Fluctuational phase-flip transitions in parametrically driven oscillators

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We analyze the rates of noise-induced transitions between period-two attractors. The model investigated is an underdamped oscillator parametrically driven by a field at nearly twice the oscillator eigenfrequency. The activation energy of the transitions is analyzed as a function of frequency detuning and field amplitude scaled by the damping and nonlinearity parameters of the oscillator. Both fourth- and sixth-order nonlinearities are taken into account. The parameter ranges where the system is bistable and tristable are investigated. Explicit results are obtained in the limit of small damping, or equivalently, strong driving, including scaling near bifurcation points. [S1063-651X(98)15405-3]

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I. INTRODUCTION

Nonlinear systems driven by a sufficiently strong periodic field often display period doubling [1]. The two emerging stable periodic states are identical, except that they are shifted in time by the period of the field $2\pi/\omega_F$. This feature is a consequence of the symmetry with respect to translation in time by $2\pi/\omega_F$, and it attracted attention to such systems as elements of digital computers [2]. Period doubling has found numerous applications, in particular in parametric amplifiers. In the presence of noise, there occur fluctuational transitions between the period-two attractors, which correspond to phase slip of the system by π. In spite of its importance, the problem of phase-flip transitions remains largely unexplored theoretically. On the experimental side, interest in such transitions has been renewed recently, because they were observed for electrons oscillating in a Penning trap [3], and also investigated, for an analog electronic circuit [4], in the context of stochastic resonance [5].

Motivated by these observations, in the present paper we develop a theory of escape rates from period-two attractors. The analysis is done for the simplest generic model that displays period doubling: an underdamped oscillator parametrically driven by a force at nearly twice the oscillator eigenfrequency ω_0 [6]. This model applies, in particular, to axial vibrations of an electron in a Penning trap [3]. Much work on a parametrically excited oscillator has been done in the context of squeezed states of light; cf. Ref. [7]. We analyze escape due to classical fluctuations, which were substantial for the systems investigated in Refs. [3,4].

A parametrically excited oscillator is an example of a system away from thermal equilibrium. Such systems usually lack detailed balance [8], they are not characterized by free energy, and escape rates depend on the system dynamics and the noise that gives rise to fluctuations in the system. In the important and quite general case where the noise is Gaussian, there has been developed a technique which reduces the problem of calculating escape rates to a variational problem [9]. The solution of this problem describes the optimal path along which the fluctuating system is most likely to move when it escapes. Such path is often called [10] the most probable escape path. The minimal value of the variational functional gives the exponent in the expression for the escape rate.

An advantageous feature of period doubling in an underdamped oscillator is that it occurs for comparatively small amplitudes F of the driving force, where the nonlinearity of the oscillator is still small: the anharmonic part of the potential energy is much less than the harmonic one, $\omega_0^2 q^2/2$, where q is the oscillator coordinate. In this case, the quantities of interest are the amplitude and phase of the vibrations at the frequency $\omega_F/2 \approx \omega_0$. They vary only a little over the time $\sim \omega_F^{-1}$. The corresponding dynamics is affected by Fourier components of the noise within a narrow band centered at $\omega_F/2$. Essentially, this means that, in the analysis of the dynamics of slow variables, the noise may be assumed to be white. A similar situation arises [12] in the problem of transitions between the stable states of forced vibrations of a resonantly driven underdamped oscillator.

Below, in Sec. II, we discuss the phase portrait of a driven Duffing oscillator (with the fourth-order nonlinearity) in the rotating frame. We then derive the properties of noise for slow variables. For low noise intensities, we formulate and solve numerically the variational problem for the activation energy of escape from period-two states. In Sec. III, explicit expressions are provided for the escape rates in the vicinities of the bifurcation points where there emerge period-two attractors (a supercritical bifurcation) or unstable period-two states. The analysis in Sec. IV refers to comparatively strong driving, where the motion in *slow* variables is underdamped. Explicit analytical results for escape activation energies are obtained in limiting cases and compared with numerical results. In Sec. V the role of sixth-order nonlinearity is discussed, and the activation energies are found near bifurcation points, and also in the range where sixth-order nonlinearity is strong and the motion in *slow* variables is underdamped. Section VI contains concluding remarks.

II. ESCAPE RATES: GENERAL FORMULATION

A. Phase portrait in slow variables

To set the scene, we will first discuss the phase portrait of a parametrically driven underdamped oscillator. A simple phenomenological equation of motion is of the form

$$\frac{d^2q}{dt^2} + 2\Gamma\frac{dq}{dt} + \omega_0^2 q + \gamma q^3 + qF\cos\omega_F t = \xi(t). \qquad (1)$$

Here Γ is the friction coefficient, γ is the nonlinearity parameter, F is the amplitude of the regular force, and $\xi(t)$ is a zero-mean noise, $\langle\xi(t)\rangle=0$. We assume that the amplitudes of the vibrations are not too large, and therefore only the fourth-order term in the coordinate q is taken into account in the oscillator potential energy (the Duffing model). In the problem of period doubling, the effect of the cubic term comes to renormalization of the parameter γ (cf. Ref. [6]). Generalization of the results to the case where, for special reasons [3,11], γ is numerically small, and it is necessary to allow for the term $\propto q^5$ in the equation of motion, is discussed in Sec. V.

We consider resonant driving, so that the oscillator eigenfrequency ω_0 is close to $\omega_F/2$,

$$\Gamma,|2\omega_0-\omega_F|\ll\omega_0. \tag{2}$$

In this case it is convenient (cf. Ref. [6]) to analyze the oscillator motion in the rotating frame. We change to slow dimensionless time $\tau=\Gamma t$ and slow dimensionless variables q_1 and q_2, respectively:

$$q(t)=\left(\frac{4\omega_F\Gamma}{3|\gamma|}\right)^{1/2}\left[q_1\cos\frac{\omega_F t}{2}-q_2\sin\frac{\omega_F t}{2}\right],$$

$$\frac{dq}{dt}=-\left(\frac{\omega_F^3\Gamma}{3|\gamma|}\right)^{1/2}\left[q_1\sin\frac{\omega_F t}{2}+q_2\cos\frac{\omega_F t}{2}\right]. \tag{3}$$

Following the standard procedure of the method of averaging (cf. Ref. [1]), and neglecting fast oscillating terms which depend on the oscillator amplitude and contain a factor $\exp(\pm in\omega_F t/2)$ with $n\neq 0$, one obtains the equations of motion for q_1 and q_2 in the forms

$$\dot q_1\equiv\frac{dq_1}{d\tau}=-q_1+\frac{\partial g}{\partial q_2}+\xi_1(\tau/\Gamma),\qquad \tau\equiv\Gamma t,$$
$$\dot q_2\equiv\frac{dq_2}{d\tau}=-q_2-\frac{\partial g}{\partial q_1}+\xi_2(\tau/\Gamma), \tag{4}$$

where $\xi_{1,2}(\tau/\Gamma)$ are random forces proportional to $\xi(t)$, and

$$g(q_1,q_2)=\tfrac{1}{2}(q_1^2+q_2^2)[\Omega-\tfrac{1}{2}(q_1^2+q_2^2)\mathrm{sgn}\ \gamma]+\tfrac{1}{2}\zeta(q_2^2-q_1^2). \tag{5}$$

In what follows we assume that $\gamma>0$; the case $\gamma<0$ can be described by replacing $\Omega\to-\Omega$ and $q_i\to-q_{3-i}$ ($i=1$ and 2).

Except for the random force, the motion of the oscillator as described by Eqs. (4) is characterized by two dimensionless parameters: the scaled frequency detuning Ω and the scaled field ζ,

$$\Omega=[(\omega_F/2)-\omega_0]/\Gamma,\qquad \zeta=F/2\omega_F\Gamma. \tag{6}$$

For $\zeta<1$ or for $\Omega<-(\zeta^2-1)^{1/2}$, the oscillator (4) in the absence of noise has only one stable state, $q_1=q_2=0$: period-two oscillations are not excited. The value $\zeta=1$ gives the threshold field amplitude $F_{\mathrm{th}}=2\omega_F\Gamma$ for their excitation. The phase portrait of the oscillator in variables q_1 and q_2, in the range where the oscillations are excited, is shown in Fig. 1.

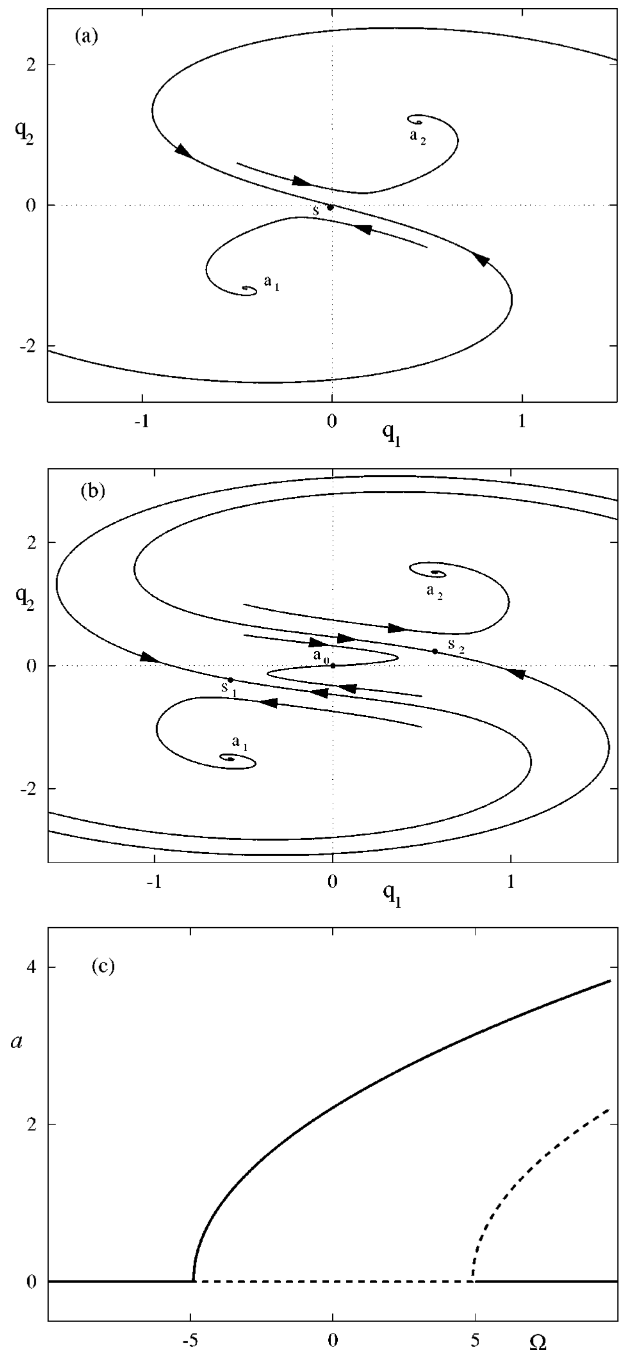

FIG. 1. Trajectories and separatrices of the oscillator in the absence of noise in slow variables q_1 and q_2 for (a) $\zeta=1.5$ and $\Omega=0.5$, where the stable states of the oscillator are period-two attractors; and (b) for $\zeta=1.5$ and $\Omega=1.5$, where the steady state $\mathbf{q}=\mathbf{0}$ is also stable. The positions of the stable states and the saddle points are denoted by the letters a_i and s, respectively. (c) The dependence of the dimensionless amplitude of the stable (solid line) and unstable (dashed line) period-two vibrations on Ω for $\zeta=5.0$.

For $\zeta>1$, with the increasing Ω there first occurs a pitchfork bifurcation for $\Omega=-(\zeta^2-1)^{1/2}$. This is a supercritical period-doubling bifurcation: the stable state $\mathbf{q}=0$ [$\mathbf{q}\equiv(q_1,q_2)$] becomes unstable, and there emerge two stable states $\mathbf{q}_{\mathrm{st}}^{(1,2)}$ which are symmetric with respect to $q_1=q_2=0$; see Fig. 1(a). These states correspond to stable period-2 vibrations which are shifted in phase by π. The vibrational amplitude $a=|\mathbf{q}_{\mathrm{st}}|$ increases monotonically with the increasing Ω [see

Fig. 1(c)]. The results of the asymptotic analysis based on Eqs. (4) apply for not too large amplitudes a,

$$\Gamma a^2, |\omega_F - 2\omega_0| a^2 \ll \omega_F. \tag{7}$$

As Ω goes through the value $(\zeta^2-1)^{1/2}$, there occurs the second pitchfork bifurcation of the state $q_1=q_2=0$: this state becomes stable again, and there emerge two unstable states $\mathbf{q}_u^{(1,2)}$, which are also symmetric with respect to the origin, and correspond to unstable period-two vibrations of the oscillator. As seen from Fig. 1(b), the phase plane of the oscillator is such that, for $\Omega > (\zeta^2-1)^{1/2}$ the domains of attraction to the stable states $\mathbf{q}_{\rm st}^{(1)}$ and $\mathbf{q}_{\rm st}^{(2)}$ are separated by the domain of attraction to the stable state $\mathbf{q}=\mathbf{0}$.

B. Fluctuations of slow variables

In the presence of noise, the amplitude and phase of the oscillator are fluctuating. The fluctuations are determined by the noise $\xi(t)$ in Eq. (1) that drives the oscillator. In many cases of physical interest, this noise is Gaussian. It may originate from the coupling of the oscillator to a thermal bath [which also gives rise to the friction force in Eq. (1)], or it may be due to an external nonthermal source. A zero-mean Gaussian noise is characterized by its power spectrum

$$\Phi_\omega(\xi(t+t'),\xi(t)) = \phi(\omega) = \int_{-\infty}^{\infty} dt'\, e^{i\omega t'} \langle \xi(t+t')\xi(t)\rangle. \tag{8}$$

For a stationary noise the power spectrum (8) is independent of time. In what follows we assume that the function $\phi(\omega)$ is *smooth* near the oscillator eigenfrequency ω_0.

Even though the noise $\xi(t)$ is stationary, the random forces $\xi_{1,2}(t)$ in Eq. (4), which give rise to fluctuations of the slow variables, are nonstationary, generally speaking. From Eqs. (1), (3), and (4), one obtains the following expressions for their power spectra:

$$\Phi_\omega[\xi_j(t+t'),\xi_j(t)] = \frac{3|\gamma|}{4\omega_F^3\Gamma^3} \sum_{\alpha k=\pm 1} \phi\left(\omega - \frac{1}{2}\alpha\omega_F\right) \times [1 + (-1)^j \exp(i\alpha\omega_F t) \quad (j=1,2),$$

$$\Phi_\omega[\xi_1(t+t'),\xi_2(t)] = -\frac{3i|\gamma|}{4\omega_F^3\Gamma^3} \sum_{\alpha=\pm 1} \alpha\phi\left(\omega + \frac{1}{2}\alpha\omega_F\right) \times [1 + \exp(i\alpha\omega_F t)]. \tag{9}$$

The dynamics of the slow variables $q_{1,2}$ is characterized by the time scales $\sim 1/\Gamma$, $1/|\omega_F - 2\omega_0|$. We assume that the power spectrum $\phi(\omega)=\phi(-\omega)$ varies only slightly in the whole frequency range where $|\omega-\omega_F/2] \lesssim \Gamma$, $\omega_F - 2\omega_0|$ [and this range does not correspond to a deep minimum of $\phi(\omega)$]. It follows then from Eq. (9) that for the characteristic frequencies $|\omega| \lesssim \Gamma$, $|\omega_F - 2\omega_0|$, the spectra of the diagonal correlators $\langle \xi_i(t)\xi_i(t')\rangle$ have both time independent components and components that oscillate quickly in time, whereas the power spectrum of the cross-correlator of ξ_1 and ξ_2 is quickly oscillating in time. Therefore, in the analysis of the effect of the noise on the slowly varying functions $q_{1,2}$, in the spirit of the averaging method, one can assume that the noise components $\xi_1(t)$ and $\xi_2(t)$ are asymptotically independent of each other; one can also leave out the terms $\propto \cos\omega_F t$ in the power spectra of the diagonal correlators. Corresponding analysis can be also done for a microscopic model of noise resulting from coupling to a bath (see Ref. [12]); rigorous mathematical results on the method of averaging in noise-driven systems were discussed in Ref. [13].

It follows from the arguments discussed above that the random functions $\xi_1(t)$ and $\xi_2(t)$ are asymptotically independent zero-mean Gaussian white noises,

$$\langle \xi_i(\tau/\Gamma)\xi_i(\tau'/\Gamma)\rangle \approx D\tilde{\delta}(\tau-\tau'), \quad i=1,2, \qquad D = (3|\gamma|/2\omega_F^3\Gamma^2)\phi(\omega_F/2). \tag{10}$$

The function $\tilde{\delta}(x)$ is δ like: it is large in a narrow domain $|x| \ll 1$, and its integral is equal to 1.

It follows from Eqs. (4) and (10) that the motion of the oscillator on the slow time scale is Brownian, i.e., the slow variables $q_{1,2}(t)$ are components of a two-dimensional Markov process. The quantity D gives the characteristic intensity of the noise in the equations of motion for q_1 and q_2. If the oscillator is coupled to a thermal bath with a correlation time much smaller than $1/\omega_0$, so that the oscillator performs a ''truly'' Brownian motion and the random force $\xi(t)$ in Eq. (1) is δ correlated, with intensity $4\Gamma kT$, we have $D = 6|\gamma|kT/\omega_F^3\Gamma$. We note, however, that the dynamics of slow variables can be described as Brownian motion even where this description does not apply to the motion of the initial oscillator, i.e., where the correlation time of the thermal bath is $\gtrsim 1/\omega_0$. In this latter case, the friction force in Eq. (1) is also retarded, but the retardation may be neglected in the equations of motion for the slow variables (cf. Ref. [12]).

C. Variational problem for the escape rate

If the dimensionless noise intensity D is small, then most of the time the oscillator is fluctuating in a small vicinity of one or the other stable state $\mathbf{q}_{\rm st}^{(n)}$ (in what follows we set $n=1$ and 2 for the stable states of period-two vibrations, and $n=0$ for the stationary state $\mathbf{q}=\mathbf{0}$ where it is stable). Only occasionally does there occur a large fluctuation which results in a transition to another stable state. The probability W_n of such fluctuations is exponentially small, and its dependence on the noise intensity is given by the activation law, $W_n \propto \exp(-S_n/D)$ (see Ref. [9] for a review). In fact, to logarithmic accuracy, W_n is determined by the probability density of the least improbable realization of the force $\boldsymbol{\xi}(\tau/\Gamma)$ which results in the corresponding transition. Therefore, one may expect that the quantity S_n is given by the solution of a variational problem. This problem is of the form (cf. Refs. [12,13])

$$W_n = C\,\exp(-S_n/D), \quad S_n = \min S_n(\mathbf{q}(\tau))[\mathbf{q}=(q_1,q_2)],$$

$$S_n(\mathbf{q}(\tau)) = \int_{-\infty}^{\infty} d\tau\, L(\dot{\mathbf{q}},\mathbf{q}), \quad L(\dot{\mathbf{q}},\mathbf{q}) = \tfrac{1}{2}[\dot{\mathbf{q}} - \mathbf{K}(\mathbf{q})]^2, \tag{11}$$

$$\mathbf{q}(\tau) \to \mathbf{q}_{\rm st}^{(n)} \quad \text{for } \tau \to -\infty, \quad \mathbf{q}(\tau) \to \mathbf{q}_u \quad \text{for } \tau \to \infty,$$

where the components of the vector $\mathbf{K}$ are given by the right-hand sides of the equations of motion (4),

$$K_1(\mathbf{q}) = -q_1 + \frac{\partial g}{\partial q_2}, \quad K_2(\mathbf{q}) = -q_2 - \frac{\partial g}{\partial q_1}. \tag{12}$$

The solution of the variational problem (11) $\mathbf{q}(\tau)$ describes the optimal, or most probable escape path from the nth stable state. This path is instantonlike (see Ref. [14]). It starts at the stable state $\mathbf{q}_{\rm st}^{(n)}$ for $\tau \to -\infty$, and for $\tau \to \infty$ it approaches the unstable state $\mathbf{q}_u$ on the boundary of the domain of attraction to $\mathbf{q}_{\rm st}^{(n)}$ (having reached the boundary, the system makes a transition to another stable state with a probability $\sim \frac{1}{2}$). It follows from Fig. 1 that, for $-(\zeta^2-1)^{1/2} < \Omega < (\zeta^2-1)^{1/2}$, where the only stable states are period-two attractors, escape from one of them means a transition to the other. For $\Omega > (\zeta^2-1)^{1/2}$, escape from one of the period-two attractors means a transition to the stationary state, where period-two vibrations are not excited, except in the case of extremely small damping, where the separatrices in Fig. 1(b) come close to each other near saddle points, so that the distance between them is less than the diffusion length. From the state where period-two vibrations are not excited, the system makes fluctuational transitions into one or the other period-two attractor.

The most probable realization of the noise is related to the optimal fluctuational path via Eq. (4), $\boldsymbol{\xi}(\tau/\Gamma) = \dot{\mathbf{q}} - \mathbf{K}$. Optimal fluctuational paths are physically real; they have been observed in experiment (see Ref. [15]).

The activation energies S_n, as defined by Eq. (11), depend on two dimensionless parameters of the driven oscillator: the scaled field strength ζ and the scaled frequency detuning Ω (6). In the general case, S_n may be calculated numerically as action of the conservative system with the Lagrangian $L(\dot{\mathbf{q}},\mathbf{q})$. Direct algorithms based on the solution of the corresponding Hamiltonian equations were discussed in Refs. [10,16] in the analysis of escape rates for fluctuating systems of other types. For the system investigated in Refs. [12,16], an alternative algorithm, based on the initial guess and subsequent iterations of the solution, was also used, and the results on the transition rates were compared with analog experiments [17]. In the present problem we used a direct method which combined algorithms [10,16]. We note that, if the initial fluctuating system is away from thermal equilibrium, the pattern of extreme Hamiltonian paths of the auxiliary problem (11) generically displays singularities; in particular there arise caustics. However, physically meaningful optimal fluctuational paths, which form a subset of the extreme paths, avoid caustics [18]. In our numerical analysis of extreme paths, we observed singularities which can be fully understood based on the general topological results [18], and we will not discuss them in the present paper.

The dependence of $S_1 = S_2$ on ζ for several values of Ω is shown in Fig. 2. It follows from this figure that the activation energy of escape increases with the increasing field $F \propto \zeta$, and $S_{1,2} \propto \zeta$ for large ζ. This behavior will be analyzed in more detail below in Sec. IV.

III. ACTIVATION ENERGY OF ESCAPE IN VICINITIES OF BIFURCATION POINTS

Explicit expressions for the activation energies S_n can be obtained in several limiting cases. In the range of comparatively small nonlinearity (7), the oscillator may experience two bifurcations with the varying field frequency, as seen from Fig. 1. In the vicinity of a bifurcation point, one of the motions of the system near the emerging stable state(s) becomes slow: there arises a ''soft mode'' [19]. Correspondingly, fluctuations near a bifurcation point have universal features [20] (see also Ref. [12]). For systems that display period doubling, the analysis of dynamics near supercritical and subcritical bifurcation points was discussed earlier, based on the normal form of the equation of motion for the slow variable with account taken of additional weak driving [21] or noise (see Ref. [22], and references therein). However, escape rates were not considered in these papers.

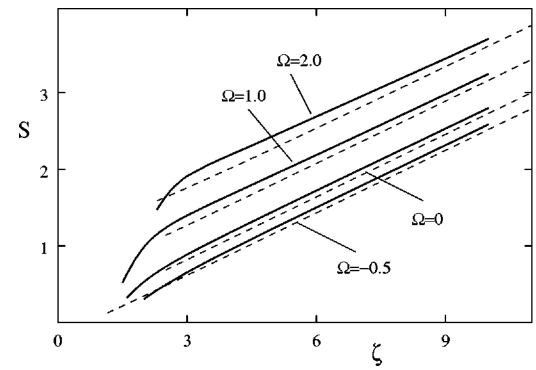

FIG. 2. Activation energies $S \equiv S_1 = S_2$ for phase-slip transitions between period-two attractors of a driven Duffing oscillator as obtained by solving numerically the variational problem (11) (solid lines). The dependence of S on the scaled field amplitude $\zeta = F/2\omega_F\Gamma$ is shown for four values of the dimensionless frequency detuning $\Omega = [(\omega_F/2) - \omega_0]/\Gamma$. The dashed lines show the low-damping (large ζ) asymptotes.

If the parameters of the nonlinear oscillator are close to a bifurcation point, one can either solve the variational problem (11) explicitly or reduce the system of equations of motion (4) to the equation for the slow variable in the normal form, which allows one to find not only the exponent, but also the prefactor in the expression for the escape rate, cf. Ref. [20].

The equation for the slow variable (the variable Q) can be derived from Eq. (4) by appropriately rotating the coordinates:

$$Q = q_1 \cos\beta + q_2 \sin\beta, \quad P = -q_1 \sin\beta + q_2 \cos\beta,$$
$$\tan 2\beta = -\Omega_B^{-1}, \quad \Omega_B = \mp(\zeta^2-1)^{1/2}. \tag{13}$$

Here Ω_B is the bifurcation value of the dimensionless frequency detuning; see Fig. 1(c).

For $|\Omega - \Omega_B| \to 0$ the dimensionless relaxation time of the variable Q goes to infinity, whereas that of P is $\frac{1}{2}$, and therefore $P(\tau)$ follows the slow variable $Q(\tau)$ adiabatically. Fluctuations in P can be neglected compared to fluctuations in Q. Using the adiabatic solution for P (i.e., neglecting $\dot{P}$ and the noise term in the equation for $\dot{P}$), we obtain the following equation for Q:

$$\dot{Q} = -\frac{dU}{dQ} + \Xi(\tau), \quad \langle \Xi(\tau)\Xi(\tau')\rangle = D\tilde{\delta}(\tau-\tau'), \tag{14}$$

$$U(Q) = \Omega_B[\tfrac{1}{2}(\Omega-\Omega_B)Q^2 - \tfrac{1}{4}\zeta^2 Q^4], \quad \Omega_B|\Omega-\Omega_B| \ll 1.$$

It is seen from Eq. (14) that, near the bifurcation point $\Omega_B = -(\zeta^2-1)^{1/2}$, the system has either one stable state (for $\Omega < \Omega_B$) or two symmetrical stable states (for $\Omega > \Omega_B$; cf. Fig. 1). The escape rates from the symmetrical states are the same, and are given by the Kramers expressions [23]

$$W_n = (\sqrt{2}/\pi)|\Omega_B|(\Omega-\Omega_B)\exp(-S_n/D) \quad (n=1,2),$$
$$S_n = 2[U(Q^{(u)}) - U(Q^{(n)})] = |\Omega_B|(\Omega-\Omega_B)^2/2\zeta^2 \tag{15}$$

($Q^{(1)} = -Q^{(2)}$ and $Q^{(u)}$ are the values of the coordinate Q in the stable and unstable states; clearly, $S_1 = S_2$).

Rate (15) is the rate at which there occur phase-slip transitions between the period-two stable states for Ω close to $\Omega_B = -(\zeta^2-1)^{1/2}$. Equation (15) applies for $\exp(-S_n/D) \ll 1$. The activation energy S_n is quadratic in the distance $|\Omega-\Omega_B|$ to the bifurcation point along the axis of the scaled driving field frequency, Ω. For a given $|\Omega-\Omega_B|$, the dependence of S_n on the dimensionless field ζ is determined by the factor $(\zeta^2-1)^{1/2}/\zeta^2$.

For $0 < \Omega - (\zeta^2-1)^{1/2} \ll 1$, Eq. (15) describes the rate of transitions from the stable state $\mathbf{q}_{st} = \mathbf{0}$, $Q_{st} = 0$ to any of the period-two stable states. The transitions occur via the appropriate unstable period-two state (the one on the boundary between the domain of attractions to the stable period-two state and the state $\mathbf{q} = \mathbf{0}$).

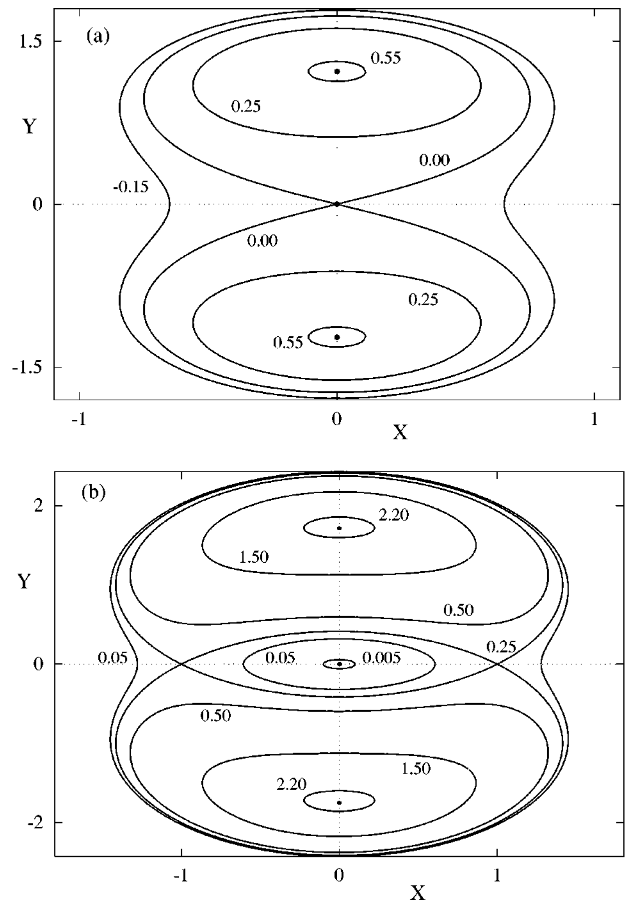

FIG. 3. Trajectories of the conservative motion (16) for (a) $\mu = 0.5$ and (b) $\mu = 2.0$. The values of the Hamiltonian function $G(X,Y)$ are shown near the trajectories. The dots show the positions of the elliptic and hyperbolic points.

IV. SMALL-DAMPING LIMIT

A. Motion in the absence of dissipation

Of special interest, particularly from the viewpoint of experiments on trapped electrons [3,11], is the case where the scaled field amplitude ζ is large enough so that the dissipation terms $-q_1$ and $-q_2$ in the right hand sides of the equations for slow variables (4) are comparatively small. In the neglect of these terms and the random force, Eqs. (4) describe conservative motion of a particle with the coordinate q_1 and momentum q_2, and with the Hamiltonian function $g(q_1,q_2)$ [Eq. (5)]. This particle moves along closed trajectories shown in Fig. 3. It is convenient to describe this motion using scaled coordinate and momentum X and Y:

$$X = q_1/\zeta^{1/2}, \quad Y = q_2/\zeta^{1/2},$$

$$\frac{dX}{d\tilde{\tau}} = \frac{\partial G}{\partial Y}, \quad \frac{dY}{d\tilde{\tau}} = -\frac{\partial G}{\partial X}, \quad \tilde{\tau} = \zeta\tau, \tag{16}$$

$$G(X,Y) = \zeta^{-2} g(\zeta^{1/2}X, \zeta^{1/2}Y) = \frac{1}{2}(\mu-1)X^2 + \frac{1}{2}(\mu+1)Y^2 - \frac{1}{4}(X^2+Y^2)^2, \quad \mu = \frac{\Omega}{\zeta}.$$

The conservative motion (16) depends only on one parameter, $\mu = \Omega/\zeta$, which characterizes the interrelation between the frequency detuning and the field strength. The shape of the effective energy $G(X,Y)$ for two different values of μ is shown in Fig. 4. The trajectories in Fig. 3 are just the cross sections of the surface $G(X,Y)$ by the planes $G = \text{const}$. The extrema of the surface $G(X,Y)$ are the fixed points of the system.

For $\mu < -1$ the surface $G(X,Y)$ has one extremum. It is located at $X = Y = 0$, and corresponds, with dissipation taken into account, to the stable state with no period-two vibrations excited. For $-1 < \mu < 1$, the function $G(X,Y)$ has two maxima at $X = 0$, $Y = \pm(\mu+1)^{1/2}$ (they correspond to two period-two attractors), and a saddle point at $X = Y = 0$. For $\mu > 1$, in addition to the above maxima the function $G(X,Y)$ has a minimum at $X = Y = 0$ (the maxima and the minimum correspond to the stable states of the oscillator), and two saddle points at $X = \pm(\mu-1)^{1/2}$, $Y = 0$. The extreme values of G, which correspond to the stable states (enumerated by the subscripts $n = 0$, 1, and 2) and the unstable periodic states (denoted by the subscript u), are given by the expressions

$$G_{1,2} = \tfrac{1}{4}(\mu+1)^2, \quad G_u = 0 \quad \text{for } \mu < 1,$$
$$G_0 = 0, \quad G_{1,2} = \tfrac{1}{4}(\mu+1)^2, \quad G_u = \tfrac{1}{4}(\mu-1)^2 \quad \text{for } \mu > 1. \tag{17}$$

We note that for $\mu > 1$ the trajectories surrounding the states at $X = 0$, $Y = \pm(\mu+1)^{1/2}$ in Fig. 3 become horse-shoe-

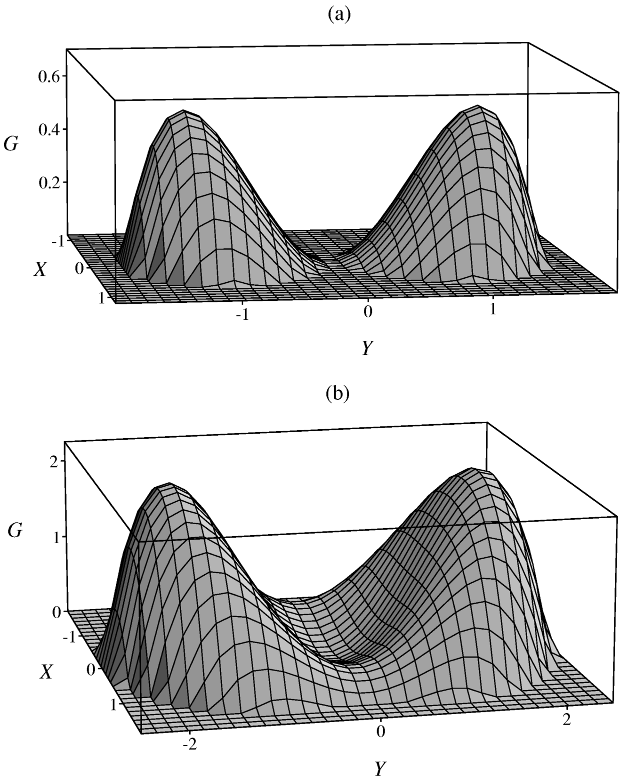

FIG. 4. The Hamiltonian function $G(X,Y)$ [Eq. (16)] (a) for $\mu=0.5$, where the function G has two maxima (which correspond to period-two attractors, with account taken of dissipation) and a saddle point at $X=Y=0$, and (b) for $\mu=2.0$ where G has two maxima, a minimum at $X=Y=0$, and two saddles which correspond to unstable period-two vibrations, with account taken of dissipation.

like for large enough G_1-G, and also that for $0\leqslant G\leqslant G_u$ there are coexisting ''internal'' and ''external'' trajectories with the same G.

B. Escape rates

The effect of small dissipation in Eqs. (4) is to transform the closed trajectories in Fig. 3 into small-step spirals which wind down to the corresponding stable states (cf. Fig. 1). The motion can be described in a standard way in terms of slow drift over the energy g toward the stable state (cf. Ref. [1]).

The random force which drives the system away from the stable state should ''beat'' this drift. It would be expected that the optimal fluctuational path corresponds to energy diffusion *away* from the stable state. A solution of the variational problem (11) for small dissipation was obtained in Ref. [12] for a different form of the function $g(q_1,q_2)$. Alternatively, one can use an approach of the type of that based on the Fokker-Planck equation, and suggested by Kramers [23] in the analysis of escape of underdamped thermal equilibrium systems, and later applied [24] to the system investigated in Ref. [12]. In this approach, one derives from Eqs. (4) an equation for $\dot{g}$ and then performs averaging of the dissipation and diffusion rates in this equation over the period of vibrations with a given g in the absence of dissipation and noise. The dissipation rate of g is determined by the expression $-\overline{[q_1(\partial g/\partial q_1)+q_2(\partial g/\partial q_2)]}$, with an accuracy to the correction $\propto D$ (here the overline means an averaging over the vibration period). The diffusion coefficient for g is given by $D\overline{[(\partial g/\partial q_1)^2+(\partial g/\partial q_2)^2]}$. The resulting first-order equation for g can be solved to give the following expression for the activation energy of escape from the state n:

$$S_n=2\zeta\int_{G_n}^{G_u}dG\,\frac{M(G)}{N(G)},\quad M(G)=\iint_{A(G)}dX\,dY,$$
$$N(G)=\tfrac{1}{2}\iint_{A(G)}dX\,dY\,\nabla^2G(X,Y). \tag{18}$$

Here the values of G_n ($n=1$ and 2) and G_u are the extreme values of $G(X,Y)$ [Eq. (17)]. The double integrals are taken over the areas $A(G)$ limited by the trajectories $G(X,Y)=G$ in Fig. 3 which surround the nth center. The expressions for $M(G)$ and $N(G)$ were obtained from the expressions for the drift and diffusion coefficients for g using the Stocks theorem, with account taken of Eqs. (16), as was done in Ref. [12].

Using the explicit form of $G(X,Y)$ for the Duffing oscillator (16), one obtains

$$N(G)=\iint_{A(G)}dX\,dY[\mu-2(X^2+Y^2)]. \tag{19}$$

The expressions for $M(G)$ and $N(G)$ [Eqs. (18) and (19)] can be further simplified by changing to polar coordinates $X=R\cos\varphi$ and $Y=R\sin\varphi$. Solving Eq. (16) for R^2 in terms of G and φ, and integrating over R^2, one then obtains that, in the problem of escape from the period-two attractors [$n=1$ and 2 in Eq. (18)],

$$M(G)=\int d\varphi\, f(G,\varphi),$$
$$N(G)=-\int d\varphi(\mu-2\cos 2\varphi)f(G,\varphi), \tag{20}$$
$$f(G,\varphi)=[(\mu-\cos 2\varphi)^2-4G]^{1/2}\quad (G\geqslant G_u).$$

The limits of the integrals over φ are determined from the conditions $f(G,\varphi)\geqslant 0$ and $\mu-\cos 2\varphi>0$.

In the problem of escape from the stable state at $\mathbf{q}=\mathbf{0}$ for $\mu>1$, we have $G\leqslant G_u$ (see Fig. 4), and

$$M(G)=\int_0^{\pi}d\varphi\, R^2(G,\varphi),$$
$$N(G)=\int_0^{\pi}d\varphi\, R^2(G,\varphi)[\mu-R^2(G,\varphi)]\quad (G\leqslant G_u), \tag{21}$$
$$R^2(G,\varphi)=\mu-\cos 2\varphi-[(\mu-\cos 2\varphi)^2-4G]^{1/2}.$$

C. Explicit expressions for escape rates in limiting cases

The expressions for the activation energies in the underdamped limit are simplified in several ranges of the single parameter of the system μ. We will start with μ close, but not too close, to the bifurcation points $\mu=\pm 1$, so that damping of the vibrations with a given g [Eq. (16)] is small compared to the vibration frequency $\sim|(\omega_F/2)-\omega_0|$ (we are talking here about vibrations of the slow variables q_1 and q_2, which are much slower than the vibrations at the oscillator eigenfrequency ω_0). We note that the dynamics of the system in the corresponding parameter range is not described by the theory of Sec. III, which applies much more closely to the bifurcation points where the slow motion of the system is overdamped.

As the increasing μ goes through the value $\mu=-1$, the maximum of the function $G(X,Y)$ at $X=Y=0$ becomes a saddle point from which there are split off two maxima of G corresponding to period two attractors (see Fig. 4). With the further increase in μ, for $\mu=1$ the point $X=Y=0$ becomes a minimum of $G(X,Y)$ from which there are split off two saddles of G. Escape from the emerging stable states is determined by small-radius orbits $G(X,Y)=G$. For such orbits $M(G)/N(G)\approx 1/\mu$ in Eq. (18), and therefore

$$S_1=S_2\approx\zeta(\mu+1)^2/2,\quad 0<\mu+1\ll 1,$$
$$S_0\approx\zeta(\mu-1)^2/2,\quad 0<\mu-1\ll 1. \tag{22}$$

For small $|\mu|$, the only stable states are period-two attractors, and for $\mu=0$ one obtains, after straightforward calculations,

$$S_1=S_2=\left(\frac{4}{\pi}-1\right)\zeta\quad(\mu=0). \tag{23}$$

This case corresponds to the exact resonance between the driving field frequency and doubled frequency of small-amplitude eigenvibrations of the oscillator, $\omega_F=2\omega_0$.

In the limit of large μ, the activation energy of escape from period-two attractors [Eq. (18)] is determined by orbits with $G_u=(\mu-1)^2/4\leq G\leq G_{1,2}=(\mu+1)^2/4$. These orbits have a shape of narrow arcs on the (X,Y) plane. It is seen from Eq. (20) that, for such orbits $M(G)/N(G)\approx 1/\mu$, and

$$S_1=S_2\approx 2\zeta,\quad \mu\gg 1. \tag{24}$$

One can also show from Eq. (21) that

$$S_0\approx\zeta\mu,\quad \mu\gg 1. \tag{25}$$

In the general case of arbitrary values of the parameter $\mu=\Omega/\zeta$, the activation energies $S_1=S_2$ and S_0 could be found by evaluating the integrals (18), (20), and (21) numerically. The results are shown in Fig. 5, and also by dashed lines in Fig. 2.

It is seen from Fig. 5 that $S_1=S_2$ is quadratic in $(\mu+1)$ for small $\mu+1$, and monotonically increases with the increasing μ. For large μ, the activation energy $S_{1,2}$ saturates at $\approx 2\zeta$ [Eq. (24)]. On the other hand, for constant $\Omega=\mu\xi$, $S_{1,2}$ becomes linear as a function of ζ for large ζ, with the slope given by Eq. (23); cf. Fig. 2. It is seen from the comparison of the asymptotic and exact results for the escape rates in Fig. 2 that the small-damping, or equivalently large-field, limit applies starting with comparatively small ζ.

The scaled activation energy S_0/ζ is quadratic in $\mu-1$ for small $\mu-1$, and then monotonically increases with the increasing μ. It is seen from Fig. 5 that $S_1>S_0$ for $\mu\lesssim 4.0$. Respectively, for such μ the escape rates from the period-two attractors are *exponentially smaller* than from the state where the vibrations are not excited. As a consequence, the stationary population of the period-two attractors $w_1=w_2$ is exponentially larger than the population w_0 of the steady state:

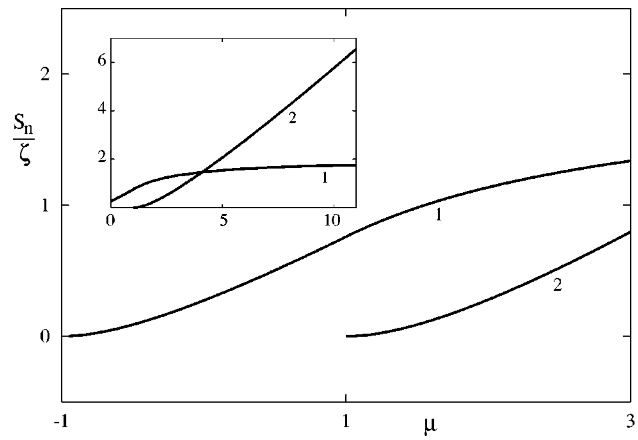

FIG. 5. The dependence of the escape activation energy $S_1=S_2$ (lines 1) and S_0 (lines 2) on the scaled frequency detuning $\mu=\Omega/\zeta\equiv 2\omega_F[(\omega_F/2)-\omega_0]/F$ in the limit of comparatively large fields or small damping, $\zeta\gg 1$.

$$w_1=w_2=(W_0/2W_1)w_0,\quad w_1/w_0\propto\exp[(S_1-S_0)/D]. \tag{26}$$

For larger frequency detuning ($\Omega=\mu\zeta$), the steady state becomes more populated, in agreement with the intuitive physical argument that, as the field is detuned further away from the resonance, it is less likely that the period-two vibrations will be excited, for the same field intensity.

V. EFFECTS OF SIXTH-ORDER NONLINEARITY

In the experiments [3], because of the structure of the electrostatic field in the trap for an oscillating electron, the sixth-order anharmonic term in the Hamiltonian of the electron vibrations could be relatively large (in fact, the fourth-order term could be relatively small), while higher-order terms remain much smaller than both the fourth- and sixth-order terms [11]. The advantage of suppressing the fourth-order term is that the amplitude of the period-two vibrations becomes larger. When the sixth-order anharmonicity is taken into account, the equation of motion takes the form

$$\frac{d^2q}{dt^2}+2\Gamma\frac{dq}{dt}+\omega_0^2q+\gamma q^3+\lambda q^5+qF\cos\omega_Ft=\xi(t). \tag{27}$$

For model (27), the equations of motion in the rotating frame are again of the form (4), but now the function $g(q_1,q_2)$ is given by the expression

$$g(q_1,q_2)=\frac{1}{2}\Omega(q_1^2+q_2^2)-\frac{1}{4}(q_1^2+q_2^2)^2\,\mathrm{sgn}\,\gamma$$
$$-\frac{\rho}{6\zeta}(q_1^2+q_2^2)^3+\frac{1}{2}\zeta(q_2^2-q_1^2), \tag{28}$$
$$\rho=5\lambda F/9\gamma^2.$$

We have neglected the renormalization $\propto\gamma^2$ of the nonlinearity parameter λ: this renormalization is substantial when γ is not small, in which case the role of the sixth-order anharmonicity is insignificant, or the whole approximation of small-amplitude vibrations does not apply. The dimensionless parameter ρ characterizes the ''strength'' of the sixth-order nonlinearity.

For Ω very close to the bifurcation values $\mp(\zeta^2-1)^{1/2}$, escape from the small-amplitude stable state(s) is determined by motion with small $q_1^2+q_2^2$. Clearly, this motion is determined by the *lowest-order* anharmonicity, and therefore neither this motion nor the bifurcation values of the parameters are affected by the higher-order nonlinear terms.

In what follows we will investigate the effect of the term $\propto\lambda$ in Eq. (27) on the escape rates in the case of weak damping, which is of utmost interest for the experiment [11]. In the neglect of dissipation and fluctuations, the motion of the oscillator [Eq. (27)] in the rotating frame can be described by Eqs. (16), with the effective energy $G(X,Y)$ of the form

$$G(X,Y)=\frac{1}{2}(\mu-1)X^2+\frac{1}{2}(\mu+1)Y^2-\frac{1}{4}(X^2+Y^2)^2 -\frac{1}{6}\rho(X^2+Y^2)^3,\quad \mu=\frac{\Omega}{\zeta}. \tag{29}$$

For $\rho>0$ [i.e., for $\gamma\lambda>0$ in Eq. (27)], the phase portrait of the conservative motion [Eqs. (16) and (29)] remains qualitatively the same as that shown in Figs. 3(a) and 3(b) for $\rho=0$, as does the topological structure of the surface $G(X,Y)$ in Fig. 4. The centers which correspond to the period-two attractors lie on the Y axis in Fig. 3, they are the projections on the (X,Y) plane of the maxima of the function $G(X,Y)$. The saddle points of $G(X,Y)$ correspond to the unstable states, and lie on the X axis.

If the motion in the rotating frame is underdamped, then the expression Eq. (18) for the activation energy of escape S_n still applies in the presence of sixth-order nonlinearity, but now the functions $M(G)$ and $N(G)$ have to be calculated with account taken of the explicit form of $G(X,Y)$ in Eq. (29). In particular,

$$N(G)=\int\int_{A(G)}dX\,dY[\mu-2(X^2+Y^2)-3\rho(X^2+Y^2)^2]. \tag{30}$$

The functions $M(G)$ and $N(G)$ can be written as single integrals over the polar angle (cf. Sec. IV B). In the problem of escape from period-two attractors, we obtain that, similar to Eq. (20),

$$M(G)=\tfrac{1}{2}\int d\varphi[R_+^2(G,\varphi)-R_-^2(G,\varphi)],$$

$$N(G)=\tfrac{1}{2}\int d\varphi[\mathcal{N}_+(G,\varphi)-\mathcal{N}_-(G,\varphi)], \tag{31}$$

$$\mathcal{N}_\pm(G,\varphi)=\mu R_\pm^2(G,\varphi)-R_\pm^4(G,\varphi)-\rho R_\pm^6(G,\varphi).$$

Here $R_\pm(G,\varphi)$ are the external (+) and internal (−) radii of the trajectories $G(X,Y)=G$ which surround the centers corresponding to the period-two attractors (cf. Fig. 3). They are given by the real roots of the equation

$$\frac{1}{6}\rho R_\pm^6+\tfrac{1}{4}R_\pm^4-\frac{1}{2}(\mu-\cos 2\varphi)R_\pm^2+G=0,$$

$$G_{1,2}>G>0\quad\text{for }\mu<1,\quad G_{1,2}>G>G_u\quad\text{for }\mu>1. \tag{32}$$

Here $G_1=G_2$ are the values of G in the period-two attractors [the maxima of $G(X,Y)$], and $G_u\equiv G_{u1}=G_{u2}$ are the saddles of $G(X,Y)$,

$$G_{1,2}=\frac{-[1+6\rho(\mu+1)]+[1+4\rho(\mu+1)]^{3/2}}{24\rho^2},$$

$$G_u=\frac{-[1+6\rho(\mu-1)]+[1+4\rho(\mu-1)]^{3/2}}{24\rho^2}\quad(\mu>1). \tag{33}$$

The limits of the integrals over φ in Eq. (31) are determined from the condition that Eq. (32) has two real roots, $R_-<R_+$. The corrections to the functions M and N for weak sixth-order nonlinearity are discussed in the Appendix.

For the case of escape from the attractor with zero vibration amplitude ($\mathbf{q}=0$), the functions $M(G)$ and $N(G)$ have the forms

$$M(G)=\tfrac{1}{2}\int_0^{2\pi}d\varphi\; R_-^2(G,\varphi),$$

$$N(G)=\tfrac{1}{2}\int_0^{2\pi}d\varphi\;\mathcal{N}_-(G,\varphi),\quad \mu>1,\quad G_u>G>0, \tag{34}$$

where the functions $R_-(G,\varphi)$ and $\mathcal{N}_-(G,\varphi)$ are defined by Eqs. (31) and (32); in the range $0<G<G_u$ Eq. (32) has only one real root R_-.

A. Activation energies near bifurcation points

For underdamped systems with sixth-order anharmonicity, the analysis of the activation energies of escape for parameter values close, but not too close to the bifurcation points is similar to that in Sec. IV C. In the range $0<\mu+1\ll1$, the centers which correspond to the period-two attractors are close to the saddle point at the origin. In this case $R_\pm\ll1$ in Eqs. (32), and the terms with X^2+Y^2 in Eq. (30) can be neglected. These terms can also be neglected in the problem of escape from the stable state $X=Y=0$ for $0<\mu+1\ll1$, since in this case the hyperbolic points are close to the stable state. Therefore, in both cases, $M(G)/N(G)\approx1/\mu$ (cf. Sec. IV C), and it follows from Eq. (18) that the activation energies of escape are

$$S_1=S_2\approx2\zeta G_1,\quad 0<\mu+1\ll1,$$

$$S_0\approx2\zeta G_u,\quad 0<\mu-1\ll1. \tag{35}$$

Here, we have taken into account that $G(0,0)=0$.

It follows from the explicit expressions for the effective energies $G_1=G_2$ and G_u [Eq. (33)] that, for weak sixth-order nonlinearity $\rho\ll1$, or just very close to the bifurcation points, so that $\rho(\mu^2-1)\ll1$ (but the effects of dissipation are still small), expressions (35) go over into the asymptotic expressions for the activation energies (22), with the scaling $S_n \propto (\mu^2-1)^2$.

Strong sixth-order nonlinearity

The activation energies (35) as functions of the distance μ^2-1 to the bifurcation points display an interesting behavior for large sixth-order nonlinearity, $\rho \gg 1$, in the range where $\mu^2-1 \ll 1$ but $\rho(\mu^2-1) \gg 1$. It follows from Eqs. (33) and (35) that in this range

$$S_{1,2} \approx \tfrac{2}{3}\zeta(\mu+1)^{3/2}\rho^{-1/2}, \quad \mu+1 \ll 1, \quad \rho(\mu+1) \gg 1,$$
$$S_0 \approx \tfrac{2}{3}\zeta(\mu-1)^{3/2}\rho^{-1/2}, \quad \mu-1 \ll 1, \quad \rho(\mu-1) \gg 1. \tag{36}$$

The dependence on the distance to the bifurcation point μ^2-1, as given by Eq. (36), is described by the power law with the exponent $\frac{3}{2}$. In contrast, for weak sixth-order nonlinearity (22) the exponent is equal to 2. Equations (33) and (35) describe both limiting behaviors and the crossover from one of them to the other.

B. Strong sixth-order nonlinearity: general case

For strong sixth-order nonlinearity, $\rho \gg 1$ and $\rho(\mu^2-1) \gg 1$, it is convenient to rescale the dynamical variables,

$$\tilde{X} = \rho^{1/4}X, \quad \tilde{Y} = \rho^{1/4}Y, \quad \tilde{G} = \rho^{1/2}G,$$
$$\tilde{G} = \tfrac{1}{2}(\mu-1)\tilde{X}^2 + \tfrac{1}{2}(\mu+1)\tilde{Y}^2 - \tfrac{1}{6}(\tilde{X}^2+\tilde{Y}^2)^3, \tag{37}$$

and the noise intensity D [Eq. (10)],

$$\tilde{D} = \rho^{1/2}D = \frac{(5\lambda F)^{1/2}}{2\omega_F^3\Gamma^2}\,\phi(\omega_F/2). \tag{38}$$

The maximal and saddle values of $\tilde{G}$, $\tilde{G}_{1,2}$, and $\tilde{G}_u$, respectively, are given by the simple expressions

$$\tilde{G}_{1,2} = \tfrac{1}{3}(\mu+1)^{3/2},$$
$$\tilde{G}_u = 0 \quad \text{for } -1<\mu<1, \quad \tilde{G}_u = \tfrac{1}{3}(\mu-1)^{3/2} \quad \text{for } \mu>1, \tag{39}$$

whereas the expressions for the trajectories in polar coordinates $\tilde{R},\varphi$ (where $\tilde{R} \equiv \rho^{1/4}R$) are of the forms

$$\tilde{R}_\pm^2(\tilde{G},\varphi) = 2(\mu-\cos 2\varphi)^{1/2}\cos\left[\frac{\theta(\tilde{G},\varphi)\mp\pi}{3}\right],$$
$$\theta(\tilde{G},\varphi) = \arccos[3\tilde{G}/(\mu-\cos 2\varphi)^{3/2}]. \tag{40}$$

With account taken of Eq. (40), expressions (31) for the functions $\tilde{M} = \rho^{-1/2}M$, $\tilde{N} = \rho^{-1/2}N$ in the problem of trajectories which surround the period-two attractors take the form

$$\tilde{M}(\tilde{G}) = \int d\varphi[3(\mu-\cos 2\varphi)]^{1/2}\sin[\theta(\tilde{G},\varphi)/3], \tag{41}$$

$$\tilde{N}(\tilde{G}) = -\int d\varphi(2\mu-3\cos 2\varphi)[3(\mu-\cos 2\varphi)]^{1/2}\sin[\theta(\tilde{G},\varphi)/3].$$

For the trajectories that surround the zero-amplitude state $\tilde{X}=\tilde{Y}=0$ we obtain, from Eqs. (34) and (40),

$$\tilde{M}(\tilde{G}) = \int_0^{2\pi} d\varphi[\mu-\cos 2\varphi]^{1/2}\cos\left[\frac{\theta(\tilde{G},\varphi)+\pi}{3}\right],$$
$$\tilde{N}(\tilde{G}) = -\int_0^{2\pi} d\varphi(2\mu-3\cos 2\varphi) \times [\mu-\cos 2\varphi]^{1/2}\cos\left[\frac{\theta(\tilde{G},\varphi)+\pi}{3}\right] + 6\pi\tilde{G}. \tag{42}$$

The expression for the escape activation energies $\tilde{S}_n = \rho^{1/2}S_n$ has the same form as Eq. (18) for S_n, with G, M, and N in Eq. (18) replaced by $\tilde{G}$, $\tilde{M}$, and $\tilde{N}$, respectively. The escape rate in the variables with tilde has the same form as in Eq. (11),

$$W_n = C\exp(-\tilde{S}_n/\tilde{D}), \quad \tilde{S}_n = \rho^{1/2}S_n. \tag{43}$$

It follows from Eqs. (28), (37), and (43) that, in the limit of large sixth-order nonlinearity, the fourth-order nonlinearity parameter γ drops out of the expressions for the activation energies $\tilde{S}_n$, the reduced noise intensity $\tilde{D}$, and the escape rates. This is in agreement with Eq. (36), which shows explicitly that $S_n \propto \rho^{-1/2}$ for large ρ, and therefore $\tilde{S}_n = \rho^{1/2}S_n$ is independent of γ.

For large frequency detuning, $\mu \gg 1$, it follows from Eqs. (18), (41), and (42) that the activation energies $\tilde{S}_{1,2}$ and $\tilde{S}_0$ are of the forms

$$\tilde{S}_{1,2} \approx \zeta\mu^{-1/2}, \quad \tilde{S}_0 \approx \zeta\mu^{1/2} \quad (\mu \gg 1). \tag{44}$$

It is clear from the asymptotic expressions (36) and (44) that the activation energy of escape from the period-two attractors $\tilde{S}_{1,2}$ is a *nonmonotonic* function of μ, i.e., of the frequency detuning $\omega_F - 2\omega_0$. The decrease of $\tilde{S}_{1,2}$ for large μ can be understood as follows. As we mentioned above, the effective reciprocal "temperature" of the distribution of the system over the energy G, which is given by $M/N = \tilde{M}/\tilde{N}$, is determined by the ratio of the rates of the drift and diffusion of the oscillator over G (for a Brownian particle this ratio is indeed equal to $1/kT$). The drift coefficient is linear in the characteristic velocities $\dot{X}$ and $\dot{Y}$ of the oscillator in the rotating frame, whereas the diffusion coefficient is quadratic in this velocity. Therefore when the velocity is large, the ratio M/N becomes small. This happens for large μ, since here the characteristic time scale for the motion with a given G is set by the reciprocal frequency detuning $1/|\omega_F - 2\omega_0|$, and thus the ratio $M/N \propto 1/|\omega_F - 2\omega_0| \propto 1/\mu$. On the other hand, the energy interval $G_1 - G_u$ for large μ is increasing with μ sublinearly if the sixth-order nonlinearity is dominating (and

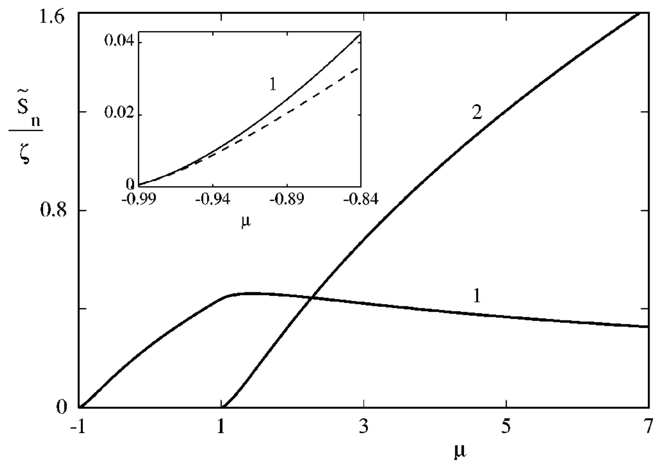

FIG. 6. The dependence of the escape activation energies $\tilde{S}_1 = \tilde{S}_2$ (line 1) and $\tilde{S}_0$ (line 2) on the scaled frequency detuning $\mu = \Omega/\zeta \equiv 2\omega_F[(\omega_F/2)-\omega_0]/F$ in the limit of comparatively large fields or small damping, $\zeta \gg 1$, for large sixth-order nonlinearity $\rho \gg 1$. Inset: $\tilde{S}_1$ near the bifurcation point; the asymptotic behavior (36) is shown by the dashed line.

linearly, if the fourth-order nonlinearity is dominating). Therefore, for large ρ and μ, the activation energy decreases with the increasing μ.

The position of the maximum of $\tilde{S}_{1,2}$ and the overall dependence of $\tilde{S}_n$ on μ can be obtained by numerical integration of Eqs. (18), (41), and (42). The results are shown in Fig. 6. It follows from the data in Fig. 6 and Eq. (26) that, in the range $\mu > 1$, the period-two attractors are populated more than the coexisting stable state $\mathbf{q}=\mathbf{0}$ up to $\mu \approx 2.3$. For higher μ the probability to find the system in the state of period-two vibrations is exponentially small.

VI. CONCLUSIONS

In the present paper we considered the rates of fluctuational transitions between coexisting vibrational states of an underdamped oscillator driven parametrically at nearly twice its eigenfrequency. Activation energies S_n have been obtained for the transitions from period-two attractors ($n=1$ and 2), and also from the stable state of period-one vibrations ($n=0$) where this state coexists with stable period-one states. We have analyzed numerically the dependence of S_1 ($S_1 = S_2$) on the dimensionless parameters of the system, the scaled field amplitude $\zeta = F/F_{\text{th}}$ and the frequency detuning $\Omega = [(\omega_F/2) - \omega_0]/\Gamma$ (the threshold value of F for the onset of period doubling is $F_{\text{th}} = 2\omega_F\Gamma$).

For comparatively large field amplitudes or small damping Γ, the appropriately scaled activation energies become functions of one dimensionless parameter. For weak sixth-order nonlinearity the exponents S_n/D in the expression for the transition rates scale as $\zeta s_n(\mu)/D$, with $\mu = \Omega/\zeta$. The function $s_1(\mu) = s_2(\mu)$ is seen from Fig. 5 first to increase, and then saturate with the increasing μ, whereas $s_0(\mu)$ is a monotonically increasing function. Therefore in the limit of small noise intensity (small effective temperature) D, the globally stable state for large μ is the period-one state, $s_0 > s_1 = s_2$, whereas for smaller μ the system is most likely to be in the period-two states. For parameter values close to the bifurcation points $\mu_B = \pm 1$ the functions $s_n(\mu)$ for the emerging stable states are equal to $(\mu - \mu_B)^2/2$.

For strong sixth-order nonlinearity, the exponents S_n/D scale with the field amplitude and frequency as $\zeta \tilde{s}_n(\mu)/\tilde{D}$, with effective temperature $\tilde{D} = \rho^{1/2} D \propto F^{1/2} D$. Close to the bifurcation points, the functions $\tilde{s}_n(\mu)$ are given by the expression $2(\mu - \mu_B)^{3/2}/3$. In contrast to the case where the sixth-order nonlinearity is small, the function $\tilde{s}_1(\mu) = \tilde{s}_2(\mu)$ is nonmonotonic.

It follows from the above results that there is a broad range of the amplitude and frequency of the driving field where the stationary populations of the period-two states are much larger than the population of the period-one state, even though this latter state may be dynamically stable. There is a narrow (with a width $\sim D$) line in the parameter space where the activation energies $S_1 = S_2$ and S_0 are close to each other, and therefore the stationary populations of all states are of the same order of magnitude. To some extent, this line is similar to the line of first-order phase transition in extended systems (cf. Ref. [12]). An interesting feature of the present system is that, because of the symmetry of period-two states, at the corresponding dynamical ''phase transition'' there are *three* rather than *two* equally populated states. New types of dynamical ''critical'' effects may be expected in the corresponding parameter range.

We note that the symmetry of period two attractors can be lifted if the system is additionally driven by a field with frequency close to the oscillator eigenfrequency $\omega_0 \approx \omega_F/2$. As a result, relative populations of the attractors can be significantly changed even by a comparatively weak field. This suggests that in a broad range of the scaled parameters of the strong field ζ and Ω, a parametrically driven nonlinear oscillator should display stochastic resonance with respect to a field at frequency close to $\omega_F/2$.

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APPENDIX: WEAK SIXTH-ORDER NONLINEARITY

For $\rho\zeta \ll 1$ the zeroth-order in $\rho\zeta$ values $M^{(0)}(G)$ and $N^{(0)}(G)$ of the functions $M(G)$ and $N(G)$ in Eqs. (31) and (34) are given by Eqs. (20) and (21), respectively. In the case of period-two attractors, to the first order in $\rho\zeta$ the functions M and N have the forms

$$M(G)\approx M^{(0)}(G)+M^{(1)}(G),\quad N(G)\approx N^{(0)}(G)+N^{(1)}(G),$$

$$M^{(1)}(G)=-\frac{4\rho\zeta}{3}\int_{\varphi_1}^{\varphi_2}d\varphi\,\frac{1}{f(G,\varphi)}(\mu-\cos 2\varphi)\times[(\mu-\cos 2\varphi)^2-3G],\tag{A1}$$

$$N^{(1)}(G)=N_1^{(1)}(G)+N_2^{(1)}(G),$$

$$N_1^{(1)}(G)=-4\rho\zeta\int_{\varphi_1}^{\varphi_2}d\varphi\; f(G,\varphi)[(\mu-\cos 2\varphi)^2-G],\tag{A2}$$

$$N_2^{(1)}(G)=\frac{\rho\zeta}{6}\int_{\varphi_1}^{\varphi_2}d\varphi\{2[(R_+^{(0)}(G,\varphi))^7+(R_-^{(0)}(G,\varphi))^7]-\mu[(R_+^{(0)}(G,\varphi))^5+(R_-^{(0)}(G,\varphi))^5]\}/f(G,\varphi).$$

Here $R_\pm^{(0)}$ are the radii of the orbits with a given G evaluated for $\rho=0$. The angles $\varphi_{1,2}$ in Eqs. (A1) and (A2) are calculated to zeroth order in $\rho\zeta$, and are given by the zeros of the function $f(G,\varphi)$ defined in Eq. (20).

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