (5.29); the variable r_0 in the classical theory corresponds to $(2\omega_0)^{-1}n$ in the quantum theory at $n \gg 1$). On the tail, however, the distribution (5.36) decreases like $\exp(-2\omega_0^2 r_0/T)$, i.e., just as the distribution of an oscillator that interacts with a medium in the absence of a nonresonant force. It is this which manifests the difference of the results of the microscopic model considered here from the results of the phenomenological model of a random force as white noise independent of the coordinates [10, 13]. In the latter model w_{st} decreases at large r_0 like $\exp(-Ar_0^2)$.

The form of the distribution (5.36) as a whole also differs from the distribution of a Van der Pol oscillator in the indicated phenomenological model, and only near the maximum do the distributions have the same form. If the maximum is narrow,

$$\frac{\omega_0^2 \mathrm{r}_0}{T + |T_h|} \gg 1, \tag{5.37}$$

then the results of the linearized theory [10, 13] are applicable to a considerable extent to the model considered here. In the general case, however, the character of the fluctuations in the microscopic model (5.2), (4.1), and (4.35) turns out to be entirely different from that in the indicated phenomenological model. In particular, near the excitation threshold ($\Gamma_h = 0$) the distribution w_{st} is described in the phenomenological model by the function $\exp(-Ar_0^2)$; in the model considered here the parameter $\mathrm{r}_0|T_h|$ in Eq. (5.36) turns out to be finite as $\Gamma_h \to 0$ ($T_h \propto \Gamma_h^{-1}$ at small Γ_h) and the distribution (5.36) differs substantially from Gaussian.

If we replace in Eq. (5.36) the parameter ζ by

$$\tilde{\zeta} = \frac{2\omega_0^2 \mathrm{r}_0}{T}\left(\frac{T_h}{T} - 1\right), \tag{5.38}$$

the expressions (5.36), (5.35), and (5.38) describe the distribution of an oscillator in a nonresonant field at $\Gamma_h > 0$. In this case the function $w_{\mathrm{st}}(r_0, \varphi_0)$ decreases monotonically with increasing r_0. At $\Gamma_h \gg \Gamma^{(2)}$, when the nonlinear friction is weak, the distribution $w_{\mathrm{st}}(r_0, \varphi_0)$ in the vital region $r_0 \ll \mathrm{r}_0 T_h/T$ is close to a Gibbs distribution with an effective temperature T_h (see Eq. (5.19)).

It follows from Eqs. (5.36) and (5.38) that both at $\Gamma_h < 0$ and at $\Gamma_h > 0$ the oscillator does not perform regular oscillations at the eigenfrequency. Even when the condition (5.37) is satisfied, and the spread of the amplitude of the eigenoscillations is small, the phase of

the oscillations remains random. The phase-diffusion coefficient is in this case

$$\frac{\Gamma^{(2)}T}{\omega_0}\left(1+\frac{|T_h|}{T}\right) \qquad (\Gamma_h<0)$$

and decreases both with decreasing temperature and with increasing nonresonant field intensity (an increase of Γ_e leads according to Eqs. (5.19) and (5.17) to a decrease of $|T_h|/T$ at $\Gamma_h<0$).

5.4 *Spectral Distribution for an Oscillator in a Nonresonant Field.*

The relaxation-parameter changes due to the nonresonant field result in alteration the time correlation functions of the singled-out oscillator and their spectral distributions. If the field $h(t)$ is strictly periodic and a stationary regime is realized, the density matrix of the "oscillator + medium" system also depends periodically on the time, with a period $2\pi/\omega$, while the two-time correlation functions can be represented in the form

$$\langle\hat{L}(t)\hat{M}(t')\rangle=\sum_{n=-\infty}^{\infty}\chi_{LM}(t-t';n)e^{in\omega t'}. \tag{5.39}$$

In this representation one can explicitly separate the periodic dependence of the double-time correlators on one of the arguments.

For a number of applications (see [30, 56]) it is of interest to calculate the correlators averaged over the period $2\pi/\omega$:

$$\langle\hat{L}(t)\hat{M}(t')\rangle_{\mathrm{av}}\equiv\chi_{LM}(t-t';0) \tag{5.40}$$

(the subscript av denotes averaging over the period). Transforming with the aid of Eq. (5.8) to the interaction representation, we can reduce these correlators to a form similar to (4.4) and reduce next the problem to a solution of the quantum kinetic equation, as was done in Sec. 4.1. In particular, the expression for the spectral representation $Q_h'(\omega')$ of a double-time correlation function of the annihilation and creation operators in the frequency region $\omega'\approx\omega_0$ can be written,

taking Eqs. (5.6), (5.7) and (5.14) into account, in the form

$$\begin{aligned}Q_h'(\omega')&=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,e^{i\omega' t}\langle\hat{a}(t+t')\hat{a}^+(t')\rangle_{\mathrm{av}}\\&=\frac{1}{\pi}\,\mathrm{Re}\int_0^{\infty}dt\,e^{i\omega' t}Q_h'(t), \qquad \omega'\approx\omega_0,\end{aligned} \tag{5.41}$$

$$Q_h'(t)=e^{-i\omega_0' t}\sum_{n=1}^{\infty}n^{1/2}\exp\left[-iV\left(n-\frac{1}{2}\right)t\right]F_{a^+}^{(h)}(n,n-1;t),$$

$$\omega_0'=\omega_0+2V|h_1|^2(\omega^2+\omega_0^2)\omega_0^{-1}(\omega^2-\omega_0^2)^{-2}.$$

The function $F_{a^+}^{(h)}(n,n-1;t)$ is here the solution of Eq. (5.15) (in which we must put $m=n-1$), with the initial condition

$$F_{a^+}^{(h)}(n,n-1;0)=n^{1/2}\tilde{\rho}_{\mathrm{st}}(n-1,n-1). \tag{5.42}$$

The quantities $F_{a^+}^{(h)}(n,n-1;t)$ are similar to the matrix elements (4.5) of the operator $F_M(t)$ ($\hat{M}=\hat{a}^+$), which describes the damping of the correlator $Q'(t)$ in the absence of an external field; at $h=0$ expression (5.42) goes over to Eq. (4.12).

From a comparison of Eq. (5.15) and Eq. (4.7) it follows that in the case of weak nonlinear friction ($\Gamma^{(2)}\ll\Gamma_h$) and at $\Gamma_h>0$, when the stationary distribution takes the form (5.19), the function $Q_h'(t)$ is described by formula (4.22), if we replace Γ, P, and $\bar{n}$ in the latter by Γ_h, P_h, and $\bar{n}_h$ (and ω_0 by ω_0'). This enables us to analyze the dependence of the form of the spectral distribution $Q_h'(\omega')$ on the field $h(t)$ and on the type of the field-induced decay processes. In the case of the decays shown in Figs. 20a and 20b, the broadening Γ_h increases with the field. If, however, $\bar{n}>1$ and $|V+V^{(2)}|\gg\Gamma_h$, then the total width of $Q_h'(\omega')$ can decrease in a cooling field ($\Gamma_+\gg\Gamma_-,\Gamma_e$), inasmuch as in this case transitions from a smaller number of levels take part in the formation of $Q_h'(\omega')$, and the nonequidistance of the oscillator energy levels has a weaker effect. At $|V+V^{(2)}|\ll\Gamma_h$ a decrease of the width of $Q_h'(\omega')$ (which is approximately equal to Γ_h) takes place in the field if $\Gamma_e\gg\Gamma_\pm$. With decreasing Γ_h, however, the effective temperature of the oscillator increases, and at $\Gamma_h\sim\bar{n}_h|V+$

$V^{(2)}|$ the distribution $Q'_h(\omega')$ begins to broaden with increasing field on account of the nonequidistance of the levels.

In the general case of arbitrary Γ_h and $\Gamma^{(2)}$ an analysis of $Q'_h(\omega')$ can be performed by taking the Fourier transform of Eq. (5.15) for $F_{a^+}^{(h)}(n, n-1; t)$ and by representing $Q'_h(\omega')$, with Eq. (5.41) taken into account, as a sum of partial spectra $\mathrm{Re}\,\varphi_h(n, \omega')$. These spectra are similar to those in Eq. (4.28) and correspond to transitions between neighboring levels of the oscillator. At $|V + V^{(2)}| \gg |\Gamma_h|, \Gamma^{(2)}$, the spectra overlap little and the intensity of the nth spectrum is proportional to the population of the $(n-1)$-level $\tilde{\rho}_{\mathrm{st}}(n-1, n-1)$. If $\Gamma_h < 0$ and population inversion is present in a definite region of n, the intensities of the partial spectra in this region increase with n. Therefore the coefficient of the absorption of the additional weak resonant field with frequency $\omega' \approx \omega_0$ by the oscillator, which is proportional to

$$\sum_n \mathrm{Re}\,\varphi_h(n, \omega)\tilde{\rho}_{\mathrm{st}}^{-1}(n-1, n-1)\left[\tilde{\rho}_{\mathrm{st}}(n-1, n-1) - \tilde{\rho}_{\mathrm{st}}(n, n)\right],$$

appears to be negative in the corresponding frequency region, i.e., the oscillator can amplify a weak resonant field at the expense of a strong nonresonant field. Since the absorption coefficient integrated over the frequency ω' (in the region $\omega' \approx \omega_0$) is positive for the oscillator, the case of a pronounced fine structure is most favorable for obtaining large amplification coefficients in a certain frequency interval.

§6 Spectral Distributions of Local and Quasilocal Vibrations

Among the physical systems described by the model of a Duffing oscillator interacting with a medium are local and quasilocal vibrations in imperfect crystals. Local vibrations arise, e.g., near light impurity atoms or those strongly bound to the crystal. The frequencies of these vibrations are outside the phonon-frequency bands of a perfect crystal and therefore, in the harmonic approximation, they do not propagate in the crystal and are localized near the defect. Quasilocal (or resonant) vibrations arise, e.g., near heavy or weakly bound impurity atoms. Their frequencies are located in the phonon-frequency band of the perfect crystal, but lie in a region with low state

density. The state density of an imperfect crystal has narrow peaks at the frequencies of the quasilocal vibrations, and the amplitude of atom vibrations at the corresponding frequencies has a sharp maximum (in space) near the defect. Quasilocal vibrations can actually be regarded as excitations that are localized near a defect and decay slowly into waves propagating through the crystal. A particular case of local or quasilocal vibrations are the intramolecular modes of weakly bound impurity molecules, whose frequencies fall respectively in forbidden or allowed regions of the spectrum of the perfect crystal.

The damping of the local and quasilocal vibrations is due to their interaction with other modes of the crystal. Greatest interest attaches to the case when the interaction is weak enough, so that the lifetimes are long. Therefore the local and quasilocal vibrations are examples of oscillators that interact weakly with the medium, and analysis given in the preceding sections can be applied to them. In most cases the amplitudes of the vibrations of the atoms that take part in the localized modes are low, and therefore the nonlinearity is likewise relatively small. This allows us to limit the expansion of the potential energy up to the terms of fourth order in the coordinates, i.e., to reduce the problem to that of Duffing oscillators. Usually several localized modes arise near the defects, so that in contrast to the single-oscillator case discussed above, we must consider the more general case of a small number of interacting nonlinear oscillators. Allowance for the nonlinearity (even if it is relatively small) appears to be necessary for the interpretation of the number of results on the spectra of infrared absorption and Raman scattering of light by local and quasilocal vibrations, since, as follows from the results of Secs. 2 and 4, even a weak nonlinearity leads to a qualitative change of the spectral distributions.

6.1 *Hamiltonian of an Imperfect Crystal in the Presence of Local and Quasilocal Vibrations*

We confine ourselves to the case when the crystal contains one defect. We shall label the local and quasilocal vibrations by the index κ and denote by q_κ and p_κ their normal coordinates and momenta. We consider first the case when the frequencies ω_κ are not close to one another and, as a consequence of the symmetry, there are no terms

cubic in q_κ in the Hamiltonian H_0 of the "isolated" localized vibrations (i.e., of those whose interaction with the vibrations of the continuous spectrum is neglected),

$$H_0 = \frac{1}{2}\sum_\kappa (P_\kappa^2 + \omega_\kappa^2 q_\kappa^2) + \frac{1}{4}\sum_\kappa \gamma_{\kappa\kappa} q_\kappa^4 + \frac{3}{8}\sum_{\kappa\neq\kappa'} \gamma_{\kappa\kappa'} q_\kappa^2 q_{\kappa'}^2 . \quad (6.1)$$

The parameters $\gamma_{\kappa\kappa'}$ with $\kappa \neq \kappa'$ determine the nonlinear interaction of the different singled-out (local or quasilocal) vibrations with one another.

In quantum theory it is convenient to express the Hamiltonian H_0 in terms of the annihilation and creation operators $\hat{a}_\kappa$ and $\hat{a}_\kappa^+$

$$H_0 = \sum_\kappa \omega_\kappa \hat{a}_\kappa^+ \hat{a}_\kappa + \frac{1}{2}\sum_{\kappa,\kappa'} V_{\kappa\kappa'} \hat{n}_\kappa \hat{n}_{\kappa'} , \quad (6.2)$$

$$\hat{n}_\kappa = \hat{a}_\kappa^+ \hat{a}_\kappa , \quad q_\kappa = (2\omega_\kappa)^{-1/2}(\hat{a}_\kappa + \hat{a}_\kappa^+), \quad V_{\kappa\kappa'} = \frac{3}{4}\frac{\gamma_{\kappa\kappa'}}{\omega_\kappa \omega_{\kappa'}} \quad (\hbar = 1).$$

Here, just as in Eq. (1.14), we have discarded nonresonant terms of the type $\hat{a}_\kappa^3 \hat{a}_{\kappa'}$, $\hat{a}_\kappa^2 \hat{a}_{\kappa'}^2$, etc., which are of no importance in what follows, and have renormalized the frequencies ω_κ to the quantities $\sim V_{\kappa\kappa'}$.

The total Hamiltonian which describes the singled-out vibrations and the continuous-spectrum vibrations can be written in the form

$$H = H_0 + H_m + H_i , \qquad H_m = \sum_k \omega_k \hat{a}_k^+ \hat{a}_k ,$$

$$H_i = H_i^{(1)} + H_i^{(2)} + H_i^{(3)} + H_i^{(4)}, \qquad H_i^{(1)} = \sum_{\kappa,k} V_{\kappa k} \hat{c}_\kappa \hat{c}_k , \quad (6.3)$$

$$H_i^{(2)} = \sum_{\kappa,\kappa',k} V_{\kappa\kappa' k} \hat{c}_\kappa \hat{c}_{\kappa'} \hat{c}_k , \qquad H_i^{(3)} = \frac{1}{2}\sum_{\kappa,k,k'} V_{\kappa k k'} \hat{c}_\kappa \hat{c}_k \hat{c}_{k'} ,$$

$$H_i^{(4)} = \frac{1}{2}\sum_{\kappa,\kappa',k,k'} V_{\kappa\kappa' k k'} \hat{c}_\kappa \hat{c}_{\kappa'} \hat{c}_k \hat{c}_{k'} , \qquad \hat{c} = \hat{a} + \hat{a}^+ .$$

We have discarded in the Hamiltonian $H_i^{(4)}$ the terms that describe the decays with participation of three singled-out vibrations or three continuous-spectrum vibrations.

If we choose the coordinates q_κ and q_k to be the true normal coordinates of an imperfect crystal, then the term $H_i^{(1)}$ in Eq. (6.3) vanishes. This choice is convenient for local vibrations. In the case of quasilocal vibrations (e.g., intramolecular vibrations whose frequencies fall in the region of low state density) it may turn out to be more convenient to start from approximate normal coordinates (e.g., those of the intramolecular vibrations of the impurity and of the crystal vibrations). In this case the term $H_i^{(1)}$ differs from zero and describes the transformation of a quasilocal vibration into a delocalized vibration having the same frequency.

6.2 *Infrared-Absorption Cross Section*

The cross section $\sigma(\omega)$ for the absorption of polarized light of frequency ψ by a single optically active impurity center is expressed in terms of the spectral representation of the time correlation function of the dipole-moment operator M (cf. [15, 57]):

$$\sigma(\omega) = \frac{4\pi^2\omega}{cn(\omega)}\left[1 - \exp\left(-\frac{\omega}{T}\right)\right]\langle M_x , M_x\rangle_\omega , \quad (6.4)$$

$$\langle M_x , M_x\rangle_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\langle M_x(t) M_x(0)\rangle .$$

Here c is the velocity of light in vacuum, $n(\omega)$ is the refractive index; the electric field in the wave is parallel to the x axis. The operator M_x can be expanded in powers of $\hat{c}_\kappa$ and $\hat{c}_k$. For the investigation of a narrow peak of $\sigma(\omega)$ in the region $\omega \approx \omega_\kappa$ it suffices to retain this expansion only the linear terms

$$M_x = \sum_\kappa m_\kappa c_\kappa + \sum_k m_k c_k , \quad (6.5)$$

and the main contribution to the cross section is made by the term with the given κ:

$$\sigma(\omega) = \frac{4\pi^2\omega}{cn(\omega)}\left[1 - \exp\left(-\frac{\omega}{T}\right)\right] m_\kappa^2 Q_\kappa'(\omega), \qquad \omega \approx \omega_\kappa , \quad (6.6)$$

$$Q_\kappa'(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\langle \hat{a}_\kappa(t)\hat{a}_\kappa^+(0)\rangle$$

6.2.1 Relaxation Parameters According to Eq. (6.6) the cross section for the light absorption is proportional to the spectral distribution $Q'_\kappa(\omega)$ of the time correlation function of the coordinates of the local or quasilocal vibrations (the parameter m_κ^2 determines the oscillator strength of the transition). In the case when there is only one singled-out mode, the function $Q'_\kappa(\omega) \equiv Q'(\omega)$ was analyzed in detail for the Duffing-oscillator model in Secs. 2 and 4. For the interaction Hamiltonian $H_i = H_i^{(1)}$ we have obtained the explicit expressions (4.22) and (4.24) for $Q'(t)$ and $Q'(\omega)$. The form of the obtained spectral distribution depends on the values of the damping parameter $\Gamma_\kappa \equiv \Gamma$ and of the nonequidistance of the oscillator levels $V_{\kappa\kappa} \equiv V$. The formulas given in Sec. 2 for Γ, which correspond to the interaction $H_i^{(1)}$ and determine the probability of decay of a quasilocal vibration with creation of one continuous-spectrum vibration, describe with sufficient accuracy the damping of the low-frequency quasilocal vibrations. In the case of local vibrations, the interaction $H_i^{(1)}$ is absent and under certain conditions the main contribution to Γ_κ is made by the interaction $H_i^{(3)}$, which describes the decay or scattering of a singled-out vibration with participation of two phonons (it can play an essential role also for high-frequency quasilocal vibrations). The resultant damping Γ_κ connected with the interaction $H_i^{(1)} + H_i^{(3)}$ which is linear in c_κ, is described according to Eq. (F.15) by the formula

$$\begin{aligned}\Gamma_\kappa = {} & \pi \sum_k V_{\kappa k}^2 \delta(\omega_\kappa - \omega_k) \\ & + \frac{\pi}{2} \sum_{k,k'} V_{\kappa k k'}^2 \big[(\bar{n}_k + \bar{n}_{k'} + 1)\delta(\omega_\kappa - \omega_k - \omega_{k'}) \\ & \qquad + 2(\bar{n}_{k'} - \bar{n}_k)\delta(\omega_\kappa - \omega_k + \omega_{k'}) \big], \\ & \bar{n}_k \equiv \bar{n}(\omega_k).\end{aligned} \tag{6.7}$$

The first term in Eq. (6.7) is independent of temperature [58, 59], while the second term [7, 53, 60] is proportional to T at high temperatures.

The term $H_i^{(2)}$ determines the nonlinear friction. Its role was analyzed in Sec. 4.4. The interaction $H_i^{(4)}$ was not considered above.

It leads to a certain renormalization of the nonlinear-friction constant (on account of decays of the type $2\omega_\kappa = \omega_k \pm \omega_{k'}$). In addition, it leads to a specific damping which is not connected with the finite lifetime (with the energy relaxation time) and is of modulation origin. This damping is due to quasielastic scattering of the vibrations of the continuous spectrum by a local or quasilocal vibration. At arbitrary values of the other relaxation parameters and nonlinearity parameters, this process causes in the expression for the time correlation function an additional factor

$$\exp(-\Gamma_\kappa^{(4)}|t|), \tag{6.8}$$

where

$$\Gamma_\kappa^{(4)} = 4\pi \sum_{k,k'} V_{\kappa k k'}^2 \bar{n}_k(\bar{n}_{k'} + 1)\delta(\omega_k - \omega_{k'}) \tag{6.9}$$

(a more complete expression for $Q'(t)$, which takes into account the renormalization of ω_κ and $V_{\kappa\kappa}$ as a result of the interaction $H_i^{(4)}$, is given in [21]). The resultant spectral distribution is therefore a convolution of the distribution $Q'(\omega)$ investigated above with a Lorentz distribution of width $2\Gamma_\kappa^{(4)}$. The broadening $\Gamma_\kappa^{(4)}$ is particularly substantial in the case of high-frequency local vibrations, when the conservation laws make decay processes resulting in the damping (6.7) impossible (see [22, 61]).

The results of Sec. 4 for $Q'(\omega)$, together with Eqs. (6.6)–(6.9), determine in explicit form the shape, the fine structure, and the temperature dependence of the light-absorption spectra in the case of one local or quasilocal vibration. When account is taken of the second term in Eq. (6.7) and of $\Gamma_\kappa^{(4)}$ (6.9), which are due to the anharmonicity of interaction with a medium, the fine structure is smeared with increasing temperature both as a result of the increase of $\bar{n}_\kappa$ (see Fig. 17) and of the increase of Γ_κ and $\Gamma_\kappa^{(4)}$. At sufficiently high temperatures $Q'_\kappa(\omega)$ is described by the results of the classical theory, i.e., by the smooth asymmetric non-Lorentzian curves of Fig. 4 in the case of linear friction or by the curves of Fig. 18 in the case of nonlinear friction.

6.2.2 Spectra for Several Interacting Nonlinear Singled-Out Vibrations

a. Nondegenerate Vibrations The structure of the spectral distri-

tion of the absorption coefficient becomes more complicated when there are several interacting local and quasilocal vibrations in the vicinity of the impurity center. If the effects of nonlinear friction can be neglected, expression (4.22) for the time correlation function can be generalized in simple fashion to include the case of interacting nonlinear oscillators [21]

$$Q'_\kappa(t) = \langle \hat{a}_\kappa(t)\hat{a}_\kappa^+(0)\rangle,$$

$$Q'^*_\kappa(t) = (\bar{n}_\kappa + 1)\exp\left[i\left(\omega_\kappa + P_\kappa - \frac{1}{2}\sum_{\kappa'} V_{\kappa\kappa'}\right)t\right]$$

$$\times \psi_{\kappa\kappa}^{-1}(t)\prod_{\kappa'}\exp(\Gamma_{\kappa'}t)\psi_{\kappa\kappa'}^{-1}(t),$$

where P_κ is the renormalization of the frequency of the oscillator κ, due to the interaction with the medium (see [21]),

$$\psi_{\kappa\kappa'}(t) = \operatorname{ch} a_{\kappa\kappa'}t + \left[1 - i\frac{V_{\kappa\kappa'}}{2\Gamma_{\kappa'}}(2\bar{n}_{\kappa'} + 1)\right]\frac{\Gamma_{\kappa'}}{a_{\kappa\kappa'}}\operatorname{sh} a_{\kappa\kappa'}t,$$

$$a_{\kappa\kappa'}^2 = \Gamma_{\kappa'}^2 - i(2\bar{n}_{\kappa'} + 1)\Gamma_{\kappa'}V_{\kappa\kappa'} - \tfrac{1}{4}V_{\kappa\kappa'}^2. \tag{6.11}$$

The cross section for the absorption of the light can be determined from Eqs. (6.6), (6.10), and (6.11) by numerical integration. In the numerical calculations it may be convenient also to solve a system of difference equations for the partial spectra, similar to the equations (4.29) for one nonlinear oscillator and considered in the case of interacting oscillators in [52].

Explicit analytic expressions for $Q'_\kappa(\omega)$ (i.e., for $\sigma(\omega)$) can be obtained for the case of several interacting oscillators in the limit of large or small ratios $V_{\kappa\kappa'}/\Gamma_{\kappa'}$. If the nonequidistance is large enough and the condition

$$|V_{\kappa\kappa'}| \gg (1 + 2\bar{n}_{\kappa'})\Gamma_{\kappa'}, \tag{6.12}$$

is satisfied, then, neglecting nonlinear-friction effects, it follows from

Eqs. (6.10) and (6.11) that

$$Q'_\kappa(\omega) = \frac{1}{\pi}\exp\left(\frac{\omega_\kappa}{T}\right)\prod_{\kappa'}(\bar{n}_{\kappa'} + 1)^{-1}$$

$$\times \sum_{\ldots, n_{\kappa'}, \ldots = 0} n_\kappa \exp\left(-\sum_{\kappa'}\frac{\omega_{\kappa'}n_{\kappa'}}{T}\right)\frac{\Gamma_{\kappa n}\cos\alpha_{\kappa n} - \Omega_{\kappa n}\sin\alpha_{\kappa n}}{\Gamma_{\kappa n}^2 + \omega_{\kappa n}^2} \tag{6.13}$$

Here

$$\Omega_{\kappa n} \equiv \Omega_\kappa(\ldots n_{\kappa'} \ldots) = \omega - \omega_\kappa - P_\kappa + \frac{1}{2}V_{\kappa\kappa} - \sum_{\kappa'} V_{\kappa\kappa'}n_{\kappa'}$$

$$- 4\sum_{\kappa'}\bar{n}_{\kappa'}(\bar{n}_{\kappa'} + 1)(1 + 2n_{\kappa'} - \delta_{\kappa,\kappa'})\Gamma_{\kappa'}^2 V_{\kappa\kappa'}^-,$$

$$\Gamma_{\kappa n} \equiv \Gamma_\kappa(\ldots n_{\kappa'} \ldots)$$

$$= \sum_{\kappa'}\left[(1 + 2n_{\kappa'} - \delta_{\kappa,\kappa'})(1 + 2\bar{n}_{\kappa'}) - 1\right]\Gamma_{\kappa'}, \tag{6.14}$$

$$\alpha_{\kappa n} \equiv \alpha_\kappa(\ldots n_{\kappa'} \ldots) = 4\sum_{\kappa'}\left[n_{\kappa'} - \delta_{\kappa,\kappa'} - \bar{n}_{\kappa'}(1 + \delta_{\kappa,\kappa'})\right]\Gamma_{\kappa'}V_{\kappa\kappa'}^{-1}.$$

The spectral distribution (6.13) corresponds to a fine structure, in which each of the lines $n_\kappa = 1, 2, \ldots$ gives rise to a series of lines with $n_{\kappa'} = 0, 1, 2, \ldots$ spaced a distance $|V_{\kappa\kappa'}|$ apart. The line widths are determined by the values of $\Gamma_{\kappa'}$ of all the singled-out vibrations and depend linearly on $n_{\kappa'}$ and $\bar{n}_{\kappa'}$. The individual lines are asymmetric in shape, and the parameters $\alpha_{\kappa n}$, which characterize the asymmetry, depend linearly on the ratios $\Gamma_{\kappa'}/V_{\kappa\kappa'}$. We note that if $|V_{\kappa\kappa'}| \ll \Gamma_{\kappa'}$ for certain singled-out vibrations, these vibrations drop out of the sums (6.13) and (6.14).

In the opposite limiting case, when

$$|V_{\kappa\kappa'}| \ll \Gamma_{\kappa'}$$

for all the vibrations, the fine structure of the spectrum vanishes and

the spectral distribution turns out to be smooth. An explicit expression for $Q'_\kappa(\omega)$ in this case (similar to Eq. (4.27)) is given in [21].

b. Effects Due to Nonlinear Friction In the case of several singled-out vibrations, the nonlinear friction described by the interaction Hamiltonian $H_i^{(2)}$, can play a more significant role than in the case of one singled-out vibration, since additional relaxation channels appear as a result of decay or scattering processes of the type $\omega_\kappa = |\omega_{\kappa'} \pm \omega_k|$. If the singled-out vibrations κ and κ' have high frequencies ($\omega_{\kappa'}\omega_{\kappa'} \gg T$), it is easy to verify (cf. [22]) that the spectral distribution is described by Eqs. (6.6), (6.10), and (6.11) for the linear friction problem. In the considered case in these equations there is added to the damping Γ_κ (6.7) the term

$$\tilde{\Gamma}_\kappa^{(2)} = \sum_{\kappa'} \Gamma_{\kappa\kappa'}^{(2)} \qquad (\omega_\kappa, \omega_{\kappa'} \gg T), \tag{6.15}$$

$$\Gamma_{\kappa\kappa'}^{(2)} = 4\pi \sum_k V_{\kappa\kappa' k}^2 \left[(\bar{n}_k + 1)\delta(\omega_\kappa - \omega_{\kappa'} - \omega_k) + \bar{n}_k \delta(\omega_\kappa - \omega_{\kappa'} + \omega_k) \right].$$

In addition, the parameters P_κ and $V_{\kappa\kappa'}$ become renormalized (see [21]; in the notation of Sec. 4, the expression of [21] for $P_\kappa^{(2)}$ corresponds to $P_\kappa^{(2)} - V_{\kappa\kappa}^{(2)}/2$). If all the singled-out oscillations have high frequencies, the spectral distribution $Q'_\kappa(\omega)$ constitutes a Lorentz peak of width $2(\Gamma_\kappa + \tilde{\Gamma}_\kappa^{(2)})$.

The interaction $H_i^{(2)}$ influences the spectral distribution in a more complicated manner if the frequencies of some of the singled-out vibrations that contribute to the nonlinear friction are of the order of T. An analytic expression for the time correlation function $Q'_\kappa(t)$ can be obtained if the high-frequency singled-out vibration κ ($\omega_\kappa \gg T$) decays into a relatively low-frequency singled-out vibration κ' ($\omega_{\kappa'} \lesssim T$) and a vibration of the continuous spectrum k. In this case, as shown in [52], the function $Q'_\kappa(t)$ is described by Eqs. (6.10) and (6.11), in which it is necessary to make the substitution

$$V_{\kappa\kappa'} \to V_{\kappa\kappa'} + i\Gamma_{\kappa\kappa'}^{(2)} \tag{6.16}$$

and multiply the resultant expression for $Q'_\kappa(t)$ by

$$\exp(-\Gamma_{\kappa\kappa'}^{(2)} t) \qquad (t > 0). \tag{6.17}$$

The corresponding spectral distribution has at large $|V_{\kappa\kappa'}|$ (compared with Γ_κ, $\Gamma_{\kappa'}$, and $\Gamma_{\kappa\kappa'}$) a pronounced fine structure consisting of the lines $n_{\kappa'} = 0, 1, 2, \ldots$ with half-widths

$$\Gamma_{\kappa n_{\kappa'}} = \Gamma_\kappa + \Gamma_\kappa^{(4)} + (1 + 2\bar{n}_{\kappa'})(1 + 2n_{\kappa'})\Gamma_{\kappa'} + (1 + \bar{n}_{\kappa'})\Gamma_{\kappa\kappa'}^{(2)} - \Gamma_{\kappa'}. \tag{6.18}$$

In the opposite limiting case $V_{\kappa\kappa'} = 0$ there is no fine structure, the spectral distribution is symmetric, but has an essentially non-Lorentz shape. In particular, at $\Gamma_{\kappa\kappa'}^{(2)} \gg \Gamma_{\kappa'}$ and $V_{\kappa\kappa'} = 0$ we have

$$Q'_\kappa(\omega) = \frac{1}{\pi}(\bar{n}_{\kappa'} + 1)^{-1} \sum_{n_{\kappa'}=0}^{\infty} \frac{\Gamma_\kappa + (n_{\kappa'} + 1)\Gamma_{\kappa\kappa'}^{(2)}}{\Omega'^2_\kappa + \left[\Gamma_\kappa + (n_{\kappa'} + 1)\Gamma_{\kappa\kappa'}^{(2)}\right]^2} e^{-\omega_{\kappa'} n_{\kappa'}/T}, \tag{6.19}$$

$$\Omega'_\kappa = \omega - \omega_\kappa - P_\kappa - \frac{1}{2} V_{\kappa\kappa}.$$

The central part of this distribution is narrower and sharper than a Lorentz curve having the same integral intensity and the same intensity on the wings.

c. Degenerate Vibrations with Cubic Symmetry The case of degenerate local or quasilocal vibrations arising near high-symmetry centers calls for a special treatment. In this case the structure of the spectral distribution becomes more complicated, since the nonlinearity not only makes the spectrum of a singled-out vibration nonequidistant, but also lifts partially the level degeneracy that takes place in the harmonic approximation. To calculate $Q'_\kappa(\omega)$ it is therefore necessary to take into account the complicated interference of the transitions between the produced multiplets of the energy levels.

In the case of triply degenerate oscillations with cubic symmetry in the presence of a symmetry center, the Hamiltonian H_0 of the oscillations in the absence of interaction with the medium takes the form

$$H_0 = \omega_0 \sum_\kappa \hat{n}_\kappa + \frac{1}{2} V_1 \sum_\kappa \hat{n}_\kappa^2 + \frac{1}{2} V_2 \sum_{\kappa \neq \kappa'} \hat{n}_\kappa \hat{n}_{\kappa'} + \frac{1}{2} V_3 \sum_{\kappa \neq \kappa'} (\hat{a}_\kappa^+)^2 \hat{a}_{\kappa'}^2 \tag{6.20}$$

Here $\kappa = 1, 2, 3$. The spectral distribution $Q'_\kappa(\omega)$, as in the case of nondegenerate oscillations, can be represented as an assembly of partial spectra (4.28). Now, however, the set of quantum numbers that enumerate the levels between which the transitions take place include, besides the principal quantum number, also additional numbers that enumerate the levels within the multiplet. A system of equations for the partial spectra, similar to (4.29), was analyzed in [62] for a problem with the Hamiltonian (6.20). For transitions between levels whose random denereracy is completely lifted by the anharmonicity, the half-widths of the lines in a spectrum with a pronounced fine structure are determined by the formula

$$\Gamma_{\kappa n} = \left[(2\bar{n}_\kappa + 1)(2n - 1) + 6\bar{n}_\kappa\right]\Gamma_\kappa, \qquad (6.21)$$

where n is the principal quantum number of the level to which the transition takes place. The spectral distributions for the general case

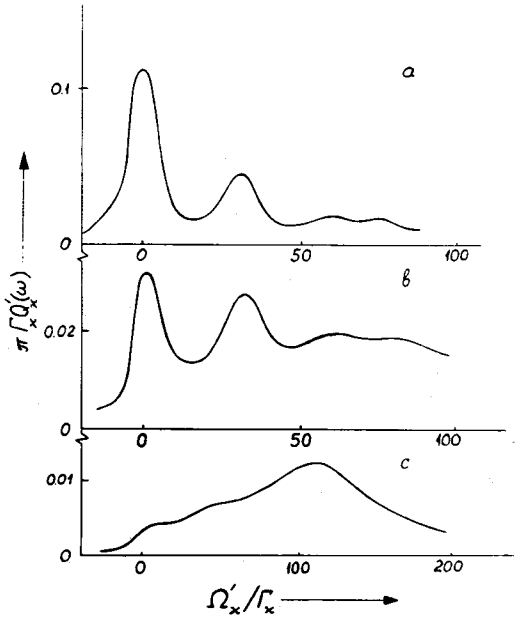

Figure 21 Spectral distribution $Q_\kappa'(\omega)$ for degenerate oscillators with spherical symmetry; $V_1/\Gamma_\kappa = 45$; $\Omega_\kappa' = \omega - \omega_\kappa - P_\kappa - \frac{1}{2}V_1$. Figures a, b, and c correspond to the values $T/\omega_0 = 0.67$, 1, and 2.

were calculated numerically. Figure 21 illustrates the smearing of the fine structure for an oscillator with spherical symmetry ($2V_1 = 3V_2 = 6V_3$) when the temperature is raised (in the figure captions of [62] the parameters V_1, V_2, and V_3 are replaced by $\frac{1}{2}V_2$, $\frac{1}{2}(V_1 - V_2)$ and $\frac{1}{2}V_3$). A comparison of Figs. 21 and 17 shows that when the nonequidistance parameters of the lowest levels have close values the structure of the spectra is richer and manifests itself more distinctly in the case of spherical oscillator than in the case of one nondegenerate oscillator.

d. Examples of Experimental Investigation of the Fine Structure and of the Broadening Due to Nonequidistance of the Levels Experimentally pronounced fine structure was observed [63] in the spectra of absorption of IR light by nondegenerate quasilocal vibrations of Eu^{2+} impurity centers in MnF_2 crystals. As seen from Fig. 22, the

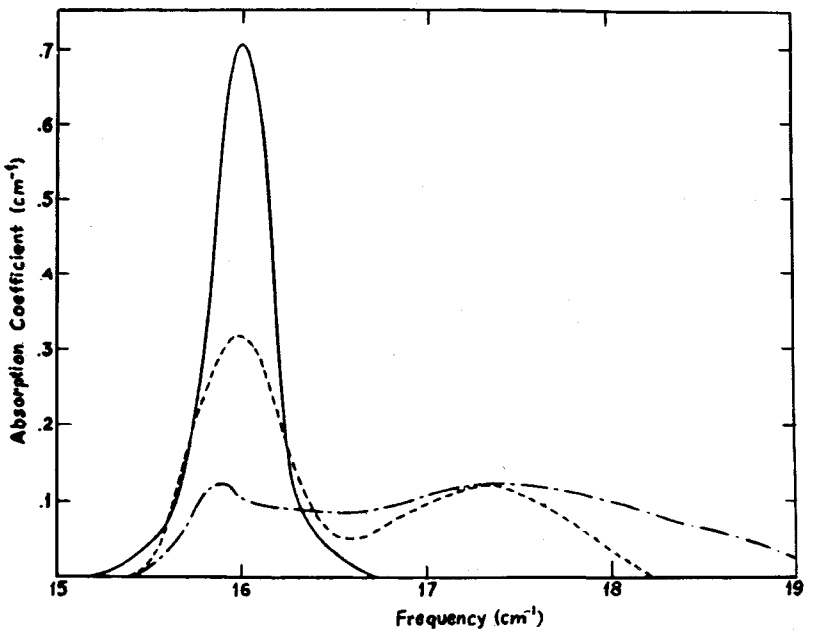

Figure 22 Spectra of infrared light absorption by quasilocal vibrations in the system $MnF_2 : Eu^{2+}$ at different temperatures. Curves I–III correspond to $T = 2$, 15, and 25 K, respectively.

absorption spectrum contains at $T = 2$ K only one line corresponding to a transition from the ground state to an excited level. At a temperature $T = 15$ K, when $\bar{n}_\kappa = 0.3$ ($\omega_0 \approx 16$ cm^{-1}) the spectrum shows, in accord with Eq. (4.25), two maxima corresponding to transitions from the levels $n = 0$ and $n = 1$, and at 25 K ($\bar{n}_\kappa \approx 0.66$) a third peak appears on the distribution wing. The transformation of the experimental spectrum in Fig. 22 with rising temperature recalls the picture shown in Fig. 17 and obtained by calculation by formulas (4.22) and (4.24). The ratio of the widths (at half-maximum) of the first and second peaks at $T = 15$ K (when the fine structure is most pronounced), calculated from Eq. (4.26), amounts to 0.4, in agreement with the experimental value 0.5 (taking into account the presence of apparatus broadening). In this case $V_{\kappa\kappa} = 1.3$ cm^{-1} exceeds Γ_κ ($\lesssim 0.1$ cm^{-1}) considerably.

Broadened smooth distributions $\sigma(\omega)$ having strongly temperature-dependent widths, much larger than the possible values of Γ_κ, were observed at $T \gtrsim \omega_\kappa$ in a number of studies (see the review [64]). They result from overlap and interference of the partial spectra of the nonlinear oscillators at relatively small values of $|V_{\kappa\kappa'}|/\Gamma_{\kappa'}$, as well as nonlinear friction processes. For example, in NaCl crystals containing Cu^+ impurity centers, the spectrum of the absorption by quasilocal vibrations becomes very broad and noticeably asymmetic in the temperature region $T \sim \omega_\kappa$ [65]. Similar effects were observed [66] in the CsI : Tl^+ system.

The modulation broadening, discussed above, of the spectral distributions of the high-frequency local vibrations due to their nonlinear interaction with low-frequency local vibrations was observed [67] in KCl crystals containing complexes of SO_4^{2-} ions with Ca^{2+} or Ba^{2+}.

6.2.3 Spectral Distributions at Multiple and Combination Frequencies Besides the already considered light-absorption peaks at the fundamental frequencies of the singled-out vibrations there exist in nonlinear systems also weaker peaks at multiple and combination frequencies. Peaks at doubled frequencies appear only for defects that have no symmetry centers [68]. The Hamiltonian H_0 of the corresponding vibrations contains terms cubic in q_κ, and the expansion of the dipole moment M_x in powers of q_κ contains quadratic terms. As a result, the formula (6.4) for $\sigma(\omega)$ contains, besides the spectral distributions of

the correlators $\langle q_\kappa(t) q_\kappa(0)\rangle$ (cf. Eq. (6.6)) also terms proportional to

$$Q_{\kappa\kappa'}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\langle q_\kappa(t) q_{\kappa'}(t) q_\kappa(0) q_{\kappa'}(0)\rangle.$$

The corresponding spectral distributions of the coefficient of light absorption at the combination frequencies $\omega_\kappa \pm \omega_\kappa'$ were considered in [68–72, 23]. Using the methods described in Secs. 2 and 4 it appears to be possible to obtain explicit expressions for $\sigma(\omega)$ for the nondegenerate problem and carry out a sufficiently complete analysis of the fine structure of the spectra at low temperatures, as well as of the smooth distributions resulting in the classical limit [71, 72]. In a number of cases, simple relations exist between the parameters of the spectra at the fundamental and at the doubled or combination frequencies. Thus, if the nonequidistance of the levels makes no contribution to the spectrum broadening and the latter is due only to the damping Γ_κ and $\Gamma_\kappa^{(4)}$ (6.7) and (6.9), the half-width $\Gamma_{\kappa\kappa}$ at double the frequency is

$$\Gamma_{\kappa\kappa} = 2\Gamma_\kappa + 4\Gamma_\kappa^{(4)}.$$

In the case of a pronounced fine structure in the spectrum of one singled-out oscillator, the distance between its lines is twice as large near the doubled frequency than for the spectrum at the fundamental frequency, the line halfwidths are larger by a factor $(1 + 2\bar{n}_\kappa)\Gamma_\kappa + 3\Gamma_\kappa^{(4)}$, and the fine structure is more strongly pronounced. In the classical limit, at $|V_{\kappa\kappa}|\bar{n}_\kappa \gg \Gamma_\kappa$ the integral width of the smooth strongly asymmetric distribution at double the frequency is $e \approx 2.72$ times larger than at the fundamental frequency.

We note that the presence of cubic terms in the Hamiltonian H_0 does not affect the shape of peaks of the spectral distributions, and only causes renormalization of the nonlinearity constants $V_{\kappa\kappa'}$.

6.2.4 Integral Intensities of Absorption Peaks Besides broadening and shifting the infrared absorption (or emission) lines at the eigenfrequencies of the local and quasilocal vibrations, the interaction with the vibrations of the continuous spectrum leads also to the appearance of relatively broad wings and changes in the integral intensities of the peaks. The absorption coefficient on the wings has a

complicated (nonmonotonic) frequency dependence. Therefore, unless the peak is δ-like, it cannot be singled out in the spectrum directly. If, however, the characteristic width of the peak is small compared with the characteristic width of the wings and with the distance from the peak to the absorption maxima on the wings, it is possible to indicate a procedure allowing one to single out the peak uniquely [73, 74]. This makes it possible to investigate the temperature dependence of the peak intensity. Analysis shows, in particular, that in a number of cases, e.g., in the case of low-frequency quasilocal vibrations of a weakly bound impurity the integral intensity of the peaks can depend weakly on the temperature, despite the strong temperature dependence of the peak widths, which is due to the nonlinearity influence considered above (such an effect was observed in experiment [6]).

§7 Conclusion

It follows from the presented results that both in classical and in quantum theory it is possible to obtain a sufficiently complete solution of the problem of fluctuations and relaxation of a Duffing oscillator interacting with a medium, in a wide range of relations between the nonlinearity and damping parameters. In the case of linear friction, the simple expressions (2.46), (2.47) and (4.20), (4.22) are obtained for the time correlation function $Q(t)$ of the coordinates. In the more general case of nonlinear friction (nonlinear interaction with the medium) the determination of the spectral representation $Q(\omega)$ reduces to solution of the linear system of difference equations (4.29), which is convenient for numerical computations.

These results enable us to investigate the influence of the nonlinearity on the spectral distribution $Q(\omega)$ and trace the gradual transition from the case of a fully degenerate (in the Bohr frequencies) linear oscillator to the case when the degeneracy no longer manifests itself and the essentially multilevel problem reduces in fact to a two-level one. Since the interference of transitions that are almost degenerate in frequency is suppressed even at the small degree of nonequidistance of the levels (of the order of their width), even a relatively small nonlinearity influences strongly the behavior of the function $Q(t)$ and

the form of the spectral distribution $Q(\omega)$. Whereas in the absence of nonlinearity $Q(\omega)$ is described by a Lorentz function, in the case when the nonequidistance of the levels exceeds their widths the spectral distribution has in a definite temperature interval a fine structure, and in the classical limit the distribution is smooth but strongly asymmetric (see Figs. 4, 7, and 17).

A nonlinear interaction with the medium (corresponding to a friction force that is cubic in the amplitude of the vibrations, i.e., to the Van der Pol oscillator model) leads also to a non-Lorentz form of the spectral distribution. The influence of the nonlinear friction force on the form of the distribution $Q(\omega)$, however, turns out to be substantially different from the influence of the nonlinearity of the restoring force (cf. Figs. 4 and 18).

The model of a Duffing oscillator interacting with a medium makes it also possible to investigate another group of problems, namely, those connected with fluctuation transitions between different stable states of a classical subsystem. Two stable states appear in this model in the presence of an external resonant field.

To solve the problem of the fluctuation transitions it is convenient to use the path-integral method. It makes it possible to show that in a quite general case the transition probability in a nonequilibrium subsystem depends exponentially on the reciprocal temperature of the medium. In the case of a Duffing oscillator in a resonant field, the corresponding activation energy can be calculated in explicit form. The transition probabilities W depend very strongly on the parameters of the problem. A smeared-out kinetic first-order transition, corresponding to an abrupt change of the populations of the equilibrium states, should therefore appear in a narrow interval near certain values of the parameters (see Figs. 11 and 16). In this range of parameters, the coefficient of absorption of the additional weak field by the oscillator has a peak of width $\sim W$.

The investigation of fluctuation transitions (and of fluctuations in general) becomes simpler (and universal) if the subsystem parameters are located in the vicinities of bifurcation points. This simplification is due to the fact that it is possible to separate in the system the slow motion, and to describe the fluctuations by using a certain adiabatic-approximation analog. This makes possible a complete analysis of the fluctuations near the bifurcation points.

In contrast to the resonant field, a nonresonant external field does

not lead to the appearance of several stable states of the oscillator. It can, however, influence substantially the relaxation parameters. The oscillator distribution in the energy levels is strongly changed thereby. Under certain conditions, population inversion can set in several low-lying levels.

Appendix A: Spectral Distribution of Linear Oscillator in the Case of Harmonic Interaction with the Vibrations of the Medium

To calculate the spectral distribution $Q(\omega)$ of the time correlation function $Q(t)$ of the oscillator coordinates it is convenient to use the method of double-time Green's functions. The Green's function of the operators $\hat{L}$ and $\hat{M}$ is determined with the aid of the relation (see, e.g., [75])

$$\langle\langle \hat{L}; \hat{M} \rangle\rangle_\omega = -\frac{i}{2\pi}\int_0^\infty dt\, e^{i\omega t}\langle [\hat{L}(t), \hat{M}(0)]\rangle, \qquad \hbar = 1. \quad \text{(A.1)}$$

It is easy to verify that the function $Q(\omega)$ is expressed in terms of the oscillator-coordinates Green's function as follows [75]:

$$Q(\omega) = -2[\bar{n}(\omega) + 1]\,\mathrm{Im}\langle\langle q; q\rangle\rangle_{\omega + i0}. \quad \text{(A.2)}$$

The Hamiltonian of an oscillator interacting with the vibrations of the medium takes in the case of the harmonic interaction (1.3) the form

$$H = H_0 + H_m + H_i, \qquad H_0 = \omega_0 \hat{a}^+\hat{a}, \qquad H_m = \sum_k \omega_k \hat{a}_k^+ \hat{a}_k,$$
$$H_i = \frac{1}{2}\sum_k \epsilon_k (\omega_0\omega_k)^{-1/2}(\hat{a} + \hat{a}^+)(\hat{a}_k + \hat{a}_k^+). \quad \text{(A.3)}$$

Here $\hbar = 1$ and the standard relation is used between the normal coordinates and the creation and annihilation operators:

$$q = (2\omega_0)^{-1/2}(\hat{a} + \hat{a}^+), \qquad q_k = (2\omega_k)^{-1/2}(\hat{a}_k + \hat{a}_k^+)$$

When Eqs. (A.1)–(A.3) are taken into account, the equations of motion for the Green's functions of the considered singled-out oscillator take the form

$$(\omega - \omega_0)\langle\langle \hat{a}; \hat{a} + \hat{a}^+ \rangle\rangle_\omega$$
$$= \frac{1}{2\pi} + \frac{1}{2}\sum_k \frac{\epsilon_k}{\sqrt{\omega_0\omega_k}}\langle\langle \hat{a}_k + \hat{a}_k^+; \hat{a} + \hat{a}^+ \rangle\rangle_\omega, \quad \text{(A.4)}$$
$$(\omega + \omega_0)\langle\langle \hat{a}^+; \hat{a} + \hat{a}^+ \rangle\rangle$$
$$= -\frac{1}{2\pi} - \frac{1}{2}\sum_k \frac{\epsilon_k}{\sqrt{\omega_0\omega_k}}\langle\langle \hat{a}_k + \hat{a}_k^+; \hat{a} + \hat{a}^+ \rangle\rangle_\omega.$$

The Green's functions in the right-hand side of Eq. (A.4) satisfy in turn the equations

$$(\omega - \omega_k)\langle\langle \hat{a}_k; \hat{a} + \hat{a}^+ \rangle\rangle_\omega = \frac{\epsilon_k}{2\sqrt{\omega_0\omega_k}}\langle\langle \hat{a} + \hat{a}^+; \hat{a} + \hat{a}^+ \rangle\rangle_\omega, \quad \text{(A.5)}$$
$$(\omega + \omega_k)\langle\langle \hat{a}_k^+; \hat{a} + \hat{a}^+ \rangle\rangle_\omega = -\frac{\epsilon_k}{2\sqrt{\omega_0\omega_k}}\langle\langle \hat{a} + \hat{a}^+; \hat{a} + \hat{a}^+ \rangle\rangle_\omega,$$

The system of Eqs. (A.4) and (A.5) yields the following exact expression for the Green's function of the singled-out oscillator (cf. [7]):

$$\langle\langle q; q\rangle\rangle_\omega = \frac{1}{2\omega_0}\langle\langle \hat{a} + \hat{a}^+; \hat{a} + \hat{a}^+ \rangle\rangle_\omega = \frac{1}{2\pi}\frac{1}{\omega^2 - \omega_0^2 - 2\omega_0 R(\omega)} \quad \text{(A.6)}$$

The polarization operator $R(\omega)$ in Eq. (A.6) is equal to

$$R(\omega) = \sum_k \frac{\epsilon_k^2}{2\omega_0(\omega^2 - \omega_k^2)} \quad \text{(A.7)}$$

Equations (A.2), (A.6), and (A.7) determine in explicit form the oscillator spectral distribution in the case of harmonic interaction with the vibrations of the medium. If the interaction is weak enough, so that the damping $\Gamma \ll \omega_0$, the peak of $Q(\omega)$ is described by a Lorentz distribution:

$$Q(\omega) = \frac{\bar{n}+1}{2\pi\omega_0} \frac{\Gamma}{(\omega - \omega_0 - P)^2 + \Gamma^2}, \qquad |\omega - \omega_0| \ll \omega_0. \quad \text{(A.8)}$$

The damping Γ and the frequency shift P are determined here by the imaginary and real parts of the polarization operator $R(\omega_0 + i0)$:

$$\Gamma = \frac{\pi}{4\omega_0} \sum_k \frac{\epsilon_k^2}{\omega_k} \delta(\omega_0 - \omega_k), \qquad P = \text{v.p.} \frac{1}{2\omega_0} \sum_k \frac{\epsilon_k^2}{\omega_0^2 - \omega_k^2} \quad \text{(A.9)}$$

(the symbol v.p. in front of the sum over k means that the principal value of the corresponding integral with respect to ω_k must be calculated). Expressions (A.9) for Γ and P coincide with the equations obtained for the case considered here in Sec. 2 and in Appendix F by other methods.

Appendix B: Transformation of Integral Operator $\Lambda_1[u_1]$

The operator $\Lambda_1[u_1]$ in Eq. (2.13) is a sum of two terms:

$$\Lambda_1[u_1] = \Lambda_1^{(1)}[u_1] + \Lambda_1^{(2)}[u_1],$$

$$\Lambda_1^{(1)}[u_1] = \frac{1}{2i\omega_0} \sum_k \frac{\epsilon_k^2}{\omega_k} \int_0^t dt_1 e^{-i\omega_0 t_1} u_1(t - t_1) \sin \omega_k t_1, \quad \text{(B.1)}$$

$$\Lambda_1^{(2)}[u_1] = \frac{1}{2i\omega_0} e^{-2i\omega_0 t} \sum_k \frac{\epsilon_k^2}{\omega_k} \int_0^t dt_1 e^{i\omega_0 t_1} u_1^*(t - t_1) \sin \omega_k t_1.$$

It can be seen from Eq. (2.13) that $\dot{u}_1$ is proportional to the small quantities γ and ϵ^2, i.e., $u_1(t)$ varies slowly with time,

$$|\dot{u}_1| \ll |u_1| t_c^{-1}. \quad \text{(B.2)}$$

At the same time the functions $u_1(t - t_1)$ and $u_1^*(t - t_1)$ in Eq. (B.1) are additionally multiplied by fast oscillating functions of t_1. In the calculation of $\Lambda_1[u_1]$ we can therefore expand $u_1(t - t_1)$ and $u_1^*(t - t_1)$ in series in t_1. Using the relation

$$\frac{1 - \exp[-i(\omega_0 \pm \omega_k)t]}{i(\omega_0 \pm \omega_k)}$$

$$= \pi\delta(\omega_0 - \omega_k)\text{sgn}\, t - i\left(\text{v.p.} \frac{1}{\omega_0 \pm \omega_k}\right), \qquad |t| \gg t_c \quad \text{(B.3)}$$

(v.p. is the principal-value symbol) and retaining only the zeroth term of the expansion of $u_1(t - t_1)$, we can represent the functional $\Lambda_1^{(1)}[u_1]$ in the form

$$\Lambda_1^{(1)}[u_1] = (-\Gamma \,\text{sgn}\, t + iP)u_1(t), \qquad |t| \gg t_c, \quad \text{(B.4)}$$

where Γ and P are defined in Eq. (2.15).

For the expansion of $u_1(t - t_1)$ in a series in t_1 to be valid it is necessary not only that the parameters Γt_c and $|P| t_c$ be small, but also that the weighted state density $g_\epsilon(\omega)$ of the vibrations of the medium be smooth in the vicinity of ω_0. Indeed, the linear term of the expansion of $u_1(t - t_1)$ in terms of t_1 adds to $\Lambda_1^{(1)}[u_1]$ a correction

$$-i\dot{u}_1(t) \frac{d}{d\omega_0} [-\Gamma \,\text{sgn}\, t + iP].$$

This correction is small compared with (B.4) if

$$\left|\frac{d\Gamma}{d\omega_0}\right| \ll 1, \qquad \left|\frac{d\beta}{d\omega_0}\right| \ll 1. \quad \text{(B.5)}$$

The condition (B.5) and the analogous conditions for the higher-order derivatives mean that ω_0 must be located far enough from singularities of the spectrum of the medium (for example, the distance from $\omega_0 + P$ to the end point of the spectrum should exceed $\Gamma(1 + |\alpha|)$ considerably).

The main difference between the functionals $\Lambda_1^{(2)}[u_1]$ and $\Lambda_1^{(1)}[u_1]$ is due to the presence of the additional factor $\exp(-2i\omega_0 t)$ in the former. This factor is contained, in particular, also in the expression

of type (B.4) for $\Lambda_1^{(2)}[u_1]$. The reason for discarding in Eq. (2.14) the fast oscillating terms proportional to the small parameters Γ, P, and γ is that upon integration with respect to time over the region δt, where

$$\Gamma^{-1}, |\alpha|^{-1}\Gamma^{-1} \gg \delta t \gg \omega_0^{-1},$$

they lead to corrections $\sim\Gamma/\omega_0$, $|P|/\omega_0$, and $|\alpha|\Gamma/\omega_0$, whereas the contributions from the smooth terms are much larger ($\sim\Gamma\delta t$, $|P|\delta t$, and $|\alpha|\Gamma\delta t$).

Appendix C: Einstein–Fokker–Planck Equation for a Weakly Nonlinear Duffing Oscillator

In the microscopic model (2.2) of the interaction with the medium the random motion of a Duffing oscillator is generally speaking not of the Markov type. However, the change of the complex amplitude $u_1(t)$ of the oscillations, described by Eq. (2.14), over times much longer than t_c, is a Markov process. This follows from the form of the distribution of the random force $f_1(t)$ that enters in Eq. (2.14).

Using the same arguments as in the analysis of the distribution of the function $\tilde{\xi}(t)$, we can easily verify that the distribution of $f_1(t)$ is Gaussian. When account is taken of Eqs. (2.13), (2.11), (2.15), and (2.9), the paired correlators for $f_1(t)$ can be expressed in terms of the δ-like function $\delta_1(t-t')$ introduced in Eq. (2.20). Since the characteristic time scale in Eq. (2.14) for the slow variable u_1 is large compared with t_c, it is necessary in accord with the idea of the averaging method, to "coarsen" the obtained expressions over a time interval much longer than the characteristic width t_c of the function $\delta_1(t-t')$ (but small compared with Γ^{-1}). As a result, the function $\delta_1(t-t')$ is transformed into $\delta(t-t')$, and neglecting the small corrections $\sim\Gamma t_c$ we have

$$\langle f_1(t) f_1^*(t')\rangle = \frac{\Gamma T}{\omega_0^2}\delta(t-t'), \qquad \langle f_1(t) f_1(t')\rangle = 0. \tag{C.1}$$

The random force $f_1(t)$ is thus δ-correlated in the "slow" time scale.

It is known from the theory of random processes (see, e.g., [10]) that the Langevin equation (2.14), in which the random force $f_1(t)$ is Gaussian and δ-correlated, describes a Markov process. Corresponding to this stochastic complex nonlinear equation is the Einstein–Fokker–Planck (EFP) equation, which takes, with allowance for Eq. (C.1), the form:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial u_1}\left\{\left[(\Gamma - iP)u_1 - \frac{3i\gamma}{2\omega_0}u_1^2 u_2\right]w\right\} + \frac{\partial}{\partial u_2}\left\{\left[(\Gamma + iP)u_2 + \frac{3i\gamma}{2\omega_0}u_1 u_2^2\right]w\right\} + \frac{\Gamma T}{\omega_0^2}\frac{\partial^2 w}{\partial u_1 \partial u_2}. \tag{C.2}$$

Here $w \equiv w(u_1, u_2; u_1(0), u_2(0); t)$ is the probability density of the transition of a Duffing oscillator from a state with coordinates $u_1(0)$, $u_2(0)$ ($u_2(0) = u_1^*(0)$) at the instant $t = 0$ into a state u_1, u_2 ($u_2 = u_1^*$) at the instant t. Since the δ-correlation condition (C.1) is satisfied for "slow" time, Eq. (C.2) is valid at $t \gg t_c$, and the function w itself describes the transition probability density "smoothed" over a time $\sim t_c$. The initial condition for Eq. (C.2) is

$$w(u_1, u_2; u_1(0), u_2(0); 0) = \delta(u_1' - u_1'(0))\delta(u_1'' - u_1''(0)), \qquad u_1 = u_2^* = u_1' + iu_1''. \tag{C.3}$$

In the case of a harmonic oscillator ($\gamma = 0$) Eq. (C.2) can be solved exactly (see, e.g., [5], where a solution of the EFP equation is given for the distribution function of a harmonic oscillator over coordinate and momentum). It is easily understood that at $\gamma = 0$ the distribution w is Gaussian: as seen from Eq. (2.14), $u_1(t)$ and $u_1^*(t)$ are linear functionals of the random forces $f_1(t)$ and $f_1^*(t)$. Since the distribution of the latter is Gaussian, the distribution of $u_1(t)$ and $u_1^*(t)$ should also be such.

In the case of a nonlinear oscillator ($\gamma \neq 0$) the distribution w differs substantially from Gaussian. To solve Eq. (C.2) in this case it is convenient to change from the complex variables u_1 and u_2 to the real variables r and φ:

$$u_1 = r^{1/2}e^{i\varphi}, \qquad u_2 = r^{1/2}e^{-i\varphi}. \tag{C.4}$$

In these variables Eq. (C.2) takes the form

$$\frac{\partial w}{\partial t} = 2\Gamma w + \frac{\Gamma T}{\omega_0^2}\left(r\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r} + \frac{1}{4r}\frac{\partial^2 w}{\partial \varphi^2}\right)$$

$$+ 2\Gamma r\frac{\partial w}{\partial r} - P\frac{\partial w}{\partial \varphi} - \frac{3\gamma}{2\omega_0} r\frac{\partial w}{\partial \varphi} \qquad \text{(C.5)}$$

The function w is obviously periodic in φ with a period 2π. The solution of Eq. (C.5) can therefore be expanded in the series

$$w(r,\varphi,t) \equiv w(r,\varphi;r(0),\varphi(0);t) = \sum_{l=-\infty}^{\infty} w_l(r,t)e^{-il\varphi}. \qquad \text{(C.6)}$$

Since the coefficients in Eq. (C.5) do not depend on φ, the equations that follow from (C.5) for $w_l(r,t)$ (with different l) are found to be uncoupled. It is convenient to solve the one-dimensional diffusion equations for the functions $w_l(r,t)$ by the generating-function method (cf. Sec. 4.1.2). We introduce the generating function

$$W_l(x,t) = \int_0^\infty dr\, e^{-xr} r^{l/2}\exp(-ilPt)w_l(r,t), \qquad l \geqslant 0 \qquad \text{(C.7)}$$

According to Eqs. (C.5)–(C.7), the function $W_l(x,t)$ satisfies the equation

$$\frac{\partial W_l}{\partial t} = -\frac{\Gamma T}{\omega_0^2} x\left[(1+l)W_l + x\frac{\partial W_l}{\partial x}\right]$$

$$- \Gamma l W_l - 2\Gamma x\frac{\partial W_l}{\partial x} - \frac{3i\gamma l}{2\omega_0}\frac{\partial W_l}{\partial x} \qquad \text{(C.8)}$$

with an initial condition

$$W_l(x,0) = \frac{1}{\pi}(r(0))^{l/2}\exp(-xr(0))\exp(il\varphi(0)),$$

$$r(0) = |u_1(0)|^2, \qquad \varphi(0) = \arg u_1(0), \qquad \text{(C.9)}$$

that follows from Eqs. (C.3), (C.6), and (C.7).

The first-order linear differential equation (C.8) can be solved by the method of characteristics. A direct check demonstrates that the solution of the equation is

$$W_l(x,t) = \frac{1}{\pi}(r(0))^{l/2}\exp(il\varphi(0))a_l^{l+1}$$

$$\times e^{\Gamma T}\left[a_l \mathrm{ch}\, a_l t + \Gamma\left(1 + \frac{Tx}{\omega_0^2}\right)\mathrm{sh}\, a_l t\right]^{-(l+1)}$$

$$\times \exp\left[-r(0)\frac{x a_l \mathrm{ch}\, a_l t - (\Gamma l + 3i\gamma l/2\omega_0)\mathrm{sh}\, a_l t}{a_l \mathrm{ch}\, a_l t + \Gamma(1 + Tx/\omega_0^2)\mathrm{sh}\, a_l t}\right], \qquad \text{(C.10)}$$

where

$$a_l = \left(\Gamma^2 - \frac{3i\gamma T\Gamma l}{4\omega_0^3}\right)^{1/2} = \Gamma(1 - 4ial)^{1/2}. \qquad \text{(C.11)}$$

Expression (C.10) for the generating function makes possible a direct determination of the mean values of various powers of the dynamic variables. For example, recognizing that according to Eqs. (C.4), (C.6), and (C.7)

$$\langle u_1^l(t)\rangle = \frac{1}{2}\int dr\, d\varphi\, r^{l/2}e^{il\varphi}w(r,\varphi,t) = \pi W_l(0,t)\exp(ilPt), \qquad \text{(C.12)}$$

we can obtain immediately the quantity $\langle u_1^l(t)\rangle$ by putting $x = 0$ in Eq. (C.10). It follows similarly from the equality

$$\langle u_1^l(t)u_2^k(t)\rangle = \pi\int dr\, r^{(l-k)/2}r^k w_{l-k} \qquad (l \geqslant k) \qquad \text{(C.13)}$$

that

$$\langle u_1^l(t)u_2^k(t)\rangle = \pi(-1)^k\left(\frac{\partial^k W_{l-k}(x,t)}{\partial x^k}\right)_{x=0}\exp\left[i(l-k)Pt\right]. \qquad \text{(C.14)}$$

It is seen in particular from Eqs. (C.14) and (C.10) that weak

nonlinearity of the oscillator does not influence the damping of the quantities $\langle u_1^l(t)u_2^l(t)\rangle$ (for example, of the energy).

The explicit expressions (C.12) for the mean values $\langle u_1^l(t)\rangle$ make it easy to determine the double-time correlation functions of the type $\langle u_1^l(t)u_2^l(0)\rangle$ for arbitrary l by integrating with respect to the initial coordinates $u_1(0)$ with the statistical weight (2.45):

$$\langle u_1^l(t)u_2^l(0)\rangle = \left(\frac{T}{2\omega_0^2}\right)^l \exp(\Gamma t + ilPt)\left[\psi_l(t)\right]^{-(l+1)},$$
$$\psi_l(t) = \operatorname{ch} a_l t + (1 - 2i\alpha l)\frac{\Gamma}{a_l}\operatorname{sh} a_l t. \tag{C.15}$$

The spectral representation of the correlator (C.15) describes, in particular, the susceptibility of a nonlinear oscillator near the lth overtone. At $l = 1$ formula (C.15) goes over into formula (2.47) for $\tilde{Q}(t)$ (the coefficient $(T/2\omega_0^2)\exp(iPt)$ is singled out in Eq. (2.47) as an additional multiplier for $\tilde{Q}(t)$). Explicit expressions can be obtained in similar fashion for correlation functions of the type $\langle u_1^l(t)u_2^k(t)u_1^k(0)u_2^l(0)\rangle$, in terms of which the double-time correlators of arbitrary powers of the coordinate and momentum, $q(t)$ and $dq(t)/dt$, are expressed according to Eq. (1.5).

Appendix D: Solution of Euler Equation for the Auxiliary Problem of the Degenerate Nonlinear Oscillator

It is convenient to construct an asymptotic solution of Eqs. (3.17) by starting from the functions X and Y satisfying the equations

$$\dot{X} = -g_Y \equiv -\frac{\partial g}{\partial Y}, \qquad \dot{Y} = g_X \equiv \frac{\partial g}{\partial X}, \qquad g = g(X,Y),$$
$$\dot{X} \equiv \frac{dX}{d\tau}, \qquad \dot{Y} \equiv \frac{dY}{d\tau}. \tag{D.1}$$

It is easy to verify that in the zeroth approximation in ϵ the substitution $u' = X$ and $u'' = Y$ yield the solution of Eq. (3.17). It is important that this substitution cancels out in the integrand that determines Q_a all the terms that are not proportional to the small parameter ϵ.

Equations (D.1) have the form of the Hamilton equations for a particle performing one-dimensional finite motion with a Hamiltonian g. This motion is periodic with a frequency $\omega(g)\sim 1$ and is described by functions that are periodic in the phase ψ:

$$X = X(g,\psi) = X(g,\psi + 2\pi), \qquad Y = Y(g,\psi) = Y(g,\psi + 2\pi),$$
$$g = \text{const}, \qquad \dot{\psi} = \omega(g).$$

We shall seek below a self-similar solution of Eqs. (3.17) (at $\epsilon \neq 0$) by supposing

$$u' = X(g,\psi), \qquad u'' = Y(g,\psi),$$

i.e., we shall assume that u' and u'' do not depend explicitly on τ, and that all the time dependences are determined by the variation of g and ψ with time. This means in fact the transformation known from the theory of nonlinear oscillations (see, e.g., [28], [44]) to the energy (g) and phase (ψ) as the variables, with both the energy and the addition to phase varying slowly with time ($\dot{g} \propto \epsilon$, $\dot{\psi} - \omega(g) \propto \epsilon$). We note that the system described by Eq. (D.1) is itself strongly nonlinear (i.e., the expansions of the functions X and Y in terms of $\exp(i\omega(g)\tau)$ contain a large number of terms of the same order of magnitude), and the self-similar solution $u' = X$ and $u'' = Y$ is degenerate at $\epsilon = 0$, namely the periods of the oscillations for u' and for u'' are identical. Allowance for the terms $\sim\epsilon$ lifts the degeneracy.

We introduce auxiliary functions A and B with the aid of the equalities

$$\dot{u}' = \dot{X} = \omega(g)X_\psi - \epsilon X + \epsilon A(g,\psi), \qquad \omega(g)X_\psi \equiv -g_Y,$$
$$\dot{u}'' = \dot{Y} = \omega(g)Y_\psi - \epsilon Y + \epsilon B(g,\psi), \qquad \omega(g)Y_\psi \equiv g_X \tag{D.2}$$

We shall obtain below for the functions A and B equations whose solution in the zeroth approximation in ϵ contain an arbitrary function (this is the manifestation of the degeneracy). This function will be

determined such that no secular terms of the type $\epsilon\psi$ appear in A and B in the higher orders of perturbation theory in ϵ.

We express $\dot{X}$ and $\dot{Y}$ in Eq. (D.2) in terms of $\dot{g}$ and $\dot{\psi}$, and take into account the identity

$$X_g Y_\psi - X_\psi Y_g = \omega^{-1}(g)(g_X X_g + g_Y Y_g) = \omega^{-1}(g). \qquad \text{(D.3)}$$

Then

$$\begin{aligned} \dot{g} &= \epsilon\omega(g)\big[(A - X)Y_\psi - (B - Y)X_\psi\big], \\ \dot{\psi} &= \omega(g) + \epsilon\omega(g)\big[(B - Y)X_g - (A - X)Y_g\big] \end{aligned} \qquad \text{(D.4)}$$

Differentiating expressions (D.2) with respect to time, with allowance for Eq. (D.4), and substituting in Eq. (3.17), we obtain equations for the functions A and B:

$$\omega(g)A_\psi - 2XYA + (3X^2 + Y^2 - 1)B = -A_g\dot{g} - A_\psi(\dot{\psi} - \omega(g)) + \epsilon A,$$

$$\omega(g)B_\psi - (X^2 + 3Y^2 - 1)A + 2XYB = -B_g\dot{g} - B_\psi(\dot{\psi} - \omega(g)) + \epsilon B. \qquad \text{(D.5)}$$

Substituting Eq. (D.2) in formula (3.16) for E, we can find an additional relation between A and B. This relation is one of the integrals of the equations of motion (D.5) with a constant corresponding to $E = 0$:

$$2g_Y A - 2g_X B = \epsilon\big[(A - X)^2 + (B - Y)^2 - X^2 - Y^2\big] \qquad \text{(D.6)}$$

We can seek the solution of Eqs. (D.5) and (D.6) in the form of a series in ϵ:

$$A = A^0 + \epsilon A' + \cdots, \qquad B = B^0 + \epsilon B' + \cdots$$

To find the lowest terms of the series, namely the functions A^0 and B^0, it is convenient to use Eq. (D.6) to eliminate from the first and second equations of (D.5) the functions B and A, respectively. We

then obtain for A^0 and B^0 linear partial differential equations of first order. The coefficients of these equations can be transformed with the aid of Eqs. (3.16) and (D.2)–(D.4). As a result we can find an explicit solution of the equations for A^0 and B^0. This solution contains one arbitrary function $F(g)$:

$$A^0 = F(g)Y_\psi, \qquad B^0 = -F(g)X_\psi. \qquad \text{(D.7)}$$

The function $F(g)$ is determined from the condition that the second-order terms in ϵA and ϵB contain no terms that are secular in ψ, i.e., that these terms remain small up to times $\tau \sim \epsilon^{-1} \gg \omega^{-1}(g) \sim 1$. In the second-order approximation it is necessary to replace A and B in the right-hand sides of Eqs. (D.5) and (D.6) by A^0 and B^0 from Eq. (D.7). Expressing B' in terms of A' with the aid of Eq. (D.6) and substituting in the first equation of (D.5), we obtain a linear equation for A':

$$\begin{aligned} &\omega(g)A'_\psi - \omega(g)Y_\psi^{-1}Y_{\psi\psi}A' = Y_\psi P, \\ &P = -Y_\psi^{-1}\big[\epsilon^{-1}A^0_g\dot{g} + \epsilon^{-1}A^0_\psi(\dot{\psi} - \omega(g)) - A\big] \\ &\quad + g_{XX}(2g_X Y_\psi)^{-1}\mathscr{R}, \\ &\mathscr{R} = (A^0)^2 + (B^0)^2 - 2A^0X - 2B^0Y \\ &\quad = F^2(g)(X_\psi^2 + Y_\psi^2) - 2F(g)(XY_\psi - YX_\psi). \end{aligned} \qquad \text{(D.8)}$$

Here $\dot{g}$ and $\dot{\psi}$ are described by formulas (D.4) and (D.7).

Equation (D.8) can be solved by the method of variation of the arbitrary constant $\mathscr{D} = \mathscr{D}(g, \psi)$

$$A' = \mathscr{D}(g,\psi)Y_\psi, \qquad \omega(g)\frac{\partial\mathscr{D}}{\partial\psi} = P. \qquad \text{(D.9)}$$

Since X and Y are periodic in ψ, it follows that all the terms in P are also periodic. For A' not to contain terms secular in ψ it is necessary, according to Eq. (D.9), to satisfy the condition $\bar{P} = 0$, where the superior bar denotes averaging over ψ from 0 to 2π. Expression (D.8)

for P can be represented, after cumbersome transformations with allowance for Eqs. (D.4), (D.7), and the identity (D.3), into the form

$$P = \frac{1}{2}\frac{\partial\omega(g)}{\partial g}\mathscr{R} - \frac{1}{2}\omega(g)\frac{\partial\mathscr{R}}{\partial g} + \frac{\partial Z(g,\psi)}{\partial\psi}. \qquad \text{(D.10)}$$

We do not need the explicit form of the function $Z(g,\psi)$, since it drops out upon averaging over ψ. Then the condition $\overline{P} = 0$ leads to a simple equation for $\overline{\mathscr{R}}$, with a solution $\overline{\mathscr{R}} = C_0\omega(g)$.

To find the constant C_0, we note that in the $\epsilon = 0$ approximation the stationarity conditions from which the positions of the foci and of the saddle point of the Duffing oscillator are determined take the form $g_X = 0$ and $g_Y = 0$. The frequency $\omega(g_f)$ of the oscillations in the vicinity of the focus equals $(g_{XX}g_{YY})_f^{1/2}$ (the subscript f means that the corresponding function is calculated at the focus f). It is easy to verify that $\omega(g_f) \neq 0$. As $g \to g_f$, the left-hand side of Eq. (D.6) tends to zero if A' and B' are finite. At the same time the right-hand side, with an accuracy to corrections of higher order of smallness in ϵ, is equal to $\epsilon\mathscr{R}$. Consequently $\mathscr{R}$ vanishes as $g \to g_f$, i.e., the coefficient C_0 in the relation $\overline{\mathscr{R}} = C_0\omega(g)$ is zero.

From the expression (D.8) for $\mathscr{R}$ it follows that

$$\overline{\mathscr{R}}(g) = F^2(g)m_1 - 2F(g)m_2, \qquad m_1 \equiv m_1(g) = \overline{X_\psi^2} + \overline{Y_\psi^2}, \qquad \text{(D.11)}$$

$$m_2 \equiv m_2(g) = \overline{XY_\psi} - \overline{YX_\psi}.$$

This shows that the condition $\overline{\mathscr{R}} = 0$ leads to two possible values of $F(g)$ and accordingly to two values of $\dot{\bar{g}}$ in Eq. (D.4):

$$F_1(g) = 0, \qquad F_2(g) = 2m_2/m_1, \qquad \dot{g}_{1,2} = \mp\epsilon\omega(g)m_2. \qquad \text{(D.12)}$$

According to Eqs. (D.7) and (D.2), the solution $F_1(g) = 0$ describes the motion in the absence of a random force from the saddle point to the focus: $\operatorname{sgn}\dot{\bar{g}}_1 = -\operatorname{sgn}(g_1 - g_f)$, inasmuch as according to Eqs. (D.3) and (D.11)

$$m_2(g) = 2\int_{g_f}^{g}\omega^{-1}(g_1)\,dg_1$$

(we have used here the relation $\overline{XY_{\psi g}} - \overline{YX_{\psi g}} = -(\overline{X_\psi Y_g} - \overline{Y_\psi X_g}) = \omega^{-1}(g)$). Obviously, the boundary conditions (3.14) are satisfied by the second solution of Eq. (D.11), $F_2(g)$, which corresponds to motion from the focus to the saddle point. For the quantity Q_a defined in Eq. (3.14) we have in this case, according to Eqs. (3.16), (D.2), (D.7), and (D.12)

$$Q_a = \frac{1}{4}\int_0^\tau d\tau_1 F^2(g)\left(X_\psi^2 + Y_\psi^2\right) = \frac{1}{\epsilon}\int_{g_f}^{g_s}\frac{m_2}{\omega(g)m_1}\,dg \qquad \text{(D.13)}$$

Here g_s is the value of g at the saddle point, and in the transition to integration with respect to g ($g \approx \bar{g}$) we used the equality (D.12).

Taking into account the definitions (D.11) for m_1 and m_2 and Eqs. (D.2) for $\omega(g)X_\psi$ and $\omega(g)Y_\psi$, we can express $m_2/\omega(g)m_1$ in Eq. (D.13) in the form

$$m_2[\omega(g)m_1]^{-1} = \oint(X\,dY - Y\,dX)\left[\oint(-g_Y\,dX + g_X\,dY)\right]^{-1}$$

$$= 2\int\int dX\,dY\left[\int\int\Delta g\,dX\,dY\right]^{-1}, \qquad \text{(D.14)}$$

$$\Delta g = g_{XX} + g_{YY} = 4R^2 - 2.$$

Here $R^2 = X^2 + Y^2$, and the double integrals are taken over the region bounded by the trajectory $g(X,Y) = \text{const}$.

Explicit expressions for $m_2[\omega(g)m_1]^{-1}$ can be obtained at small β. At $\epsilon^2 \ll \beta \ll 1$, according to Eq. (D.1) (with allowance for expression (3.16) for g), we have $g_{f_1} \approx 1/4$, $g_s \approx -g_{f_2} \approx \sqrt{\beta}$ (the positions of the foci and of the saddle are shown in Fig. 23). At $\beta \ll 1$ the trajectories $g(X,Y) = \text{const}$, that correspond to values of g between g_{f_1} and g_s, are almost circular (see curve 1 of Fig. 23). The square of the trajectory radius depends on the polar angle φ in the following manner:

$$R^2 = 1 - 2\left[g + \sqrt{\beta}\left(1 - 2\sqrt{g}\right)^{1/2}\cos\psi\right]^{1/2}$$

At $g_{f_2} < g < g_s$ the trajectories have the shape of a very narrow horseshoe surrounding the circle $R = 1$ (see curve 2 of Fig. 23).

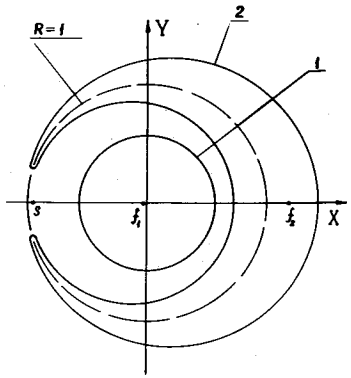

Figure 23 Phase trajectories (solid lines) of auxiliary nonlinear system with effective Hamiltonian $g(X, Y)$. The circle $X^2 + Y^2 = 1$ is shown dashed. The positions of the foci f_1 and f_2 and of the saddle s for a Duffing oscillator are indicated.

Therefore, transforming in Eq. (D.14) to polar coordinates, we obtain for trajectories 1 and 2, respectively,

$$\left(\frac{m_2}{\omega(g)m_1}\right)_1 = -\pi\left[\int_0^{2\pi} d\varphi\left(g+\sqrt{\beta}\cos\varphi\right)^{1/2}\right]^{-1}$$
$$\left(\frac{m_2}{\omega(g)m_1}\right)_2 = 1 \qquad \text{(D.15)}$$

(the expression for $m_2[\omega(g)m_1]^{-1}$ is written for the region $g \ll 1$, which is the most significant at $\beta \ll 1$).

Appendix E: Adiabatic Approximation for the Einstein–Fokker–Planck Equation Near Bifurcation Points

The EFP equation for the transition probability density $\tilde{w}(\ldots x_i \ldots, t)$ of the multidimensional random process (3.21) is of the form

$$\frac{\partial \tilde{w}}{\partial t} = \sum_{i,j} \alpha_{ij} \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_j} - \sum_i \frac{\partial}{\partial x_i}(P_i \tilde{w}). \qquad \text{(E.1)}$$

It is assumed here that $\alpha_{ij}(0,0) \equiv \alpha_{ij} \neq 0$ and, at the same time, that the coefficients α_{ij} are small enough and depend smoothly on $\mathbf{x}$ and $\mathbf{x}'$ in the region where x and x' are small (the dependence of the distribution function $\tilde{w}$ on $\mathbf{x}$ is much stronger). The terms proportional to the derivatives of $\alpha_{ij}(\mathbf{x}, \mathbf{x}')$ have therefore been discarded.

It was indicated in Sec. 2.5 that in the case when the coefficients P_i are of the form (3.22), the motion with respect to the variables x_i and $i > 1$ is rapid and have a characteristic time $t_0 = \max(\mathscr{A}_i')^{-1}$. It follows from Eqs. (E.1) and (3.22) that the distribution over variables with $i > 1$ also evolves within a time $\sim t_0$, and acquires then a characteristic width $\sim \alpha_1^{1/2}$ (it is assumed that the parameters α_{ij} do not exceed $\alpha_1 \equiv \alpha_{11}$ in order of magnitude). The maximum of the distribution corresponds to the quasistationary values $x_i = X_i(\mathbf{c}, x_1)$, where the functions $X_i(\mathbf{c}, x_1)$ are defined by the equations

$$P_i(\mathbf{c}, x_1, X_j) = 0, \qquad |X_i| \ll |x_1| \ll 1 \qquad \text{(E.2)}$$

Here and elsewhere i and j take on the values $2, 3, \ldots$ (it can be seen from Eqs. (3.21) and (E.2) that $x_i = X_i(\mathbf{c}, x_1)$ are solutions of the equations $\dot{x}_i = 0$ at fixed x_1 in the absence of a random force).

The distribution in x_1

$$w(x_1, t) = \int \tilde{w}(x, t)\, dx_2\, dx_3 \ldots \qquad \text{(E.3)}$$

evolves within a time much longer than t_0, and is substantially broader. To determine this distribution in the time region $t \gg t_0$ and to check on the validity of the statements made concerning the distributions in x_i $(i > 1)$, we can use the method of moments. Integrating Eq. (E.1) with respect to $x_2, x_3, \ldots$, we obtain

$$\frac{\partial w}{\partial t} = \alpha_1 \frac{\partial^2 w}{\partial x_1^2} - \frac{\partial}{\partial x_1}\left[P_1(\mathbf{c}, x_1, 0)w + \sum_{i>1}\left(\frac{\partial P_1}{\partial x_i}\right)_{x_2 = x_3 = \cdots = 0} \langle x_i \rangle + \cdots\right], \qquad \text{(E.4)}$$

$$\langle x_i \rangle = \int x_1 \tilde{w}(x, t)\, dx_2\, dx_3 \ldots, \qquad \alpha_1 \equiv \alpha_{11}.$$

Multiplication of Eq. (E.1) by x_i and integration lead to the following equations for the moments $\langle x_i \rangle$:

$$\frac{\partial \langle x_i \rangle}{\partial t} = \alpha_1 \frac{\partial^2 \langle x_i \rangle}{\partial x_1^2} - 2\alpha_{1i} \frac{\partial w}{\partial x_1} + P_i(\mathbf{c}, x_1, 0) w - \mathscr{A}_i \langle x_i \rangle - \cdots$$

$$- \frac{\partial}{\partial x_1} \Bigg[P_1(\mathbf{c}, x_1, 0) \langle x_i \rangle$$

$$+ \sum_{j>1} \left(\frac{\partial P_1}{\partial x_j} \right)_{x_2 = x_3 = \cdots = 0} \langle x_i x_j \rangle + \cdots \Bigg]. \quad \text{(E.5)}$$

We have used here the expansion (3.22) for $P_i(\mathbf{c}, \mathbf{x})$ in powers of x_j. The equations for the moments $\langle x_i x_j \rangle$ etc. are similar in form.

The chain of equations for the moments can be decoupled at times $t \gg t_0$ in the region of small $|x_1| \ll 1$ and for a sufficiently smooth function $w(x_1, t)$:

$$|x_1| \ll 1, \qquad \alpha_1 \left| \frac{\partial^2 w}{\partial x_1^2} \right| \ll w, \qquad \left| \frac{\partial w}{\partial t} \right| \ll w \qquad (t \gg t_0 \sim 1). \quad \text{(E.6)}$$

It can be seen from Eqs. (E.5) that the moments $\langle x_i \rangle$ relax after a characteristic time $t_0 = \mathscr{A}_i'^{-1}$, and at $t \gg t_0$ it suffices to restrict oneself to the quasistationary solution of the equations (if w changes little over the time $\sim t_0$). The approximate form of this solution is

$$\langle x_i \rangle = \mathscr{A}_i^{-1} \left[P_i(\mathbf{c}, x_1, 0) w - 2\alpha_{1i} \frac{\partial w}{\partial x_1} \right]$$

$$= X_i(\mathbf{c}, x_1) w - \frac{2\alpha_{1i}}{\mathscr{A}_i} \frac{\partial w}{\partial x_1}. \quad \text{(E.7)}$$

Account was taken here of Eq. (E.2). It is easy to verify with the aid of a simple iteration that when the conditions (E.6) and (E.2) are satisfied the terms discarded in Eq. (E.5) lead to a small correction in Eq. (E.7).

Substituting Eq. (E.7) in Eq. (E.4) and discarding the term with α_{1i}, which is small as compared with the first term in the right-hand side

of Eq. (E.4), we obtain the EFP equation for one-dimensional motion:

$$\frac{\partial w}{\partial t} = \alpha_1 \frac{\partial^2 w}{\partial x_1^2} - \frac{\partial}{\partial x_1} (P(x_1) w),$$

$$P(x_1) = P_1(\mathbf{c}, x_1, X_i(\mathbf{c}, x_1)). \quad \text{(E.8)}$$

Near a marginal bifurcation point (singularity of the codimension 1), according to Eqs. (3.22) and (E.2), we can put $X_i = 0$ in $P(x_1)$ neglecting quantities of higher order of smallness. A transformation from x_1 and t to new variables z and τ', and allowance for Eq. (3.22), enables us to write the EFP equation (E.8) in the form

$$\frac{\partial w}{\partial \tau'} = \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z} \left[(B_{11} z^2 - \epsilon_1 \alpha_1^{-2/3}) w \right],$$

$$z = \alpha_1^{-1/3} x_1, \qquad \tau' = \alpha_1^{1/3} t. \quad \text{(E.9)}$$

In the vicinity of the spinode point K in Fig. 10 (singularity of the codimension 2) we can determine the functions $X_i(\mathbf{c}, x_1)$ and $P(x_1)$ by expanding $P_1(\mathbf{c}, \mathbf{x})$ and $P_i(\mathbf{c}, \mathbf{x})$ in powers of $\mathbf{x}$ and $\mathbf{c} - \mathbf{c}_K$ and taking into account the relations (3.30) for the expansion coefficients. As a result it follows from Eqs. (E.2), (E.8), and (3.30) that

$$\frac{\partial w}{\partial \tau'} = \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z} \left[(d_3 z^3 - d_1 z - d_0) w \right],$$

$$z = \alpha_1^{-1/4} x_1, \qquad \tau' = \alpha_1^{1/2} t, \quad \text{(E.10)}$$

where

$$d_0 = \alpha_1^{-3/4} \frac{\partial P_1}{\partial \mathbf{c}} (\mathbf{c} - \mathbf{c}_K),$$

$$d_1 = \alpha_1^{-1/2} \frac{\partial^2 P_1}{\partial \mathbf{c} \partial x_1} (\mathbf{c} - \mathbf{c}_K)$$

$$+ \alpha_1^{-1/2} \sum_i \frac{1}{\mathscr{A}_i} \frac{\partial^2 P_1}{\partial x_1 \partial x_i} \frac{\partial P_i}{\partial \mathbf{c}} (\mathbf{c} - \mathbf{c}_K), \quad \text{(E.11)}$$

$$d_3 = -\frac{1}{6} \frac{\partial^3 P_1}{\partial x_1^3} - \frac{1}{2} \sum_i \frac{1}{\mathscr{A}_i} \frac{\partial^2 P_1}{\partial x_1 \partial x_i} \frac{\partial^2 P_i}{\partial x_1^2} > 0.$$

All the derivatives are taken here at $\mathbf{c} = \mathbf{c}_K$ and $\mathbf{x} = 0$. The condition $d_3 > 0$ is the condition for the existence of a stable equilibrium state for the point K.

From the adiabaticity condition $|\partial w/\partial t| \ll w$ it follows according to Eq. (E.8) that $|P(x_1)| \ll 1$ and $|dP(x_1)/dx_1| \ll 1$, i.e., that for the adiabatic approximation to hold the condition $|\mathbf{c} - \mathbf{c}_B| \ll 1$ must be satisfied.

Appendix F: Quantum Kinetic Equation for a Nonlinear Oscillator

In the case when the interaction of the nonlinear oscillator with the medium is weak enough, a quantum kinetic equation for the operator $F_M(t)$ (4.4) can be obtained at a sufficiently general form of the interaction Hamiltonian H_i. If the characteristic relaxation parameters Γ and P are small compared with T, we can replace $\exp(-H/T)$ by $\exp(-\mathscr{H}_0/T)$ in expression (4.4) for $F_M(t)$. Here $\mathscr{H}_0 = H_0 + H_m$ (see Eq. (4.1)). Then $F_M(t)$ takes the form

$$F_M(t) = \mathrm{Tr}_m \mathscr{F}_M(t),$$
$$\mathscr{F}_M(t) = Z^{-1}\hat{U}(t)\hat{M}\exp(-\mathscr{H}_0/T)\hat{U}^+(t). \tag{F.1}$$

From the definition (4.3) of the operators $\hat{U}(t)$ follow the relations

$$\begin{aligned}\mathscr{F}_M(t) &= \mathscr{F}_M(0) - i\int_0^t dt_1 \left[\tilde{H}_i(t_1), \mathscr{F}_M(t_1)\right] \\ &= \mathscr{F}_M(0) - i\int_0^t dt_1 \left[\tilde{H}_i(t_1), \mathscr{F}_M(0)\right] \\ &\quad - \int_0^t dt_1 \int_0^{t_1} dt_2 \left[\tilde{H}_i(t_1), \left[\tilde{H}_i(t_2), \mathscr{F}_M(t_2)\right]\right] \\ &= \ldots\end{aligned} \tag{F.2}$$

Since the operators $\hat{M}$ and H_0 do not depend on the dynamic variables of the medium, it can be seen from Eqs. (F.1) and (F.2) that

averaging over these variables reduces to calculation of mean values of the type

$$\int dt_1 \ldots dt_n \langle \tilde{H}_i(t_1) \ldots \tilde{H}_i(t_n)\rangle_m, \tag{F.3}$$

where

$$\langle \hat{O}\rangle_m \equiv Z_m^{-1}\mathrm{Tr}_m \hat{O}\exp(-H_m/T), \qquad Z_m = \mathrm{Tr}_m \exp(-H_m/T). \tag{F.4}$$

Without loss of generality it can be assumed that

$$\langle H_i\rangle_m = 0. \tag{F.5}$$

Since the off-diagonal matrix elements of the operator $\tilde{H}_i(t)$ oscillate rapidly with a characteristic time $\sim t_c$ ($t_c = \max(\omega_0^{-1}, \omega_e^{-1})$, see Eq. (2.6)), the pair correlators

$$\langle \tilde{H}_i(t_1)\tilde{H}_i(t_2)\rangle_m$$

also oscillate rapidly as functions of $t_1 - t_2$, and the main contribution to the integral

$$\int\int dt_1\, dt_2 \langle \tilde{H}_i(t_1)\tilde{H}_i(t_2)\rangle_m$$

is made by the region $|t_1 - t_2| \lesssim t_c$ (the region of integration with respect to each of the arguments t_1 and t_2 is of the order $t \gg t_c$). Analogously, the main contribution to the multiple integrals (F.3) is made by integration regions in which the times t_i break up into pairs such that the time differences within the pairs are $\sim t_c$, while the intervals between different pairs are $\sim t$. Since the correlation vanishes after times $\sim t_c$, it is possible to replace in this case the mean value of the product of n operators in (F.3) by the product of $n/2$ mean values of the separate pairs. This approximation, which is asymptotically valid at

$$t \gg t_c, \qquad \Gamma t_c \ll 1, \tag{F.6}$$

makes possible the decoupling

$$\mathrm{Tr}_m\int_0^t dt_1\,\tilde{H}_i(t)\tilde{H}_i(t_1)\mathscr{F}_M(t_1)=\int_0^t dt_1\langle\tilde{H}_i(t)\tilde{H}_i(t_1)\rangle_m\mathrm{Tr}_m\mathscr{F}_M(t_1). \tag{F.7}$$

In the language of diagram technique, this approximation corresponds to allowance for diagrams with lines that do not intersect and are not imbedded in one another [76, 77].

The main contribution to the integral in the right-hand side of Eq. (F.7) is made by the region $|t-t_1|\sim t_c$, in which the operator $\mathrm{Tr}_m\mathscr{F}_M(t_1)=F_M(t_1)$ remains practically unchanged, and can consequently be replaced by $F_M(t)$. Therefore, applying the operation Tr_m to Eq. (F.2) and taking the condition (F.5) into account, we obtain the integral operator equation

$$F_M(t)=F_M(0)-\int_0^t dt_1\int_0^{t_1}dt_2 \times\mathrm{Tr}_m\left\{\left[\tilde{H}_i(t_1),\left[\tilde{H}_i(t_2),Z_m^{-1}\exp\left(-\frac{H_m}{T}\right)F_M(t_1)\right]\right]\right\} \tag{F.8}$$

Corresponding to it is the differential equation

$$\frac{\partial F_m}{\partial t}=-\int_0^t dt_1\,\mathrm{Tr}_m\left\{\left[\tilde{H}_i(t),\left[\tilde{H}_i(t_1),Z_m^{-1}\exp\left(-\frac{H_m}{T}\right)F_M(t)\right]\right]\right\}, \tag{F.9}$$

which is the analog of the quantum kinetic equation. Obviously, this is a Markov equation.

Neglecting corrections of order $\sim|P|T^{-1}$, the initial condition for Eq. (F.9) is according to Eq. (F.1)

$$F_M(0)=Z_0^{-1}\hat{M}\exp\left(-\frac{H_0}{T}\right),\qquad Z_0=\mathrm{Tr}_0\exp\left(-\frac{H_0}{T}\right). \tag{F.10}$$

In the considered long-time region (F.6), the integration with respect to t_1 and the averaging over the variables of the medium in Eq.

(F.9) can be carried out explicitly for various concrete forms of the interaction Hamiltonian. We shall do this for the case when the interaction is linear in the coordinate of the singled-out oscillator and has an arbitrary dependence on the dynamic variables of the medium:

$$H_i=(\hat{a}+\hat{a}^+)\hat{h}_i. \tag{F.11}$$

Here $\hat{h}_i$ is an arbitrary function of the operators $\hat{a}_k^+$ and $\hat{a}_k$ (or of the creation and annihilation operators of other quasiparticles of the medium) and does not depend on $\hat{a}^+$ or $\hat{a}$.

To integrate with respect to time in Eq. (F.9), we use an explicit expression for the evolution of the operator $\tilde{a}(t)$ in the interaction representation

$$\tilde{a}(t_1)=\tilde{a}(t)\exp\left\{i\left[\omega_0+\left(\tilde{n}-\tfrac{1}{2}\right)V\right](t-t_1)\right\}$$
$$\left(\tilde{a}(t)\equiv\exp(iH_0t)\ddot{a}\exp(-iH_0t),\ \tilde{n}=\tilde{a}^+\tilde{a}\right), \tag{F.12}$$

which follows from Eq. (4.1) for the Hamiltonian H_0. If the actual occupation numbers of the oscillator and the parameter of its nonlinearity are not too large, i.e.,

$$t_c|V|(2\bar{n}+1)\ll 1,$$

we can neglect in the region $|t-t_1|\sim t_c$ the terms proportional to V in the argument of the exponential in Eq. (F.12). The integrals with respect to time in Eq. (F.9) (in the case of H_i in the form (F.11)) are then simplified. For example,

$$\int_0^t dt_1\langle\tilde{a}^+(t)\tilde{h}_i(t)\tilde{a}(t_1)\tilde{h}_i(t_1)\rangle_m$$
$$=\tilde{a}^+(t)\tilde{a}(t)\int_0^t dt_1\,e^{i\omega_0(t-t_1)}\langle\tilde{h}_i(t)\tilde{h}_i(t_1)\rangle_m$$
$$=\tilde{a}^+(t)\tilde{a}(t)\int_0^t dt_1\,e^{i\omega_0 t_1}\langle\tilde{h}_i(t_1)\tilde{h}_i(0)\rangle_m$$
$$\left(h_i(t)\equiv\exp(iH_mt)h_i\exp(-iH_mt)\right). \tag{F.13}$$

Since the integrand in Eq. (F.13) decays within a time $\sim t_c$, the upper

integration limit in the last expression in (F.13) can be allowed to tend to infinity.

When account is taken of the relations such as (F.13), the kinetic equation (F.9) is transformed into

$$\frac{\partial F_M}{\partial t} = -\big[(\bar{n}+1)(\tilde{a}^+\tilde{a}F_M - 2\tilde{a}F_M\tilde{a}^+ + F_M\tilde{a}^+\tilde{a}) + \bar{n}(\tilde{a}\tilde{a}^+F_M - 2\tilde{a}^+F_M\tilde{a} + F_M\tilde{a}\tilde{a}^+)\big]\Gamma - i\big[\tilde{a}^+\tilde{a}, F_M\big]P. \tag{F.14}$$

All the operators are referred here to the instant of time t, while Γ and P are given by

$$\Gamma = \mathrm{Re}\int_0^\infty dt\, e^{i\omega_0 t - \epsilon t}\langle\big[\tilde{h}_i(t), \tilde{h}_i(0)\big]\rangle_m, \qquad P = \mathrm{Im}\int_0^\infty dt\, e^{i\omega_0 t - \epsilon t}\langle\big[\tilde{h}_i(t), \tilde{h}_i(0)\big]\rangle_m \quad (\epsilon \to +0). \tag{F.15}$$

Expressions (F.15) coincide with those obtained in [4] for the damping parameter and the frequency shift, while Eq. (F.14) itself (without the shift term $\propto P$) for the density matrix of a harmonic oscillator was considered phenomenologically in [11, 12].

In the particular case of the bilinear interaction Hamiltonian (4.1) (the $\hat{h}_i$ depend linearly on $\hat{a}_k + \hat{a}_k^+$), Eqs. (F.15) go over into formulas (2.15) of the classical theory. Another derivation of the generalized kinetic equation for this case is given in [21]. A convenient method (based on the use of superoperator theory) of obtaining the quantum kinetic equation for a nonlinear oscillator, and of analyzing the correlation functions, has been developed in [78].

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