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QUANTUM COMPUTING WITH PERPETUALLY COUPLED QUBITS: ON-SITE LOCALIZATION OF EXCITATIONS

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We demonstrate that, in a quantum computer with perpetually coupled qubits, all excitations can be confined to their sites (qubits) even without refocusing. The on-site localization is obtained by constructing a sequence of qubit energies that efficiently suppresses resonant hopping. The time during which a many-excitation state remains strongly localized in an infinite chain can exceed the reciprocal hopping frequency by $\gtrsim 10^5$ already for a moderate bandwidth of qubit energies. The proposed energy sequence is also convenient for performing quantum operations on the qubits.

Keywords: perpetually coupled qubits, localization, many-particle transitions, lifetime
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1 Introduction

In many proposed physical implementations of a quantum computer (QC) the qubit-qubit interaction is never turned off [1]- [7]. A generic consequence of the interaction is hopping of excitations between the qubits. Preventing hopping is a prerequisite for quantum computation. In all proposed schemes, control of a QC and measurement are done assuming that excitations remain localized between operations. For perpetually coupled qubits localization can be accomplished with refocusing techniques [1]. Recently there were also proposed approaches that do not require ongoing resonant pulsing [8, 9]. At the same time, localization has been one of the central problems of condensed-matter physics [10]. One-particle localization often results from disorder caused by randomness of particle energies on different sites and/or inter-site hopping integrals. The problem of localization is particularly challenging for many-body systems, where only a limited number of results has been obtained [11].

In this paper we provide an overview and further extend our recent work [12, 13] on on-site many-particle localization. Such localization means strong confinement of excitations to the qubits (sites) where they were created. It is a stronger requirement than just exponential

decay of the wave function at large distances, and it is this requirement that must hold in a QC. Throughout the paper we use the term "particles" and "sites" for excitations and qubits, respectively. For 1D chains there is one-to-one mapping of the excitations onto fermions via the Jordan-Wigner transformation.

On-site localization does not arise in a disordered many-particle system with bounded random on-site energies [14]. Indeed, consider an N-particle state where each particle is fully localized on its site, with no more than one particle per site. We call this an on-site state or a quantum register. For short-range hopping each on-site N-particle state is directly coupled to $\sim N$ other such states. With probability $\propto N$ one of them will be in resonance with the initial state, provided the on-site energies are uniformly distributed over a finite-width band. For large N this leads to state hybridization over time $\sim J^{-1}$, where J is the intersite hopping integral (we set $\hbar = 1$).

In a QC, the quantity J is determined by the qubit-qubit interaction and often characterizes the rate of two-qubit operations. The on-site excitation energies are interlevel distances of the qubits. In many cases they can be individually controlled, which makes it possible to construct an arbitrary energy sequence. However, since the qubit tuning range is limited, so should be the energy bandwidth. This imposes an important constraint that has to be met when localization is sought. A smaller bandwidth leads also to a higher speed of quantum gate operations, particularly if they involve changing qubit energies [3].

We discuss localization from two points of view. One is based on the analysis of *stationary* states of a many-particle system. The other is based on studying the system dynamics. As a result of hopping, an on-site many-particle state can hybridize with another on-site state with nearly the same energy. We will study the time it takes for resonant hybridization to happen, which we call the localization lifetime $t_{\rm loc}$.

In a QC all states have a finite coherence time due to coupling to the environment and external noise. For successful QC operation, delocalization should not occur during this time. For most of the proposed models of a QC, the coherence time is $\lesssim 10^5 J^{-1}$. Therefore it is sufficient to have the localization lifetime $\gtrsim 10^5 J^{-1}$. Such lifetime-based formulation of the many-particle localization problem is relevant to condensed-matter systems as well, because of finite decay and decoherence times of quasiparticles for nonzero temperatures.

We show in this paper that, within the lifetime-based formulation, many-particle localization of *all* states can be obtained in an *infinite* chain by constructing a sequence of on-site energies. For the proposed narrow-band sequence, all many-particle states remain confined for a time that largely exceeds J^{-1}. We find that all stationary many-particle states in moderately long chains are also strongly on-site localized. This makes the proposed system a potential candidate for a quantum memory device.

To localize one particle, the difference between excitation energies on neighboring sites should be much larger than J. The energies of remote sites should also differ. This is obvious for long-range coupling, but is also true for nearest-neighbor coupling, which is considered in this paper. For such coupling, hopping between remote states occurs via sequential virtual transitions to intermediate states. Because the effective hopping amplitude decays with the intersite distance, the further away the sites are, the smaller their energy difference can be. We use this idea to obtain strong on-site single-particle confinement for a bounded energy bandwidth.

For many-particle localization one has to suppress not only single-particle, but also combined resonances, where several interacting excitations make a transition simultaneously. There is no known way to eliminate all such resonances. However, the effective many-particle hopping integral quickly falls off with the increasing number of involved excitations and intermediate nonresonant sites, which gives the effective "order" of a transition. To obtain a desired lifetime of a localized state it is sufficient to eliminate resonances up to a certain order. We show how to do it to a fairly high order for an arbitrary number of excitations and for an infinite system [13].

2 The model and the on-site energy sequence

An array of qubits can be modelled by a one-dimensional chain of $S = 1/2$ spins in a magnetic field. The excitation energy of a qubit is the Zeeman energy of a spin. The qubit-qubit interaction is the exchange spin coupling. For many proposed realizations of QC's [1] - [6] it has a form $\frac{1}{2}\sum' J^{\mu\mu}_{nm} S^{\mu}_{n} S^{\mu}_{m}$, where n, m are spin sites, $\mu = x, y, z$ are spin projections, and $J^{xx}_{nm} = J^{yy}_{nm}$ for the effective magnetic field in the z-direction. The 1D spin system can be mapped, via Jordan-Wigner transformation, onto a system of fermions. For nearest neighbor coupling, the fermion Hamiltonian is

$$H = \sum\nolimits_n \varepsilon_n a_n^\dagger a_n + \frac{1}{2} J \sum\nolimits_n \left(a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n \right) + J\Delta \sum\nolimits_n a_n^\dagger a_{n+1}^\dagger a_{n+1} a_n. \tag{1}$$

Here, $a_n^\dagger, a_n$ are the fermion creation and annihilation operators. Presence of a fermion on site n corresponds to the nth spin (qubit) being in the excited state. The on-site energies ε_n in Eq. (1) are the Zeeman energies counted off from the characteristic central energy, $J \equiv J^{xx}_{n\,n+1}$ is the hopping integral, and $J\Delta \equiv J^{zz}_{n\,n+1}$ is the fermion interaction energy; we set $J, \Delta > 0$.

To give an example, for a QC based on electrons on helium, the typical distance between the energy levels (in frequency units) is $\sim 100 - 200$ GHz, the hopping integral J is ~ 0.1 GHz, and Δ varies from ~ 3 to ~ 28 depending on the operation conditions. The dynamical range over which the frequencies ε_n can be tuned without causing an increase of the relaxation rate is limited by ~ 10 GHz [15].

Localization of stationary states in a many-particle system can be conveniently characterized by the inverse participation ratio (IPR), which shows over how many sites the wave function spreads. For an N-particle eigenstate $|\psi_{N\lambda}\rangle$ (λ enumerates the eigenstates) it is given by

$$I_{N\lambda} = \left(\sum\nolimits_{n_1 < \ldots < n_N} \left| \langle \Phi_{n_1 \ldots n_N} | \psi_{N\lambda} \rangle \right|^4 \right)^{-1}, \tag{2}$$

where $|\Phi_{n_1 \ldots n_N}\rangle$ is an on-site N-particle wave function (quantum register) with particles on sites $n_1, \ldots, n_N$.

For fully localized stationary states $I_{N\lambda} = 1$. For delocalized states $I_{N\lambda} \gg 1$. Strong on-site localization that we are interested in corresponds to $I_{N\lambda}$ being close to 1 for all states.

Along with $I_{N\lambda}$ we will be interested in its average value $\langle I_N \rangle$, where averaging is performed over all states λ of the N-particle system.

Localization requires that the on-site energies ε_n be tuned away from each other. For nearest neighbor coupling a natural first step is to separate ε_n's into two subbands, for even

and odd n, with the inter-subband distance h that significantly exceeds J. Then we further split each subband into two subbands to detune next nearest neighbors. Here the splitting can be smaller, because next-nearest-neighbor hopping occurs via a nonresonant site, and the effective hopping integral is $\sim J^2/h$. The procedure of band splitting is continued, with higher-order splitting being smaller and smaller.

A simple sequence of ε_n that implements the above idea has the form

$$\varepsilon_n = \frac{1}{2}h\left[(-1)^n - \sum\nolimits_{k=2}^{n+1}(-1)^{\lfloor n/k \rfloor}\alpha^{k-1}\right], \quad n \geq 1 \tag{3}$$

($\lfloor\cdot\rfloor$ is the integer part). The energies (3) are illustrated in Fig. 1(a). Besides the scaling factor h, they are characterized by one dimensionless parameter $\alpha < 1$. One can see from Eq. (3) and Fig. 1(a) that sites with close energies are indeed spatially separated and that the energy spectrum has a subband structure. Analytical estimates of the energy difference can be obtained for small α. We have $|\varepsilon_{n+m} - \varepsilon_n| \sim h$ for odd m and $\sim \alpha h$ for odd $m/2$. In general, the larger is m the higher may be the order in α of the leading term in $|\varepsilon_{n+m} - \varepsilon_n|$. For $\alpha \gtrsim 0.4$ all subbands overlap and the subband structure disappears.

It is important for localization that the sequence (3) has no simple symmetry. It is neither self-similar nor quasi-periodic (quasi-periodicity is another example of "constructed" disorder [16]). For analytical estimates it is essential that the coefficients at any given power α^q are repeated with period $2(q+1)$ [13].

2.1 Single-particle localization

Spatial decay of the single-particle stationary states can be characterized by the amplitude of a particle transition from site n to site $n+m$. To the lowest order in J it has the form

$$K_n(m) = \prod\nolimits_{k=1}^{m} J/\left[2(\varepsilon_n - \varepsilon_{n+k})\right]. \tag{4}$$

It can be shown using some results from number theory that $K_n(m)$ decays with m nearly exponentially [13]. For small α and large $|m|$ we have

$$K_n(m) = \alpha^{-\nu|m|}\,(J/2h)^{|m|}. \tag{5}$$

The decrement ν depends on n, m. However, it is limited to a narrow region around $\nu = 1$ with $0.89 < \nu < 1.19$, cf. Fig. 1(b). For estimates one can use $\nu = 1$, i.e., set $K_n(m) = K^m, K = J/2\alpha h$.

Equation (5) describes the tail of the transition amplitude for $J/2h\alpha \ll 1$. On-site single-particle confinement occurs for $\alpha \gg \alpha_{\rm th}$, where the threshold value of α is $\alpha_{\rm th} \approx J/2h$. The condition $\alpha_{\rm th} \ll \alpha < 0.4$ can be satisfied for a moderately large ratio of the energy bandwidth h to the hopping integral J.

Strong on-site single-particle localization for $h/J = 20$, as evidenced by $I_{1\lambda}$ being very close to 1, is seen from Fig. 1(c). The data are obtained by diagonalizing the Hamiltonian (1) for open chains with different numbers of sites L.

In the limit $\alpha \to 0$ the stationary single-particle states are sinusoidal, which gives $\langle I_1\rangle \approx L/3$, cf. Fig. 1(c). As α increases, the bands are split into more and more subbands, and $\langle I_1\rangle$ decreases. It sharply drops to ≈ 1 in a narrow region, which can be conditionally associated with a smeared transition to on-site localization. The center of the transition region gives

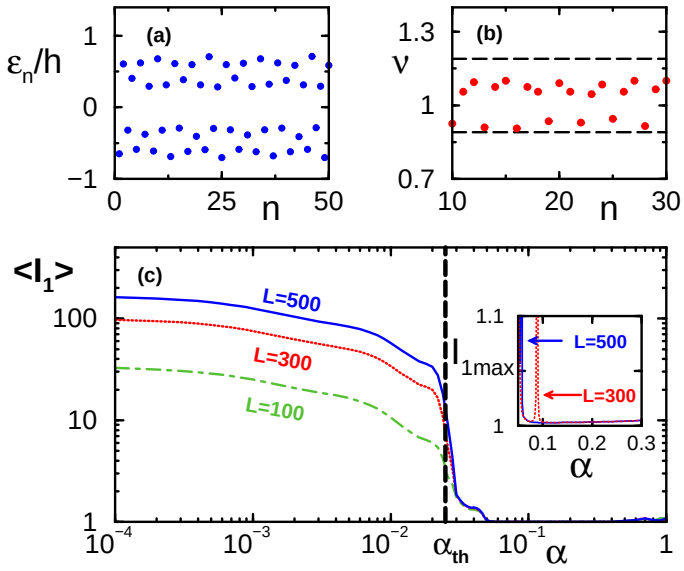

Fig. 1. Single-particle localization for the on-site energy sequence (3). (a) The energies ε_n/h for $\alpha = 0.3$. (b) The decrement ν of the α-dependence of the transition amplitude $K_n(m)$ (5) for $m = 200$ as function of n. The dashed lines show the analytical limits on ν. (c) The mean single-particle inverse participation ratio $\langle I_1 \rangle$ vs. α for $h/J = 20$ and for different chain lengths L. The vertical dashed line shows the analytical estimate for the threshold of strong on-site localization. The inset shows the maximal IPR, $I_{1\,\max} = \max_\lambda I_{1\lambda}$, demonstrating strong confinement.

$\alpha_{\rm th}$. It appears to be independent of the chain length L. The estimate $\alpha_{\rm th} = J/2h$ is in good agreement with the numerical data for different h/J.

When $\alpha \gg \alpha_{\rm th}$, all states are strongly confined. The wave function tails are small and limited mostly to nearest neighbors. At its minimum over α for given h/J, for all states $I_{1\lambda} - 1 \approx J^2/h^2$, see inset in Fig. 1(c). We note that the peak of $I_{1\,\max}$ for $L = 300$ near $\alpha = 0.1$ is due to accidental degeneracy at the boundary. It occurs because the state on the boundary has less neighbors than inside the chain, and therefore the hopping-induced shift of the boundary energy $\sim J^2/h$ differs from that inside the chain; one can show that, for $\alpha \approx 0.1$, the renormalized energies ε_{296} and ε_{300} are resonating. For $\alpha \gtrsim 0.4$, when the bands of ε_n start overlapping, the IPR increases with α.

3 Many-particle localization

The difference in the localization problems for many-particle and single-particle systems stems from the interaction term $\propto J\Delta$ in the Hamiltonian (1). For nonzero Δ (i) the energy levels are shifted depending on the occupation of neighboring sites, potentially leading to many-particle resonances, and (ii) there occur interaction-induced many-particle transitions.

To analyze many-particle effects, it is convenient to change from $a_n^\dagger, a_n$ to new creation and annihilation operators $b_n^\dagger, b_n$ that diagonalize the single-particle part of the Hamiltonian (1), $a_n = \sum_k U_{nk} b_k$. The interaction part of the Hamiltonian becomes

$$H_i = J\Delta \sum V_{k_1 k_2 k_3 k_4} b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3} b_{k_4}, \tag{6}$$

where the sum runs over $k_{1,2,3,4}$, and

$$V_{k_1k_2k_3k_4} = \sum\nolimits_p U^*_{pk_1} U^*_{p+1\,k_2} U_{p+1\,k_3} U_{pk_4}. \tag{7}$$

The Hamiltonian (6) describes the interaction of the exact single-particle excitations.

If all single-particle stationary states are strongly localized, the off-diagonal matrix elements U_{nk} are small. They are determined by the decay of the wave functions and fall off exponentially, $U_{nk} \sim K^{|k-n|}$ for $|k-n| \gg 1$. At the same time, the diagonal matrix element is $U_{nn} \approx 1$. Therefore the major terms in the matrix $V_{k_1k_2k_3k_4}$ are those with $\varkappa = 0$, where

$$\varkappa = \min_p(|k_1 - p| + |k_2 - p - 1| + |k_3 - p - 1| + |k_4 - p|). \tag{8}$$

The terms with $\varkappa = 0$ lead to an energy shift $\propto J\Delta$ from each pair of occupied neighboring sites in a given many-particle state.

The meaning of the parameter $\varkappa$ (8) can be understood by noticing that the terms $\propto V_{k_1k_2k_3k_4}$ in Eq. (6) describe two-particle inter-site transitions $(k_4, k_3) \leftrightarrow (k_1, k_2)$ of the renormalized fermions. For a given transition, $\varkappa$ is simply the number of virtual steps that have to be made by the original fermions. The steps are counted off from the configuration where two such fermions occupy neighboring sites, and each step is a transition by one of the fermions to a nearest site. In other words, the original fermions go first from sites (k_4, k_3) to sites $(p, p+1)$ and then to (k_1, k_2) (we assume for concreteness that $k_3 > k_4$ and $k_2 > k_1$); the value of p is chosen so as to minimize the number of steps.

To make the meaning of $\varkappa$ even more intuitive we give examples of some $\varkappa = 4$ transitions. For the initial and final states $(n, n+1)$ and $(n-2, n+3)$ one of the sequences of steps of the original fermions is $(n, n+1) \to (n, n+2) \to (n-1, n+2) \to (n-1, n+3) \to (n-2, n+3)$, whereas for the initial and final states $(n, n+2)$ and $(n-1, n+3)$ one of the sequences is $(n, n+2) \to (n, n+1) \to (n-1, n+1) \to (n-1, n+2) \to (n-1, n+3)$ [the energy denominators must be obtained directly from Eqs. (3), (6), (7)].

It follows from the above argument that, for $\alpha \gg \alpha_{\rm th}$,

$$V_{k_1k_2k_3k_4} \sim K^{\varkappa} \qquad \text{for} \qquad \varkappa \gg 1. \tag{9}$$

Transitions of renormalized fermions are not limited to nearest neighbors. However, from Eq. (9), the amplitudes of transitions over many sites are small and rapidly decrease with the number of involved virtual steps.

In higher orders of the perturbation theory, the interaction (6) leads also to many-particle transitions. The overall transition amplitude is determined by the total number of involved virtual single-particle steps.

In order to localize many-particle excitations, one has to suppress combinational many-particle resonances keeping in mind that, for localization, the effective hopping integral must be smaller than the energy detuning of the initial and final on-site states. Because of the large number of possible resonances, we do not have an analytical proof of many-particle localization in an infinite system for our energy sequence (3). However, our numerical results demonstrate strong on-site localization of all stationary states in a chain of a limited size.

Numerical results on the many-particle IPR are shown in Fig. 2. We have studied chains of length $L = 10, 12$, and 14 with $L/2$ excitations, which have the largest number of states

for given L ($\propto 2^L$ for large L). The results for all L were similar, and we present the data for $L = 12$, in which case the total number of states is 924.

For small α, the IPR is independent of α and is large because of the large number of resonating on-site states $|\Phi_{n_1 \ldots n_6}\rangle$. It is reduced by the interaction $\propto J\Delta$ that splits the energy spectrum into subbands depending on the number of occupied neighboring sites. On the whole, the IPR decreases with increasing α as long as $\alpha \lesssim 0.4$. In the region $0.2 \lesssim \alpha \lesssim 0.4$ and for $h/J = 20$ we have $\langle I_6 \rangle \approx 1.01$ except for narrow peaks. This indicates that away from the peaks the stationary states are strongly localized. Strong on-site localization of all states is indicated by the data on $I_{6\,\max} = \max_\lambda I_{6\lambda}$.

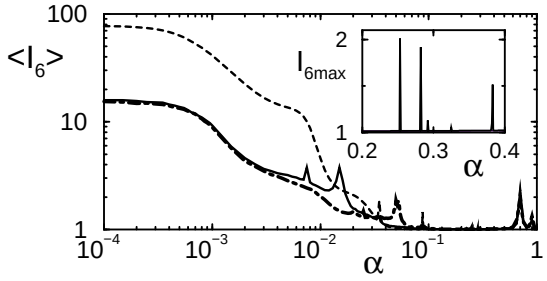

Fig. 2. Many-particle localization for a chain of length $L = 12$ with 6 excitations. The data refer to the first 12 sites of the chain (3), the reduced bandwidth is $h/J = 20$. The dashed, solid, and dot-dashed curves give the mean IPR for the coupling parameter $\Delta = 0, 0.3$, and 1, respectively. The inset shows the maximal I_6 for $\Delta = 1$. Sharp isolated peaks for $\Delta \neq 0$ result from the hybridization of many-particle on-site states that are in resonance for the corresponding α. The peaks for $\Delta = 0$ are due to the boundaries.

A distinctive feature of the many-particle IPR as function of α are multiple resonant peaks, a part of which is resolved in Fig. 2. They occur when two on-site states resonate. Two-site resonances lead to $I_{6\,\max} \lesssim 2$. The strongest peaks of $I_{6\,\max}$ happen when the two-particle energy difference

$$\delta\varepsilon = |\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{k_3} - \varepsilon_{k_4}| \tag{10}$$

is close to $MJ\Delta$ with $M = 0, 1, 2$.

As we increase α starting from $\alpha = 0$, pronounced peaks of $\langle I_6 \rangle$ appear first for $\delta\varepsilon \approx s\alpha h \approx J\Delta$ with $s = 1, 2$. They are due to hybridization of pairs on sites $(n, n+1)$ and $(n, n+3)$ for $s = 1$, and $(n, n+1)$ and $(n-1, n+2)$ for $s = 2$, for example ($\varkappa = 2$-transitions).

For larger α, resonances occur when $s\alpha^n h \approx MJ\Delta$ with $n \geq 2$. In most cases (see, however, below for a notable exception) such resonances require more intermediate steps, with $\varkappa \geq 4$. The widths of the IPR peaks are small and are in good agreement with simple estimates based on Eq. (6) [12, 13]. In between the peaks $I_{6\,\max} = 1.02$ for $0.2 \lesssim \alpha \lesssim 0.4$ and $h/J = 20$.

For the sequence (3), a special role is played by two-particle resonances where $\delta\varepsilon \ll J$ for all $\alpha < 0.4$. They emerge already for $\varkappa = 2$-transitions $(n, n+1) \leftrightarrow (n-1, n+2)$. Here, if n and $n+2$ are prime numbers, $\delta\varepsilon \sim \alpha^{n-1} h$ is extremely small for large n and small α. Strong resonance occurs for all $n = 6k - 1$, in which case $\delta\varepsilon/h \propto \alpha^\xi$ with $\xi \geq 4$. For example, for a transition $(23, 24) \leftrightarrow (22, 25)$ we have $\delta\varepsilon/h \propto \alpha^4$, whereas for a transition $(29, 30) \leftrightarrow (28, 31)$

we have $\delta\varepsilon/h \propto \alpha^{28}$. As a result of these resonances the condition $\delta\varepsilon \gg J\Delta K^{\varkappa}$ may not be satisfied in an infinite chain.

For different sections of the chain (3) we found that the resonances $(n, n+1) \leftrightarrow (n-1, n+2)$ for $n = 6k-1$ increase $\langle I_6 \rangle$ up to 1.15 between the peaks, for $h/J = 20, 0.2 < \alpha < 0.4$, and $\Delta = 1$. These resonances can be eliminated by modifying the sequence (see the next section). This modification brings $\langle I_6 \rangle$ and $I_{6\,\max}$ back to ≈ 1.01 and ≈ 1.02, respectively [12, 13]. We emphasize that such small values of $\langle I_6 \rangle - 1$ and $I_{6\,\max} - 1$ were obtained for all 12-site long sections of the chain that we have tested.

4 Lifetime of strongly localized states

The problem of strong localization can be viewed also from a different perspective. In the context of quantum computing, it suggests a more appropriate formulation then the one based on the analysis of stationary states. It is also relevant for condensed-matter systems at nonzero temperatures.

First we note that excitations in quantum computers and in condensed-matter systems have a finite coherence time $t_{\rm coh}$. For QC's, this time has to be compared with the duration of a single- or two-qubit operation and measurement. The duration of a two-qubit swap operation is of order of the time it takes to resonantly transfer an excitation between the qubits, which is $\sim J^{-1}$. A single-qubit operation is often faster; however, the measurement can sometimes be slower or even much slower. In most proposed realizations of a QC the coherence time exceeds the gate operation time by a factor $\lesssim 10^5$.

We define the localization lifetime $t_{\rm loc}$ as the time it takes for excitations to leave occupied sites. Localization of excitations is only relevant on times $\sim t_{\rm coh}$. Then to have meaningful on-site localization it suffices that $t_{\rm loc} \gtrsim t_{\rm coh}$. It follows from the estimate for $t_{\rm coh}$ that the latter condition is met if

$$t_{\rm loc} \gtrsim 10^5 J^{-1}. \tag{11}$$

The condition (11) must be satisfied for all on-site many-particle states. It is this condition that the energy sequence ε_n must meet in an infinite system.

The time $t_{\rm loc}$ is determined by hopping between resonant on-site states. It occurs through virtual transitions via nonresonant sites. For a two-particle resonant transition, the minimal number of the needed virtual steps is given by the parameter $\varkappa$ (8). Then from Eqs. (6), (9) the hopping integral for a resonant transition $(k_4, k_3) \leftrightarrow (k_1, k_2)$ is $J\Delta V_{k_1 k_2 k_3 k_4} \sim J\Delta K^{\varkappa}$ for $\varkappa \gg 1$. Here, K is defined by Eq. (5), $K \approx J/2\alpha h$, and $K \ll 1$ in the region $\alpha/\alpha_{\rm th} \gg 1$.

In the case of the energy sequence (3) and for $\Delta \lesssim 1$, up to a fairly high number of virtual steps (≤ 5), of interest for the estimate of $t_{\rm loc}$ are resonances between two-particle on-site states. This applies to systems with an *arbitrary* number of particles; only those transitions matter in which up to two particles change sites. Indeed, transitions where three particles change sites emerge in the second order in the two-particle Hamiltonian (6), (7), and their amplitude contains an extra nonresonant denominator.

For resonant two-particle transitions

$$t_{\rm loc} \sim \left(J\Delta K^{\varkappa_{\min}}\right)^{-1},$$

where $\varkappa_{\min}$ is the minimal value of $\varkappa$ for all pairs of resonating initial and final on-site states. To have $t_{\rm loc}J$ that exceeds a given value, we must have an appropriate $\varkappa_{\min}$. This means that we should eliminate resonances between all states connected by $\varkappa < \varkappa_{\min}$ virtual transitions.

A two-particle transition with odd $\varkappa = 1, 3, 5, \ldots$ involves a change of the total number of occupied sites with n of given parity. Therefore, for the sequence (3) with $\alpha \ll 1$, the energy change in such a transition is $\delta\varepsilon \sim h$. If $J\Delta \ll h$, then $\delta\varepsilon$ significantly exceeds the change of the interaction energy $J\Delta$, $2J\Delta$. As a result, resonant two-particle transitions may occur only for even $\varkappa$.

We will modify the sequence (3) to eliminate resonances with $\varkappa = 2$ and $\varkappa = 4$. This will allow us to have the localization time $t_{\rm loc} \sim J^{-1}K^{-4}$ and $> J^{-1}K^{-6}$, respectively, for $\Delta \lesssim 1$.

4.1 Eliminating second order many-particle resonances

The potentially resonant transitions with $\varkappa = 2$ are

$$(n, n+1) \leftrightarrow (n, n+1 \pm 2),\ (n \pm 2, n+1)$$
$$(n, n+1) \leftrightarrow (n-1, n+2). \tag{12}$$

In the transitions listed in the first line of this equation, one of the particles in the pair moves by two sites in one or the other direction, whereas for the transition shown on the second line both particles move by one site.

We note that Eq. (6) describes "direct" transitions over several sites, with amplitude $\propto K^{\varkappa}$, which is a result of the diagonalization of the single-particle Hamiltonian; for example, a transition $(n, n+1) \leftrightarrow (n, n-1)$ does not involve double-occupancy on site n. Alternatively, one can think of this transition as a sequence of steps $(n, n+1) \leftrightarrow (n-1, n+1) \leftrightarrow (n, n-1)$.

The number of occupied nearest sites in the transitions (12) can change by one or remain unchanged. Therefore the maximal change of the interaction energy is $J\Delta$. Second-order resonances will be eliminated if the detuning of the on-site energy differences $\delta\varepsilon$ for the transitions (12) is

$$\delta\varepsilon > J\Delta.$$

This means that we need a zero-energy gap of an appropriate width in $\delta\varepsilon$.

We note that this is a sufficient, not the necessary condition. In principle, it would suffice to have narrow gaps at $\delta\varepsilon = 0$ and $J\Delta$. These gaps should just be broader than the tunneling matrix element and than the energy shifts due to occupation of next nearest neighbors. For a specific finite-length section of a chain this may be more practical. However, here we are interested in an infinite chain, and we want to demonstrate that even for such a chain all resonances with $\varkappa < 4$ can be eliminated.

To create the zero-energy gap, sequence (3) has to be modified. The modification has to eliminate, in the first place, the "anomalous" broad-band resonances for transitions $(n, n+1) \leftrightarrow (n-1, n+2)$ with $n = 6k-1$ discussed before. A simple and sufficient modification is a constant shift of ε_n for each 6th site,

$$\varepsilon_n^{\rm md} = \varepsilon_n + (h/2)\alpha' \qquad \text{for} \qquad n = 6k, \tag{13}$$

while $\varepsilon_n^{\rm md} = \varepsilon_n$ for $n \neq 6k$.

For the modified sequence (13), the gap in the on-site energies for the 2nd-order transitions (12) is $\delta\varepsilon \sim \alpha^2 h, \alpha' h/2$ to leading order in α. A more accurate estimate is $\min \delta\varepsilon \approx (\alpha^2 - \alpha^3)h, \alpha' h/2$. We assume that $\alpha^2 \lesssim \alpha' \ll \alpha$, in which case no new resonances are created for the transitions (12) as a result of the modification (13).

It follows from the above estimate that, for an infinite chain and an arbitrary number of particles, all resonant transitions with $\varkappa < 4$ will be eliminated provided $J\Delta/h < \alpha^2 - \alpha^3, \alpha'/2, .$ Then the localization time $t_{\rm loc} \sim 10^5 J^{-1}$ already for $h/J = 30$, $\alpha = 0.3$, $\alpha' \approx 0.1 - 0.2$, and $\Delta \lesssim 1$.

4.2 Eliminating fourth order resonances

The localization time is further dramatically increased if $\varkappa = 4$ resonances are eliminated. The potentially resonant 4th order transitions are

$$\begin{aligned}
&(n, n+1) \leftrightarrow (n-2, n+3),\\
&(n, n+1) \leftrightarrow (n+2, n+3),\\
&(n, n+3) \leftrightarrow (n-1, n+2),\\
&(n, n+3) \leftrightarrow (n-2, n+1),
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&(n, n+1) \leftrightarrow (n-1, n+4),\\
&(n, n+1) \leftrightarrow (n-3, n+2),\\
&(n, n+1) \leftrightarrow (n, n+1 \pm 4),\ (n \pm 4, n+1).
\end{aligned} \tag{15}$$

In the last line of Eq. (15) we list transitions where one of the particles in the pair moves by 4 sites, whereas in all other transitions both particles move away from their sites.

For the modified sequence (13), the minimal energy change in the transitions (14), (15) is $\min \delta\varepsilon \sim \alpha^3 h$, to leading order in α, provided $\alpha' \gg \alpha^3$. The value of α' has to be in such a range that the modification (13) does not lead to extra resonances between the on-site energies for the states (14) and (15). The "dangerous" combinations in $\delta\varepsilon/h$ are $|\alpha - \alpha'/2|, |2\alpha - \alpha'/2|, |\alpha^2 - \alpha'/2|, |2\alpha^2 - \alpha'/2|$, to leading order in α [13]. We will choose α, α' so that all of them exceed $\min \delta\varepsilon/h \approx \alpha^3$.

Fig. 3 shows how the modification (13) leads to a zero-energy gap in $\delta\varepsilon$. We plot $\delta\varepsilon_n^{\rm md}$ for all transitions (12), (14), (15), with n being the smallest number of the site involved in a transition, $n > 2$. Therefore we show all potentially resonant transitions with $\varkappa \leq 5$.

The left panel in Fig. 3 shows that, for the initial sequence (3), there is practically no gap in the values of $\delta\varepsilon$ at low energies. The right panel demonstrates that the correction (13) leads to a zero-energy gap. The gap depends on the values of α and α'. For the specific parameter values in Fig. 3 we have $\delta\varepsilon/h \geq 0.01$. It can be shown that this result applies for an infinite chain [13].

It follows from the discussion above that, for $2J\Delta \lesssim 0.01h$, all particles will remain localized on their sites for the time $t_{\rm loc} \sim (J\Delta)^{-1} K^{-6}/\alpha$ [we have taken into account here that the hopping integral for transitions with $\varkappa = 6$ is limited by $\sim J\Delta(J/2h)^6\alpha^{-5}$ rather than $J\Delta(J/2h)^6\alpha^{-6}$, as would be expected from the asymptotic expression (9)]. For $h/J = 50$ and $\alpha = 0.25$ this gives an extremely long localization time, $t_{\rm loc} J \gtrsim 10^{10}$. This estimate holds

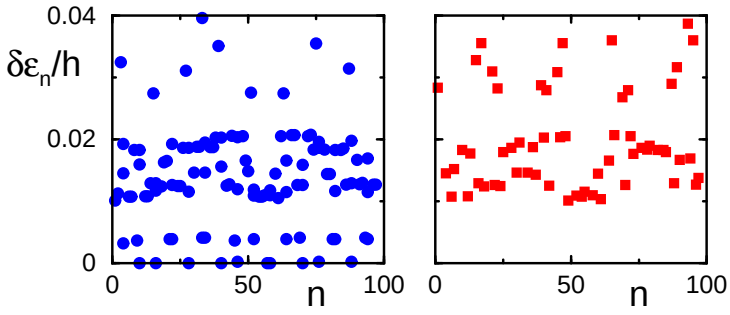

Fig. 3. The low-energy part of the two-particle energy differences $\delta\varepsilon_n/h$ (10) for all transitions with $\varkappa \leq 5$; n is the smallest number of the site involved in a transition, $n > 2$. The data refer to $\alpha = 0.25$. The left panel corresponds to the sequence (3). The right panel refers to the modified sequence (13) with $\alpha' = 0.22$; it shows the zero-energy gap.

provided the coupling is weak, $\Delta \lesssim 0.25$ for the used parameter values. It is important that, as a proof of principle, the results on a gap apply for arbitrary coupling. However, as we show below, for stronger coupling an alternative approach becomes more practical.

4.3 Strong coupling: an alternative approach

For strong coupling, where $\Delta \gg 1$, the condition $2J\Delta < 0.01h$ may become too restrictive. Such situation is of interest for several models of QC's, in particular for a QC based on electrons on helium, as it follows from the estimate of Δ given above. The results described in the previous sections apply for large Δ, but the ratio h/J required for localization becomes very large. Large bandwidth of ε_n may limit the speed of quantum operations where qubit energies are tuned in resonance with an external microwave field or with each other [3], because in an operation ε_n has to be varied over a broad range $\propto h$. It may also be simply incompatible with the bounds on the range of ε_n imposed by physical constraints in a system.

An alternative approach for obtaining long localization time is to have an energy sequence with sufficiently broad gaps in the spectra of the combination energies, and to use separate gaps for the transitions with the change of interaction energy $\delta E = 0$ and $\delta E = J\Delta, 2J\Delta$.

The occurrence of gaps in the combination energies $\delta\varepsilon$ is seen from Fig. 4. From Eqs. (3), (10), $\delta\varepsilon$ form bands centered at $0, h$, and $2h$. To lowest order in α, the bandwidths are $2\alpha h$, $4\alpha h$, and $2\alpha h$. The interband gap between the lowest and first band is broad, $\sim (1 - 4\alpha)h$.

For strong coupling, one can adjust the parameters so that the coupling energies $J\Delta, 2J\Delta$ lie inside the gap between the bands of $\delta\varepsilon$ centered at $\delta\varepsilon = 0, h$. The transitions where the number of nearest neighbors is not changed should lie inside a zero-energy gap (which is not seen in Fig. 4). The corresponding conditions can be reduced in the limit of small α to

$$(2\Delta)^{1/3} < 2\alpha h/J < \Delta$$

(here we have taken into account that the zero-energy gap $\sim \alpha^3 h$ should exceed the transition matrix element $\propto J\Delta(J/2h)^2$). The above condition allows using a much smaller bandwidth h for large Δ than the condition $2J\Delta < 0.01h$. However, for very large Δ one may have to take into account transitions in which more than two particles are involved; this may modify the constraint on h/J.

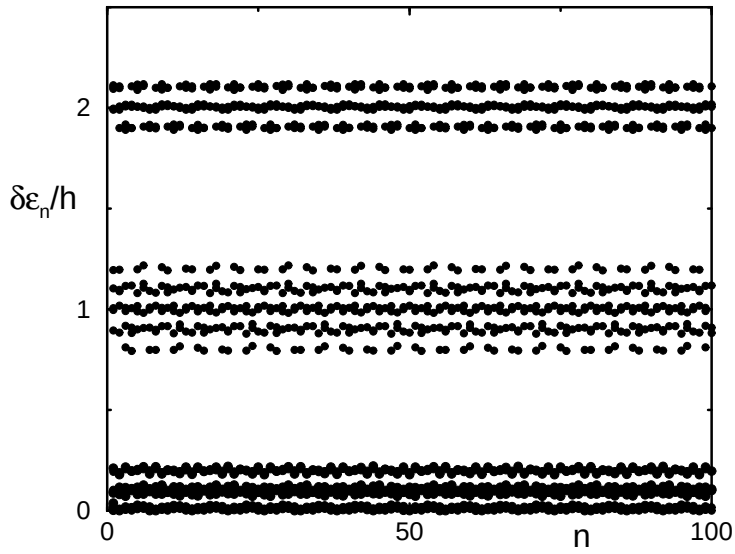

Fig. 4. The two-particle energy differences $\delta\varepsilon_n/h$ for two-particle transitions with $\varkappa \leq 4$; n is the smallest number of the site involved in a transition. The data refer to the energy sequence (3) with $\alpha = 0.1$. The energy differences form bands centered at $\delta\varepsilon/h = 0, 1$, and 2. The bandwidths are $\approx 2\alpha h$, $4\alpha h$, and $2\alpha h$, respectively.

5 Robustness with respect to errors in on-site energies

In a real system, it will be impossible to implement sequence of on-site energies (3) precisely. This is because these energies contain high powers of the small parameter α, while the precision to which they can be set and/or measured is limited. Therefore it is necessary to study localization in the presence of errors in ε_n and to find how large these errors can be before they cause delocalization.

We will address this problem by looking at the zero-energy gap in the energy differences $\delta\varepsilon$ in the presence of errors in $\varepsilon_n^{\rm md}$ (13). This gap is most sensitive to errors. For weak to moderate coupling, $\Delta \lesssim 1$, as long as the gap remains larger than $2J\Delta$ for all resonant transitions with $\varkappa \leq 5$, the localization lifetime $t_{\rm loc}$ will remain large.

The effect of errors on the zero-energy gap can be modelled by adding a random term to on-site energies, i.e., replacing $\varepsilon_n^{\rm md}$ with

$$\varepsilon_n^{\rm err} = \varepsilon_n^{\rm md} + \frac{1}{2} D h r_n. \tag{16}$$

Here, r_n are random numbers uniformly distributed in the interval $(-1, 1)$, and D characterizes the error amplitude. It should be compared with α^s with different exponents $s \geq 1$. When $D \sim \alpha^s$ it means that the energies ε_n are well controlled up to terms $\sim \alpha^{s-1}$, to leading order in α.

From the above arguments it follows that, for $\alpha \gg \alpha' \gtrsim \alpha^2$ the gap should remain unchanged if $D \ll \alpha^4$. This is because, for the modified energies $\varepsilon_n^{\rm md}$, the terms $\sim \alpha^4$ drop out from the energy differences that we discuss. For $D \sim \alpha^4$ the gap should be somewhat reduced. For $D \sim \alpha^3$ it should become significantly smaller than for $D = 0$, and it should ultimately disappear with increasing D.

Numerical results on the gap $\delta\varepsilon$ as a function of $\log D$ are shown in Fig. 5 [13]. The gap is calculated for two-particle transitions with $\varkappa \leq 5$, as in Fig. 3. In the lower panel the gap is scaled by its value in the absence of errors,

$$R = \min_n \delta\varepsilon_n^{\rm err} / \min_n \delta\varepsilon_n^{\rm md}. \tag{17}$$

The data refer to the same α, α' as in Fig. 3. They are in full agreement with the above estimate.

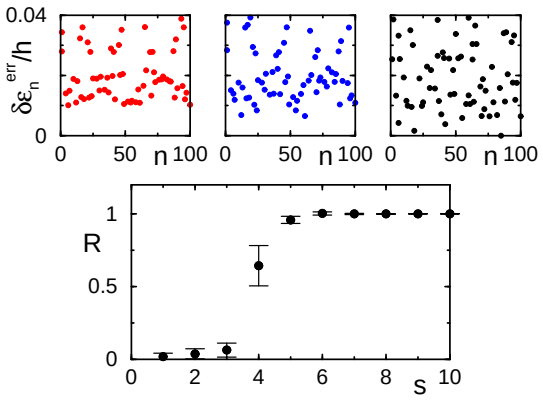

Fig. 5. Upper panels: all energy differences $\delta\varepsilon_n^{\rm err}/h = |\varepsilon_n^{\rm err} + \varepsilon_{n_1}^{\rm err} - \varepsilon_{n_2}^{\rm err} - \varepsilon_{n_3}^{\rm err}|/h$ for the transitions (12), (14), (15) that correspond to the number of intermediate steps $\varkappa \leq 5$. The data refer to $\alpha = 0.25$, $\alpha' = 0.22$, and to a specific realization of the random numbers r_n in Eq. (16). The boxes from left to right correspond to the values of the noise intensity $D = \alpha^s$ in Eq. (16) with $s = 5, 4$, and 3. Lower panel: the scaled minimal gap R (17) as a function of the exponent $s = \ln D/\ln\alpha$; the value of R is averaged over 10 realizations of noise.

The results of Fig. 5 demonstrate that the localization persists even for relatively large errors in the on-site energies. At least for the chosen α and α', errors in ε_n up to $\sim 0.4\%$ (when $D = \alpha^4$) lead to a change in the width of the energy gap by $\sim 50\%$.

The observed dependence on the noise strength suggests that, in the presence of noise, sequence (3), (13) can be cut so that the terms $\propto \alpha^s$ with $s > s_{\rm cutoff}$ are disregarded. The value of $s_{\rm cutoff}$ depends on the noise, $s_{\rm cutoff} = \ln D/\ln\alpha$. As a result of the cutoff, the energies ε_n become polynomials in α of power $\leq s_{\rm cutoff}$. From Eq. (3), these polynomials are periodic in n, with the period determined by twice the least common multiple of $(2, 3, \ldots, s_{\rm cutoff}+1)$. For example, for $s_{\rm cutoff} = 5$ the period in n is 120. For such a long period and short-range hopping, excitations will stay on their sites for a long time compared to J^{-1}.

6 Conclusions

In this paper we have reviewed and extended the results [12,13] on on-site localization of excitations in a quantum computer with perpetually coupled qubits. We proposed a sequence of on-site energies that is extremely effective, in terms of localization. Two aspects of the localization problem have been addressed. One is localization of stationary states. For one-particle (one-excitation) states, it has been studied analytically. We found that the wave functions decay quasi-exponentially and obtained the bounds on the decay length. The numerical results on strong localization are in agreement with the theory.

For many-particle (many-excitation) stationary states, the localization has been analyzed numerically for a chain of a finite length. It was found that, already for a relatively small bandwidth of on-site energies, the inverse participation ratio becomes very close to its value for the case of fully localized states.

A different approach is based on studying the lifetime of on-site states. For a quantum computer, it is sufficient to have a localization lifetime $t_{\rm loc}$ that exceeds the coherence time of the excitations. We have shown that such $t_{\rm loc}$ can be achieved in a chain of an arbitrary length and with an arbitrary number of excitations. For the explicit construction of on-site energies (3), (13), resonant transitions that lead to delocalization require at least 4 or even 6 virtual transitions. Even in the 4-step case this gives the ratio of the delocalization rate to the inter-site hopping integral $\sim 10^{-5}$ for $K \sim 0.06$ and for the coupling parameter $\Delta \lesssim 1$.

An advantageous feature of the suggested on-site energy sequence (3) and its modification (13) is that one radiation frequency can be used to resonantly excite different qubits. This can be achieved by selectively tuning them to this frequency without bringing neighboring qubits in resonance with each other, or by sweeping the frequency of the targeted qubit through the radiation frequency and having a Landau-Zener-type interstate transition. A two-qubit gate can be conveniently done by selectively tuning neighboring qubits in resonance with each other and having a Landau-Zener type excitation swap [3].

We note that, in fact, implementing localization does not require operations on qubits, and therefore does not require a fully operational quantum computer. An important example of such system is quantum memory. Of course, many-excitation localization is a necessary ingredient of a quantum memory device.

Localization is also a prerequisite for a projective measurement of the states of individual qubits. To enable such measurement, $t_{\rm loc}$ should exceed the measurement time. In our approach, localization does not require refocusing [1], which is not always easy to implement and which is sometimes incompatible with slow measurement.

A potential advantage of our approach compared to an elegant idea [8] is that the interaction does not have to be ever turned off, and no multi-qubit encoding is necessary for operating a QC. Another distinction from the approach [8] is that the presented scheme can be extended to systems with long-range coupling. For several proposed QC's the interqubit coupling is dipolar for a few near neighbors and becomes quadrupolar or falls down even faster for remote neighbors [2, 3]. Long-range interaction makes transitions over several sites more probable. We leave detailed analysis for a separate paper.

In this paper we have not addressed the question of optimization of the energy sequence, so that maximal localization lifetime could be obtained for a minimal bandwidth of on-site energies. For a finite-length chain the optimization problem can be approached using Eq. (3) as an initial approximation and adjusting energies of several specific sites.

In conclusion, we have provided proof of principle of long-lived strong on-site localization of all states of a quantum computer, independent of its size and the number of excitations. We have also demonstrated numerically on-site localization of all stationary states in comparatively long chains of perpetually coupled qubits. The localization does not require using refocusing techniques. The proposed sequence of on-site energies (3) and its modification (13) have low symmetry, which allows eliminating resonances between the states to a high order in the hopping integral. When second-order resonances are eliminated, the lifetime exceeds

the reciprocal hopping integral by 5 orders of magnitude provided the bandwidth of on-site energies is larger than the inter-site hopping integral by a factor ~ 40. We show that it can be further significantly increased by eliminating fourth-order resonances and propose an approach that gives long lifetime for strong coupling without large increase of the bandwidth. The proposed energy sequence is stable with respect to errors.

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