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LARGE FLUCTUATIONS IN A PERIODICALLY DRIVEN DYNAMICAL SYSTEM

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Fluctuations in a periodically driven overdamped oscillator are studied theoretically and experimentally in the limit of low noise intensity by investigation of their prehistory. It is shown that, for small noise intensity, fluctuations to points in coordinate space that are remote from the stable states occur along paths that form narrow tubes. The tubes are centered on the *optimal paths* corresponding to trajectories of an auxiliary Hamiltonian system. The optimal paths themselves, and the tubes of paths around them, are visualized through measurements of the prehistory probability distribution for an electronic model. Some general features of fluctuations in nonequilibrium systems, such as singularities in the pattern of optimal paths, the corresponding nondifferentiability of the generalized nonequilibrium potential, and the feasibility of their experimental investigation, are discussed.

1. Introduction

Large fluctuations, although infrequent, play a fundamental role in a wide range of important processes, from earthquakes to nucleation at phase transitions, mutations in DNA sequences, and failures of electronic devices. It has been recognized recently that their role can sometimes be creative: large occasional fluctuations can be used to enhance signals in nonlinear systems and improve signal processing through stochastic resonance [Moss *et al.*, 1993; Dykman *et al.*, 1995; Bulsara & Gammaitoni, 1996]; they may also give rise to unidirectional transport in periodic structures (ratchets) [Magnasco, 1993, 1994; Millonas & Dykman, 1994; Leibler, 1994; Doering, 1994].

In many cases of interest large fluctuations occur in systems far from thermal equilibrium. Examples are systems driven by strong time-dependent fields, lasers, electronic devices, and biological systems.

Whereas, for systems in thermal equilibrium, the stationary distribution and the probabilities of fluctuations are known at least in principle (although the problem of evaluating them is often extremely complicated, cf. [Kagan & Leggett, 1992]), for nonequilibrium systems there are no general relations from which these probabilities can be obtained.

A powerful approach to the analysis of large fluctuations in classical systems is based on the concept of optimal path [Ventcel' & Freidlin, 1970;

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Freidlin & Wentzell, 1984; Ludwig, 1975; Dykman & Krivoglaz, 1984; Jauslin, 1987; Dykman, 1990; Dykman *et al.*, 1994a, 1994b; Maier & Stein, 1996, 1997]. This is the path along which a system moves, with overwhelming probability, when it fluctuates away from a stable state (an attractor in phase space) to a given remote state. Dykman *et al.* [1992] suggested how optimal paths might be observed experimentally, and demonstrated the physical reality of such paths through an analog electronic experiment.

There are general arguments and experimental evidence [Dykman *et al.*, 1992] showing that, in thermal equilibrium systems, the optimal path to a given state is the time-reversed dynamical path along which the system moves from this state to the stable state if fluctuations are neglected. So, in thermal equilibrium, the pattern of optimal paths is known at least in principle from an analysis of relaxation in the absence of fluctuations [Luchinsky & McClintock, 1997].

The above arguments do not apply to nonequilibrium systems. There are general theoretical arguments [Dykman *et al.*, 1994b; Maier & Stein, 1996] strong numerical evidence [Jauslin, 1987; Day, 1987, 1992; Maier & Stein, 1992, 1993; Chinarov *et al.*, 1993] and recently some direct experimental evidence [Dykman *et al.*, 1996] showing that the pattern of optimal paths in nonequilibrium systems differs qualitatively from that in the equilibrium case. Theoretical and experimental investigations of the pattern of optimal paths present an intriguing and important problem.

Its importance arises because knowledge of the pattern of optimal paths provides a convenient way — often the only known one — of calculating basic statistical quantities such as stationary probability distributions and transition probabilities (see below); furthermore it illuminates the dynamical processes underlying general properties of fluctuations in nonequilibrium systems such as nondifferentiability of the generalized thermodynamic potential (see [Graham & Tel, 1984a, 1984b, 1985]). This understanding promises to pave the way towards the development of techniques for *controlling* large fluctuations because, even for a small change of parameters, a system may fluctuate to the same point in configuration space along completely different paths.

An advantageous feature of optimal fluctuational paths is that they can be directly observed through investigations of the prehistory probability distribution ([Dykman *et al.*, 1992, 1996], see also [Schulman, 1991]). Moreover, we expect that it will thereby be possible to shed light on the topology of the instantons which are considered in quantum field theory.

In the present paper we analyze optimal paths for large fluctuations in a dynamical system driven by white noise and by a periodic force, and we present the first experimental results demonstrating the existence of a pattern of optimal paths in a thermally nonequilibrium system.

2. Formulation of the Problem

We consider an overdamped system driven by a periodic force $K(q;\,\phi)$ and white noise $\xi(t)$, with equation of motion

$$\dot{q} = K(q;\,\phi) + \xi(t), \quad K(q;\,\phi) = K(q;\,\phi + 2\pi)\,, \qquad (1)$$
$$\phi \equiv \phi(t) = \omega t + \phi_0; \quad \langle \xi(t)\xi(t')\rangle = D\delta(t-t')\,.$$

A simple example to bear in mind is the familiar *overdamped* bistable oscillator driven by a periodic force:

$$\dot{q} = -U'(q) + A\cos\omega t + \xi(t)\,, \qquad (2)$$
$$U(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4\,.$$

We consider a situation that is both *nonadiabatic* and *nonlinear*: neither Ω nor A need be small; only the noise intensity D will be assumed small.

In what follows we investigate fluctuations in the domain of attraction of a single attractor. In this regime, before a large fluctuation to the point $(q_f,\,\phi_f)$ occurs, the system fluctuates for a long time (compared to the relaxation time τ_r) about the stable stationary state whose basin of attraction contains the point $(q_f,\,\phi_f)$. The position of the stable state $q^{(0)}(t)$ is a periodic function of time,

$$\dot{q}^{(0)} = K(q^{(0)};\,\phi), \quad q^{(0)}\left(t + 2\pi\omega^{-1}\right) = q^{(0)}(t)\,. \qquad (3)$$

The equations for optimal paths can be found using the eikonal approximation to solve the corresponding Fokker–Plank equation, or by using a path integral formulation and evaluating the path integral over the fluctuational paths in the steepest descent approximation (for details and discussion see [Freidlin & Wentzel, 1984; Ludwig, 1975; Graham, 1989]). The optimal path of a periodically driven system gives rise to a maximum in the prehistory probability density, $p_h(q,\,\phi|\,q_f,\,\phi_f)$ [Dykman *et al.*, 1992, 1996]. This is the probability density for a system arriving at the point $(q_f,\,\phi_f)$ at the instant t_f $(\phi(t_f) = \phi_f)$ to have passed through the point $q,\,\phi$ at the instant $t(t < t_f)$. The important advantage of this formulation is that p_h is the physical quantity that can be measured in an experiment. This approach can be extended to include analysis of singular points in the pattern of optimal paths.

The expression for p_h can be written in path integral form ([Dykman *et al.*, 1992],

$$\begin{aligned} &p_h(q,\,\phi|q_f,\,\phi_f) \\ &\quad = C\int_{q(t_i)\approx q^{(0)}(t_i)}^{q(t_f)=q_f} \mathcal{D}q(t')\,\delta(q(t)-q) \\ &\quad \times \exp\left[-\frac{S[q(t)]}{D} - \frac{1}{2}\int_{t_i}^{t_f} dt'\frac{\partial K}{\partial q}\right], \quad t_i \to -\infty \\ &\qquad \phi \equiv \phi(t), \quad \phi_f \equiv \phi(t_f)\,. \end{aligned} \tag{4}$$

Here, C is a normalization constant determined by the condition

$$\int dq p_h(q,\,\phi|q_f,\,\phi_f) = 1\,.$$

$S[q(t)]$ has the form of an action functional for an auxiliary dynamical system with time-dependent Lagrangian $L(\dot{q},\,q;\,\phi)$:

$$\begin{aligned} S[q(t)] &= \int_{t_i}^{t_f} dt\,L(\dot{q},\,q;\,\phi)\,, \\ L(\dot{q},\,q;\,\phi) &= \frac{1}{2}[\dot{q} - K(q;\,\phi)]^2\,. \end{aligned} \tag{5}$$

A theoretical analysis of the prehistory probability density (4) was given by Dykman *et al.* [1996]. In the range of small noise intensities D the optimal path $q_{\mathrm{opt}}(t|q_f,\,\phi_f)$ to the point $(q_f,\,\phi_f)$, where p_h has a peak, is given by the condition that the action S be minimal. The variational problem for S to be extremal gives Hamiltonian equations of motion for the coordinate q and momentum p of the auxiliary system:

$$\begin{aligned} &\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dS}{dt} = \frac{1}{2}p^2\,, \\ &H \equiv H(q,\,p;\,\phi) = \frac{1}{2}p^2 + pK(q;\,\phi)\,, \\ &H(q,\,p;\,\phi) = H(q,\,p;\,\phi + 2\pi)\,. \end{aligned} \tag{6}$$

The boundary conditions for the extreme paths (6) follow from (4) and (5):

$$\begin{aligned} &q(t_f) = q_f\,; \\ &q(t_i) \to q^{(0)}(t_i), \quad p(t_i) \to 0, \quad S(t_i) \to 0 \\ &\qquad\text{for} \quad t_i \to -\infty\,. \end{aligned} \tag{7}$$

Since the Hamiltonian $H(q,\,p;\,\phi)$ is periodic in ϕ, the set of paths $\{q(t),\,p(t)\}$ is also periodic: the paths that arrive at a point $(q_f,\,\phi_f + 2\pi)$ are the same as the paths that arrive at the point $(q_f,\,\phi_f)$, but shifted in time by the period $2\pi/\omega$. The action $S(q_f,\,\phi_f)$ evaluated along the extreme paths is also periodic as a function of the phase ϕ_f of the final point $(q_f,\,\phi_f)$. The function $S(q,\,\phi)$ satisfies the Hamilton–Jacobi equation:

$$\begin{aligned} &\omega\frac{\partial S}{\partial \phi} = -H\left(q,\,\frac{\partial S}{\partial q};\,\phi\right), \quad p \equiv \frac{\partial S}{\partial q}\,, \\ &S(q,\,\phi) = S(q,\,\phi + 2\pi)\,. \end{aligned} \tag{8}$$

It is straightforward to see that the extreme paths obtained by solving (6) form a one-parameter set $\{q(t;\,\Delta),\,p(t;\,\Delta)\}$. The parameter Δ can be chosen as the distance from the extreme path to the attractor $q^{(0)}(t)$ for a certain large negative value of the initial phase $\phi(t_i)$ (we note that to identify the path we have to specify the value of $\phi(t_i)$ for which Δ has been chosen). To find the value of the momentum p and of the action S for a given Δ we note from (7) that, for $t \to -\infty$, the extreme paths are confined to the immediate neighborhood of the attractor. Therefore Eqs. (6) can be linearized about $q^{(0)}(t)$. If for a certain large negative t_i the difference between $q(t_i)$ and $q^{(0)}(t_i)$ is Δ, then the momentum $p(t_i)$ is also proportional to Δ:

$$\begin{aligned} q(t_i) &\equiv q(t_i;\,\Delta) = q^{(0)}(t_i) + \Delta\,, \\ p(t_i) &\equiv p(t_i;\,\Delta) = a(\phi(t_i))\Delta\,, \\ S(t_i;\,\Delta) &\equiv S(q(t_i;\,\Delta),\,\phi(t_i)) = \frac{1}{2}a(\phi(t_i))\Delta^2\,, \end{aligned} \tag{9}$$

The function $a(\phi)$ can be found from the linearized Hamilton–Jacobi equation (6), noting that the periodicity of $S(q,\,\phi)$ also implies the periodicity of $a(\phi)$:

$$\begin{aligned} a(\phi) &= \omega(\tilde{\Upsilon} - 1)\left[\int_0^{2\pi} d\varphi\Upsilon(\phi,\,\varphi)\right]^{-1}\,, \\ \Upsilon(\phi,\,\varphi) &= \exp\left[-2\omega^{-1}\int_{\phi}^{\phi+\varphi} d\phi'\left(\frac{\partial K}{\partial q}\right)^{(0)}\right]\,, \\ \tilde{\Upsilon} &= \Upsilon(\phi,\,\phi + 2\pi), \quad a(\phi) \equiv a(\phi + 2\pi) \end{aligned} \tag{10}$$

(the derivative $(\partial K/\partial q)^{(0)}$ is evaluated for $q = q^{(0)}(t')$, $t' \equiv \omega^{-1}(\phi' - \phi_0)$).

The formulation of the problem of fluctuational paths in terms of the Hamiltonian dynamics of an auxiliary system makes it possible to apply powerful analytical methods developed in the theory of Hamiltonian systems to the theory of large fluctuations.

3. Discussion of Results

It is known from the theory of dynamical systems (see for instance [Arnold, 1978]) that trajectories emanating from a stationary state lie on a Lagrangian manifold (LM) in phase space (q, ϕ, $p = \partial S/\partial q$) (the unstable manifold of the corresponding state) and form a one-parameter set. The action $S(q, t)$ is a smooth single-valued function of position on the LM. It is a Lyapunov function: it is nondecreasing along the trajectories of the initial system in the absence of noise $\dot{q} = K(q; \phi)$. Therefore $S(q, t)$ may be viewed as a generalized nonequilibrium thermodynamical potential for a fluctuating dynamical system (see [Graham, 1989]). The projections of trajectories in phase space onto configuration space form the extreme paths. Optimal paths are the extreme paths that give the minimal action to a given point in the configuration space. These are the optimal paths that can be visualized in an experiment via measurements of the prehistory probability distribution.

The pattern of extreme paths, LM, and action surfaces for an overdamped periodically driven oscillator (2) are shown in Fig. 1. The figure illustrates generic topological features of the pattern in question. It can be seen from Fig. 1 that, although there is only one path to a point (q, ϕ, p) in phase space, several different extreme paths may come from the stationary periodic state to the corresponding point (q, ϕ) in configuration space. These paths cross each other. This is a consequence of folding of the Lagrangian manifold.

A generic feature related to folding of LMs is the occurrence of *caustics* in the pattern of extreme

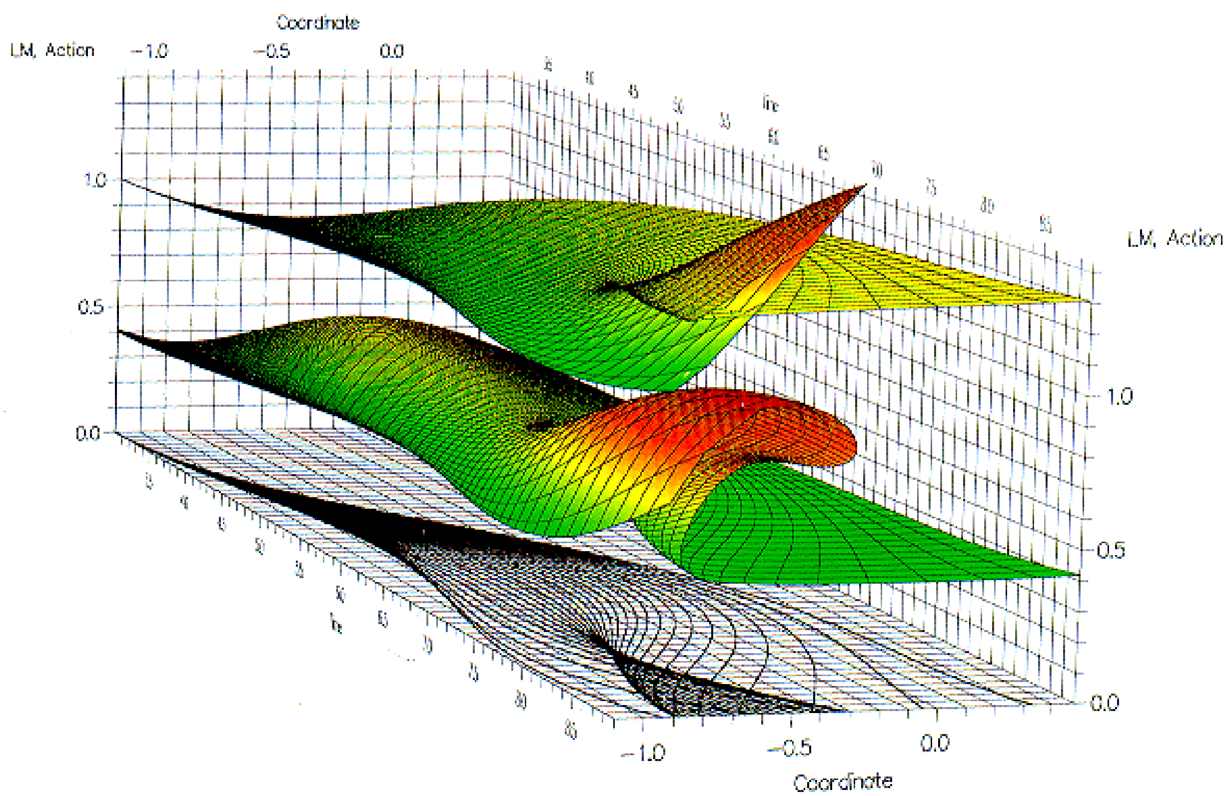

Fig. 1. From top to bottom: action surface, Lagrangian manifold (LM) and extreme paths calculated for the system (2) using Eq. (6) with initial conditions (9). Parameters for the system were: $A = 0.264$, $\omega = 1.2$. To clarify interrelations between singularities in the optimal paths pattern, action surface, and LM surface, they are shown in one figure: the action surface has been shifted up by one unit, and the LM scaled by a factor 1/2 and shifted up by 0.4.

paths. Caustics are projections of the folds of an LM. They start at cusp points. It is clear from Fig. 1 that the structure of an LM with two folds merging at the cusp give rise to a locally swallowtail-type singularity of the action surface. The spinode edges of the action surface correspond to the caustics. A *switching line* emanates from the cusp point at which two caustics meet. This is the projection of the line in phase space along which the two lowest sheets of the action surface intersect. The switching line separates regions which are reached along different *optimal* paths, and the optimal paths intersect on the switching line. The intersection occurs *prior* to a caustic being encountered by the optimal path. The formation of the singularities, avoidance of caustics, and formation of switching lines were analyzed numerically by Jauslin [1987], and a complete theory was given by Dykman *et al.* [1994b]. The generic topological features of the pattern of optimal paths have not previously been observed in experiments; but we now describe briefly a method of investigating the pattern of optimal paths in thermally nonequilibrium systems, based on analogue electronic experiments. We present our initial experimental results, and discuss the possibility of developing the technique to reveal singularities in the pattern of optimal paths.

The experimental investigation of large fluctuations and optimal paths is complicated by two factors. First, by definition, these fluctuations occur only occasionally. Secondly, in general the coordinate space has two or more dimensions. An additional problem occurs when transitions between stable states are investigated: *reaching* a given stable state does not necessarily means that the system has actually *switched* to this state: as the fluctuation of the random force decays, the decreasing

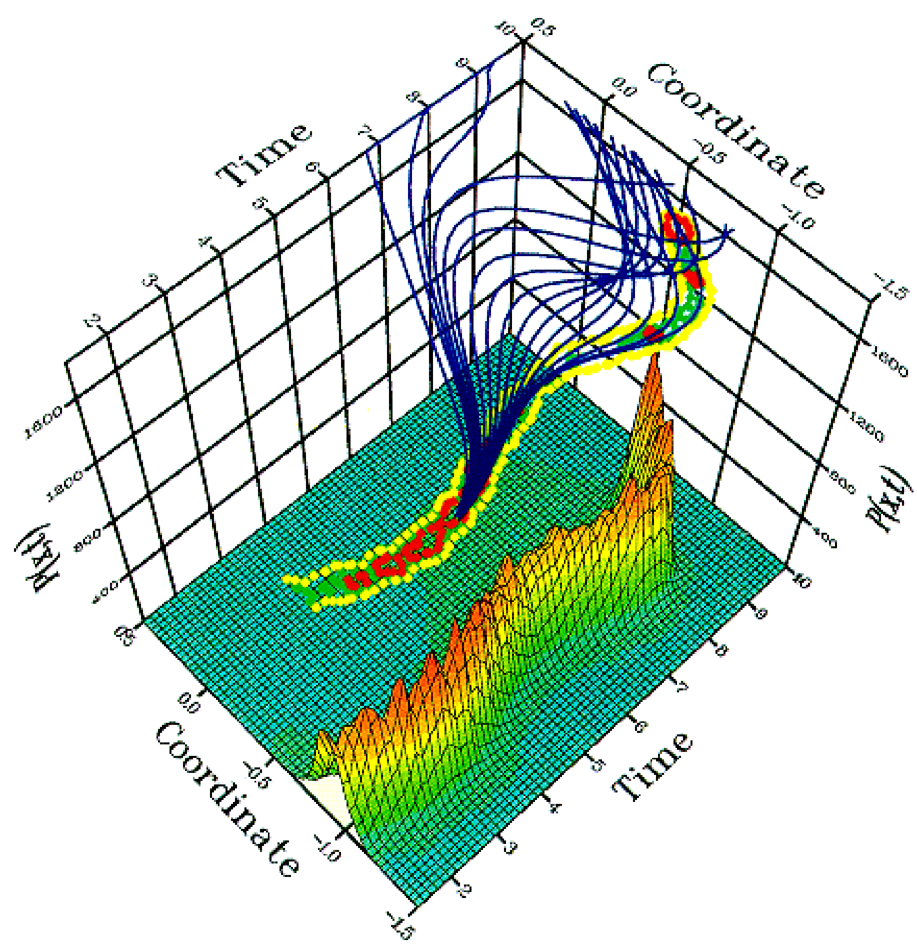

Fig. 2. Distribution of fluctuational paths which terminate inside the rectangle $-0.7 \le q_f \le -0.65$, $9.0 \le t_f \le 9.2$ for the system (2) with $A = 0.264$, $\omega = 1.2$. The maxima of the distribution represent the optimal paths. The theoretical extreme paths and the contour plot of the distribution are shown at the top.

(but still large) force can "pull" the system back to the initially occupied state.

The problem of statistics can be overcome, at least in part, by using a multichannel technique. Several dynamical variables of the system and the external force are recorded simultaneously, and then the statistics of all actual trajectories along which the system moves in a particular subspace of the coordinate space is analyzed. It is clear that information about stochastic processes obtained in this way is much more detailed than that obtained by the standard method of measuring stationary probability distributions. In our technique, not only do we count rare events (i.e. arrivals of the system at a given point in configuration space), but we also learn how each of these events comes about.

The technique has been tested with an analogue electronic circuit model of the system (2), designed in accordance with standard techniques [Fronzoni, 1989; McClintock & Moss, 1989]. It was driven by weak quasiwhite noise from a noise-generator, and a periodic force from a frequency synthesizer. The fluctuating voltage representing $q(t)$ was digitized and analyzed in discrete blocks of 32768 samples using a Nicolet NIC-1180 data-processor. The input sweeps were triggered by the frequency synthesizer so that information about the phase of the periodic force could be retained. The analysis algorithm enabled an 8×8 matrix of 64 termination squares, each centred on particular $(q_f,\,\phi_f)$ values, to be scanned. Whenever $q(t)$ entered one of these squares, the immediately preceding section of the trajectory was collected and stored. The trajectories that had arrived in any chosen square could subsequently be ensemble-averaged together to create the prehistory probability distribution $p_h(q,\,\phi|q_f,\,\phi_f)$ corresponding to the chosen $(q_f,\,\phi_f)$. Because the fluctuations of interest are by definition rare, it was necessary to

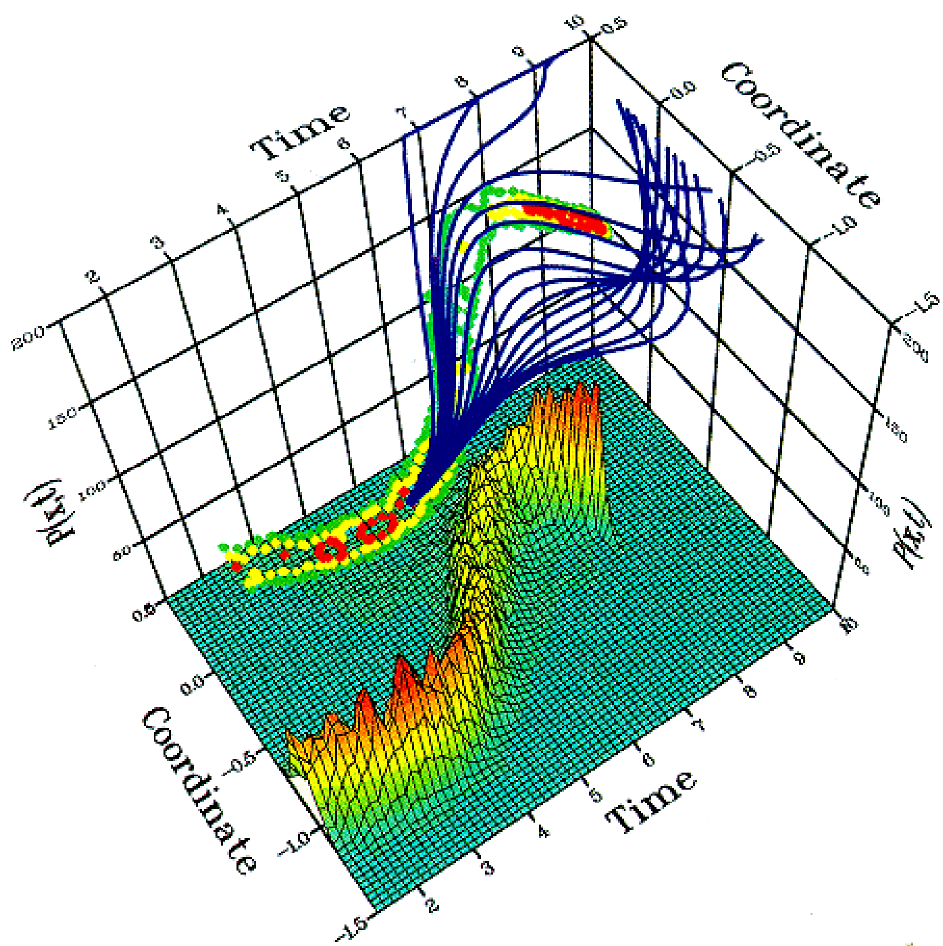

Fig. 3. Distribution of fluctuational paths which terminate inside the rectangle $-0.4 \le q_f \le -0.35$, $7.81 \le t_f \le 8.08$ for the system (2) with $A = 0.264$, $\omega = 1.2$. The maxima of the distribution represent the optimal paths. The theoretical extreme paths and the contour plot of the distribution are shown at the top.

continue the experiments for a considerable period, typically weeks, to be able to build up reasonably smooth distributions.

Some experimentally measured prehistory probability distributions for arrivals at two different points in configuration space are shown in Figs. 2 and 3. It is clear from the figures: (i) that the prehistory probability distributions are sharp and have a well defined ridges; (ii) that the shapes of the ridges are very different for different final points; and (iii) that the ridges follow very closely the theoretical trajectories obtained by solving numerically the equations of motion for the optimal paths (6).

4. Conclusion

The good quantitative agreement between theory and experiment shows that our experimental technique makes it possible to investigate optimal paths for thermally nonequilibrium systems, and to reveal the singularities in the pattern of optimal paths. These include in particular switching lines and strong (nonanalytic in the noise intensity) smearing of the prehistory probability distribution near cusp points. The system we have investigated has the least number of degrees of freedom necessary to observe these singularities, and therefore it is most appropriate for the analysis. Detailed experimental data related to the singularities in the pattern of optimal paths, and the theory of self-similar Lagrangian manifolds, will be presented elsewhere.

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