PHYSICAL REVIEW B **71**, 140508(R) (2005)

Multiphoton antiresonance

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(Received 21 December 2004; published 26 April 2005)

We show that nonlinear response of a quantum oscillator displays antiresonant dips as the field frequency passes adiabatically through multiphoton resonance. This coherent quantum effect has no analog in two-level systems. Its emergence is a consequence of special symmetry of a weakly nonlinear oscillator. We discuss the possibility to observe the antiresonance and the associated multiphoton Rabi oscillations in Josephson junctions.

DOI: 10.1103/PhysRevB.71.140508 PACS number(s): 74.50.+r, 05.45.−a, 05.60.Gg, 33.80.Wz

Many resonant nonlinear phenomena are described by the model of a nonlinear oscillator in a resonant field. Examples range from laser-induced dissociation of molecules[1] to recently studied hysteresis in resonantly driven Josephson junctions[2,3] and nanomechanical resonators.[4] The generality of the oscillator as a model system and the current interest in quantum computing and coherent phenomena lead to a question: Does a resonantly driven nonlinear oscillator display coherent quantum effects that would qualitatively differ from those in two-level systems? In the present paper we provide an example of such an effect.

A specific feature of a nonlinear oscillator is that its energy levels E_n are nearly equidistant. Therefore, a periodic force of frequency ω_F can be nearly resonant for many transitions at a time, i.e., $\hbar\omega_F$ can be close to the interlevel distance $E_{n+1}-E_n$ for many n. This makes an oscillator convenient for studying multiphoton Rabi oscillations. They arise when the spacing between remote energy levels n and m coincides with the energy of $n-m$ photons, $E_n-E_m=(n-m)\hbar\omega_F$.[1] The multiphoton transition amplitude is resonantly enhanced, because the $m\to n$ transition occurs via a sequence of virtual field-induced transitions $k\to k+1$ (with $m\leqslant k\leqslant n-1$), all of which are almost resonant.

In this paper we show that multiphoton transitions in the oscillator are accompanied by an unexpected effect, *antiresonance* of the response. When the frequency of the driving field adiabatically passes through a resonant value, the vibration amplitude displays a sharp minimum or maximum, depending on the initial conditions. We argue that the antiresonance and the multiphoton Rabi oscillations can be observed in Josephson junctions.

The antiresonance is a consequence of a special degeneracy of a weakly nonlinear oscillator. In the neglect of multiphoton mixing, the amplitudes of forced vibrations in the resonating states coincide with each other, as seen from the intersection of the dashed lines in Fig. 1(b). Mixing of the resonating states lifts the degeneracy, leading to the amplitude "repulsion," which increases with increasing field. In turn, it leads to pronounced dips (peaks) in the vibration amplitude as ω_F adiabatically passes through resonances.

In the semiclassical picture, resonant multiphoton transitions correspond to tunneling between Floquet states of the oscillator with equal quasienergies [the quasienergy ε gives the change of the wave function $\psi(t)$ when time is incremented by the modulation period τ_F, $\psi(t+\tau_F)=\exp(-i\varepsilon\tau_F/\hbar)\psi(t)$]. Tunneling of a driven oscillator is a carefully studied[5] example of dynamical tunneling.[6] As we show, the WKB analysis gives an important insight into the origin of the antiresonance, which goes beyond the perturbation theory in the driving field.

The Hamiltonian of a driven nonlinear oscillator with mass $M=1$ has the form

$$H(t)=\frac{1}{2}p^2+\frac{1}{2}\omega_0^2q^2+\frac{1}{4}\gamma q^4-qA\cos(\omega_F t). \qquad (1)$$

We assume that the driving field is nearly resonant, i.e., the frequency detuning $\delta\omega$ is small,

$$|\delta\omega|\ll\omega_F, \quad \delta\omega=\omega_F-\omega_0. \qquad (2)$$

We consider not too large amplitudes of the driving field A, so that the oscillator anharmonicity is small, and in particular $|\gamma|q^2\ll\omega_0^2$ for typical q. We also assume that γ and $\delta\omega$ have the same sign. If there is a cubic term $\alpha q^3/3$ in the potential energy, its major effect of interest for this paper is the renormalization $\gamma\to\gamma-(10/9)(\alpha/\omega_0)^2$ (Ref. 7).

To study quantum dynamics, we will write the Hamiltonian in terms of the raising and lowering operators of the oscillator $a^\dagger, a$, and switch to the rotating frame with a canonical transformation $U(t)=\exp(-i\omega_F a^\dagger a t)$. The transformed Hamiltonian $H_0=U^\dagger(t)H(t)U(t)-i\hbar U^\dagger(t)\dot{U}(t)$ is time independent in the rotating wave approximation (RWA),

$$H_0=-\delta\omega\hat{n}+\frac{1}{2}V\hat{n}(\hat{n}+1)-f(a+a^\dagger), \quad \hat{n}=a^\dagger a,$$

$$V=3\hbar\gamma/4\omega_0^2, \quad f=(8\hbar\omega_0)^{-1/2}A. \qquad (3)$$

In the expression for H_0 and in what follows $\hbar=1$.

The eigenvalues of the Hamiltonian (3) ε_n give the quasienergies of the driven oscillator. In the limit of weak driving their spectrum is particularly simple,

$$\varepsilon_n=-n\delta\omega+Vn(n+1)/2, \quad f\to 0. \qquad (4)$$

We will study multiphoton resonance for the ground state of the oscillator, $E_N-E_0=N\omega_F$, or equivalently $\varepsilon_0=\varepsilon_N$. From

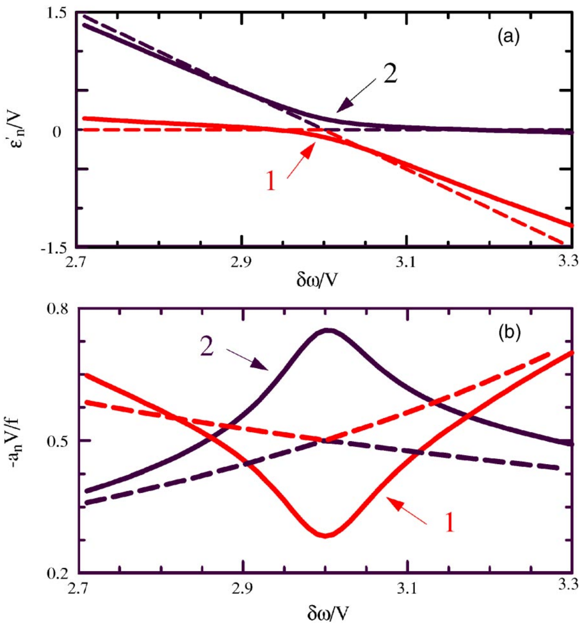

FIG. 1. (Color online) Anticrossing of the quasienergy levels (a) and reduced susceptibilities (b) with varying frequency $\delta\omega$ near five-photon resonance. The lines 1 and 2 refer to the states $n=0$ and $n=5$. The solid and dashed lines refer to the reduced driving force $f/f_5=0.75$ and $f/f_5=0.075$, respectively, with f_5 given by Eq. (8); $\varepsilon_n'=\varepsilon_n-f^2/2V$. In the limit $f\to 0$ the levels cross for $\delta\omega_5=3V$. For $(\delta\omega_5-\delta\omega)/\Omega_R \gg 1$ the adiabatic states $|0\rangle$ and $|5\rangle$ are close to the corresponding Fock states of the oscillator.

Eq. (4), for small f and given N the resonance occurs for

$$\delta\omega = \delta\omega_N \equiv V(N+1)/2.$$

Remarkably, at resonance *all* quasienergy levels (4) with $n \leq N$ are pairwise degenerate, $\varepsilon_{N-n}=\varepsilon_n$.[8] Equivalently, $E_{N-n}-E_n=(N-2n)\omega_F$. One can see that the degeneracy is not lifted by the lowest-order ($\propto f^2$) field-induced level shift [except for the levels $n=(N\pm 1)/2$ for odd N and $n=(N/2)\pm 1$ for even N]. As shown below, it persists for all f in the WKB approximation in the neglect of tunneling.

The response of the oscillator to the field is characterized by the expectation value of its coordinate q. For an eigenstate $|n\rangle$ of the Hamiltonian (3), this value is

$$q_n = (2\omega_0)^{-1/2} a_n e^{-i\omega_F t} + \text{c.c.}, \quad a_n = \langle n|a|n\rangle. \tag{5}$$

To first order in the field, the reduced amplitude of forced vibrations a_n is

$$a_n = -f\,\delta\omega/([\delta\omega - Vn][\delta\omega - V(n+1)]). \tag{6}$$

For $\delta\omega=\delta\omega_N$ the vibration amplitudes in the resonating states coincide with each other, $a_{N-n}=a_n$ for $0\leq n < N/2$, cf. the dashed lines in Fig. 1(b).

Multiphoton mixing leads to splitting of the quasienergy levels and the vibration amplitudes. It can be calculated by diagonalizing the Hamiltonian (3) and is shown in Fig. 1 as a function of frequency detuning $\delta\omega$. One of the involved resonating states is the ground state of the oscillator $n=0$ in the limit $f\to 0$.

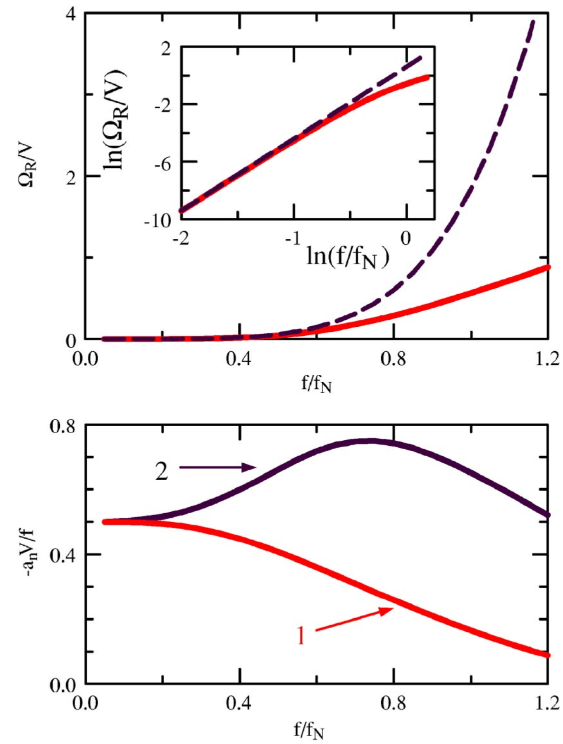

FIG. 2. (Color online) Upper panel: Field-induced splitting of the quasienergy levels $n=0$ and $n=5$ for resonant driving frequency, $\delta\omega=\delta\omega_5=3V$. The splitting gives the five-photon Rabi frequency Ω_R. The dashed line shows the weak-field perturbation theory (7). Lower panel: Splitting of the reduced amplitudes of forced vibrations in the corresponding Floquet states. The curve labeling coincides with that in Fig. 1.

The minimal splitting of the levels ε_0 and ε_N is given by the multiphoton Rabi frequency Ω_R. For weak field it can be obtained from Eq. (3) by perturbation theory.[1] To the lowest order in $f/\delta\omega_N$

$$\Omega_R = 2f|2f/V|^{N-1} N^2 (N!)^{-3/2}. \tag{7}$$

For $N \gg 1$ this expression becomes

$$\Omega_R = V(f/f_N)^N N^{5/4} (2\pi)^{-3/4},$$

$$f_N = |V| N^{3/2} \exp(-3/2)/2. \tag{8}$$

The Rabi frequency depends on N exponentially, $\Omega_R \propto f^N$. In the case $N=5$ it is shown in Fig. 2. One can see from this figure that Eq. (7) works well in the whole range of the field amplitudes $f/f_N \lesssim 0.5$. For larger fields Ω_R depends on f much weaker than the asymptotic expression (8) (Ref. 5).

The most interesting feature of Fig. 1 is the antiresonant splitting of the amplitudes. It occurs at the adiabatic passage of $\delta\omega$ through resonance, where the system switches between the ground and excited states. In particular, the amplitude displays an antiresonant dip if the oscillator is mostly in the ground state for $(\delta\omega-\delta\omega_N)/V < 1$ or in the state N for $(\delta\omega-\delta\omega_N)/V > 1$. The magnitude and sharpness of the dip are determined by Ω_R/V and depend very strongly on the field and N. With decreasing Ω_R/V the dip (and peak) start looking like cusps located at resonant frequency. The amplitude splitting as a function of the field is shown in Fig. 2.

The dip in the oscillator response has no analog in two-level systems. There, for nearly resonant driving, the coherent response in the two adiabatic states differs only in sign. It displays a peak when the radiation frequency adiabatically passes through the transition frequency.

To explain the antiresonance we note that, for $\delta\omega=\delta\omega_N$, the field leads to two major effects. One is mixing of the wave functions of the resonating Fock states $|n\rangle_0$, $|N-n\rangle_0$ into symmetric and antisymmetric combinations $|n,N-n\rangle_\pm =(|n\rangle_0 \pm |N-n\rangle_0)/\sqrt{2}$ with quasienergies $\varepsilon_{n\pm}$. The second effect is nonresonant mixing of the states $|n,N-n\rangle_\pm$ with adjacent n, which determines the expectation values of the vibration amplitudes in these states.

To first order in f, the vibration amplitudes $a_{0\pm} = {}_\pm\langle 0,N|a|0,N\rangle_\pm$ are determined by mixing of the states $|0,N\rangle_\pm$ with $|1,N-1\rangle_\pm$ and $|N+1\rangle_0$. For comparatively weak fields, the level splitting $\varepsilon_{1+}-\varepsilon_{1-}\propto \Omega_R(\delta\omega/f)^2$ largely exceeds the splitting $\varepsilon_{0+}-\varepsilon_{0-}=\Omega_R$. Setting $\varepsilon_{0\pm}=\varepsilon_0$, from perturbation theory we obtain

$$a_{0+}-a_{0-}\propto f[(\varepsilon_0-\varepsilon_{1+})^{-1}-(\varepsilon_0-\varepsilon_{1-})^{-1}]\propto (f/V)^{N-1}.$$

This scaling describes the resonant small-field amplitude splitting in Fig. 2 extremely well (the prefactor is determined by the admixture of states $|n,N-n\rangle_\pm$ with $n>1$ and will be discussed elsewhere).

The simultaneous degeneracy of quasienergies and vibration amplitudes for many pairs of states in a broad field range can be shown analytically in the case where the oscillator dynamics is described by the WKB approximation. This approximation applies for

$$\lambda \ll 1, \quad \lambda = V/(2\delta\omega). \tag{9}$$

It is convenient to introduce the reduced coordinate and momentum of the oscillator in the rotating frame

$$Q=(V/4\delta\omega)^{1/2}(a+a^\dagger), \quad P=-i(V/4\delta\omega)^{1/2}(a-a^\dagger)$$

with the commutator $[P,Q]=-i\lambda$. In these variables, in the neglect of terms $\propto\lambda$, the Hamiltonian (3) becomes $H_0 =2(\delta\omega)^2V^{-1}[g(Q,P)-1/4]$, where

$$g(Q,P)=(Q^2+P^2-1)^2/4-\beta^{1/2}Q. \tag{10}$$

Here $\beta=f^2V/(\delta\omega)^3$ is the reduced field intensity.

The function $g(Q,P)$ is illustrated in Fig. 3; $\delta\omega\, g(Q,P)$ is the classical Hamiltonian in the RWA, it gives the quasienergy of the oscillator;[5,9] Q, P are the canonical variables. The minimum and local maximum of $g(Q,P)$ correspond to the stable states of forced vibrations. They coexist for $0<\beta<4/27$. For such β, in a certain range of g there are two Hamiltonian trajectories with the same g, one on the internal "dome" and the other on the external part of the surface $g(Q,P)$. We call them, respectively, internal and external trajectories.

The external trajectory for given $g(Q,P)=g$ has the form $Q(t)=\beta^{-1/2}[X^2(t)-g]$, with

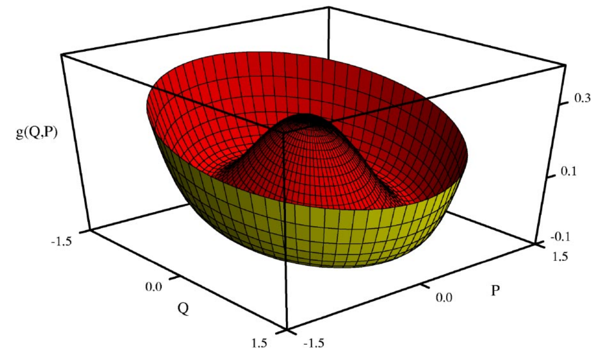

FIG. 3. (Color online) The reduced classical quasienergy of the oscillator (10). The plot refers to the reduced field $\beta=2/27$.

$$X(t)=\frac{c_2(c_1-c_3)-c_3(c_1-c_2)\mathrm{sn}^2u}{c_1-c_3-(c_1-c_2)\mathrm{sn}^2u}. \tag{11}$$

Here, sn u is the Jacobi elliptic function; the elliptic modulus is $m=(c_1-c_2)(c_3-c_4)/(c_1-c_3)(c_2-c_4)$, and $u=[(c_1-c_3)(c_2-c_4)]^{1/2}\delta\omega\, t/2$ is the appropriately scaled time. The coefficients $c_1>c_2>c_3>c_4$ are the roots of the polynomial $\beta(1+2x)-(x^2-g)^2$.

The internal trajectory $Q(t)$ is given by Eq. (11) with $u\to u+K+iK'$, where $K\equiv K(m)$ is the complete elliptic integral, and $K'\equiv K(1-m)$.

An immediate consequence of the analytical interrelation between the external and internal trajectories is that their periods $2\pi/\omega(g)$ are the same.[10] When the motion is quantized, $\omega(g)$ gives the distance between the energy levels. Therefore if, for some $\delta\omega$ and β, two levels that correspond to the external and internal trajectories coincide with each other, many levels will coincide pairwise as well. Level splitting (anticrossing with varying $\delta\omega$) is due to tunneling between the external and internal parts of the surface $g(Q,P)$.

In the WKB approximation, the expectation value a_n of the vibration amplitude in a quantum state $|n\rangle$ is given by the period-averaged coordinate Q on the appropriate classical trajectory, $\langle Q(g_n)\rangle$. It can be shown, using the analytical properties of the elliptic functions, that the values of $\langle Q(g)\rangle$ turn out to be the same on the internal and external trajectories with the same g. Therefore a_n for resonating states are the same, in the neglect of tunneling.

In order to observe coherent multiphoton quantum effects, the Rabi frequency Ω_R should exceed the relaxation rate. Relaxation of a nonlinear oscillator can often be described by the quantum kinetic equation.[9] Evolution of the density matrix ρ due to decay into excitations of the medium and fluctuational modulation of the oscillator energy is given by the equation $(\dot\rho)_{\mathrm{dm}}=-\hat\Gamma\rho-\hat\Gamma_\varphi\rho$, with

$$\hat\Gamma\rho=\Gamma(\hat n\rho-2a\rho a^\dagger+\rho\hat n), \quad \hat\Gamma_\varphi\rho=\Gamma_\varphi[\hat n,[\hat n,\rho]]. \tag{12}$$

The parameters Γ and Γ_φ characterize the rates of decay and phase diffusion (energy modulation), respectively, and we have assumed that $\exp(\hbar\omega_F/k_BT)\gg 1$.

The relaxation rate Γ_N relevant for an N-photon resonance, $\delta\omega=\delta\omega_N$, is given by the damping rate of the population difference $\rho_{++}-\rho_{--}$ of the symmetric and antisymmetric adiabatic states $|0,N\rangle_{\pm}=(|0\rangle_0 \pm |N\rangle_0)/\sqrt{2}$, $\rho_{\nu\nu'} = {}_{\nu}\langle 0,N|\rho|0,N\rangle_{\nu'}$ $(\nu,\nu'=\pm)$. From Eq. (12) we obtain $\Gamma_N = \Gamma N + \Gamma_{\varphi}N^2/2$. The first term is just the decay rate of the population of the Nth Fock state of the oscillator, whereas $\Gamma_{\varphi}N^2$ is the diffusion rate of the phase difference of the Fock states $|0\rangle$ and $|N\rangle$.

The rate Γ_N quickly increases with increasing N. Therefore the coherence condition $|V| \gtrsim \Omega_R \gg \Gamma_N$ imposes a limitation on N from above. From our analysis, strong antiresonance in the susceptibility is pronounced already for $N = 3-5$. We note that this coherent quantum effect is qualitatively different from the nonmonotonic field dependence of the stationary amplitude of a driven damped oscillator for nonzero temperatures.[11]

The antiresonance in the vibration amplitude can be observed in Josephson junctions (JJs) and JJ-based systems and can provide a spectroscopic tool for studying their energy spectrum. Resonant dynamics of JJs is well described by the model of a nonlinear oscillator (3). The measurement could be similar to the one in which forced vibrations of a JJ and their bistability were studied.[3] The antiresonance requires that the system be in the quantum regime, $\exp(\hbar\omega_0/k_BT) \gg 1$, and the level nonequidistance exceed damping, $|V| \gg \Gamma_N$.

The JJ energy spectrum is controlled by the dc bias current I_{dc}. For $I_{\mathrm{dc}}=0$ we have $|V|=\hbar\omega_p^2/E_J$, where ω_p is the JJ plasma frequency and $E_J=\hbar I_0/2e$ (I_0 is the critical current). Such $|V|$ is often small; e.g., $|V| < \Gamma$ for the parameters of Ref. 3. It can be largely increased if I_{dc} is close to I_0, i.e., $\eta=(I_0-I_{\mathrm{dc}})/I_0 \ll 1$. Then the effective potential of the JJ near a local minimum is a cubic parabola, and in Eq. (3)

$$\omega_0 = \omega_p(2\eta)^{1/4}, \quad V = -5\hbar\omega_p^2/48E_J\eta. \tag{13}$$

A limitation on η is imposed by the condition that there are several levels in a metastable potential well and their tunneling decay rates $\gamma_n \ll |V|$ for $n \leqslant N$. The rates γ_n depend on η exponentially, $\ln[\gamma_0/\omega_p] \approx -c_t E_J \eta^{5/4}/\hbar\omega_p$, with $c_t \approx 11.4$, and $\ln[\gamma_{N+1}/\gamma_N] \approx 2\pi$. Experimentally, the ratio $|V|/\omega_p \sim 0.1$ was obtained for a three-level JJ, with $\omega_p/2\Gamma \sim 10^3$ (Ref. 12). For such damping, even smaller $|V|/\omega_p$ will allow observing the antiresonance. From Eq. (8) the needed rf current is $\lesssim I_N = (5/48)I_c \exp(-3/2)(2N\hbar\omega_p/E_J)^{3/2}(2\eta)^{-7/8}$.

Along with adiabatic passage it may be interesting also to study multiphoton Rabi oscillations between the Fock states $|0\rangle_0$ and $|N\rangle_0$ when a resonant field is turned on. For $\delta\omega = \delta\omega_N$ the oscillation frequency is given by Ω_R. In JJ-based systems, oscillations of state populations can be detected from tunneling decay, which is much faster in the excited Fock states. This approach has been used to detect single-photon Rabi oscillations in strongly nonlinear JJs with a small number of metastable states.[12,13]

The susceptibility should also display multiphoton Rabi oscillations. Their amplitude is close to half the distance between the branches 1 and 2 in Fig. 2. It *increases* with the field for small f/f_N. This is in contrast with the case of two-level systems, where the oscillation amplitude decreases with the increasing field at resonance.

In this paper we have shown that multiphoton response of a quantum oscillator displays antiresonant dips (peaks) as a function of frequency. This coherent quantum effect is specific for multilevel systems; it does not arise in two-level systems and is related to resonant pairwise mixing of several oscillator states at a time. The effect provides a means of coherent nonlinear spectroscopy of excited vibrational states. The shape and magnitude of the dips (peaks) of the response strongly depend on the field. We discuss the possibility to observe the predicted antiresonant response and the multiphoton Rabi oscillations in Josephson junctions.

We are grateful to B.L. Altshuler for the discussion. This research was supported in part by the Institute for Quantum Sciences at Michigan State University (MSU). The work of M.D. was also supported by the NSF ITR-0085922. M.V.F. thanks the hospitality of MSU, where this work was started, and the financial support of SFB 491.

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