Activated Escape of Periodically Modulated Systems

M. I. Dykman and D. Ryvkine
Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48823, USA
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The rate of noise-induced escape from a metastable state of a periodically modulated overdamped system is found for an arbitrary modulation amplitude A. The instantaneous escape rate displays peaks that vary with the modulation from Gaussian to strongly asymmetric. The prefactor ν in the period-averaged escape rate depends on A nonmonotonically. Near the bifurcation amplitude A_c it scales as $\nu \propto (A_c - A)^\zeta$. We identify three scaling regimes, with $\zeta = 1/4$, -1, and $1/2$.

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Thermally activated escape from a metastable state is often investigated in systems driven by time-dependent fields. Recent examples are activated transitions in modulated nanomagnets [1–3] and Josephson junctions [4–6]. Modulation changes the activation barrier. This enables efficient control of the escape rate and accurate measurement of the system parameters [7]. Most frequently used types of modulation are slow ramping of a control parameter, when the system remains quasistationary, and periodic modulation. In the latter case the system is away from thermal equilibrium, which complicates the theoretical formulation of the escape problem [8].

In the present Letter we extend to periodically modulated systems the analysis of the escape rate done by Kramers for systems in thermal equilibrium [9]. Our approach gives the full time-dependent escape rate $W(t)$ as well as the period-averaged rate $\overline{W} = \nu \exp(-R/D)$, where R is the activation energy of escape and D is the noise intensity, $D = k_B T$ for thermal noise.

For comparatively small modulation amplitude A escape of an overdamped Brownian particle was studied in Ref. [10]. The range of intermediate A and intermediate modulation frequencies ω_F was analyzed in Refs. [11,12]. Here we find $W(t)$ for an arbitrary A and an arbitrary interrelation between ω_F and the relaxation time of the system t_r. We show that the prefactor ν depends on A strongly and nonmonotonically. It displays scaling behavior near the bifurcational modulation amplitude A_c for which the metastable state disappears.

In the spirit of Kramers's approach, we relate the instantaneous escape rate $W(t)$ to the current *well behind* the boundary $q_b(t)$ of the basin of attraction to the initially occupied metastable state (q is the system coordinate). This is the current usually studied in experiments. Because of the oscillations of $q_b(t)$, it has a different functional form from the current at $q_b(t)$ calculated in Refs. [11,12]. We find $W(t)$ by matching the probability distribution $\rho(q,t)$ near $q_b(t)$ and inside the basin of attraction. This can be done without a complete calculation of $\rho(q,t)$ near $q_b(t)$, using singular features of the dynamics of large fluctuations.

For a periodically modulated overdamped Brownian particle, the distribution $\rho(q,t)$ is given by the Fokker-Planck equation (FPE)

$$\partial_t \rho = -\partial_q[K(q,t)\rho] + D\partial_q^2\rho. \tag{1}$$

Here, $K(q,t)$ is the periodic force driving the particle, $K(q,t) = K(q, t+\tau_F) \equiv -\partial_q U(q,t)$, where $\tau_F = 2\pi/\omega_F$ is the modulation period and $U(q,t)$ is the metastable potential. The equation of motion of the particle in the absence of noise is $\dot{q} = K(q,t)$. The metastable state $q_a(t)$, from the vicinity of which the system escapes due to noise, and the basin boundary $q_b(t)$ are the stable and unstable periodic solutions of this equation, respectively.

We assume that the noise intensity D is small. Then in a broad time range $t_\mathrm{r} \ll t \ll 1/\overline{W}$ the distribution $\rho(q,t)$ is nearly periodic in the basin of attraction to $q_a(t)$. The current away from this basin, and thus the escape rate $W(t)$, are also periodic.

The distribution ρ is maximal at $q_a(t)$ and falls off exponentially away from it. In the presence of periodic driving it acquires singular features as $D \to 0$ [13]. The singularities accumulate near $q_b(t)$. In order to find $W(t)$ one has to understand how they are smeared by diffusion.

In the absence of noise the motion of the system close to the periodic states $q_i(t)$ ($i = a, b$) is described by the equation $\dot{q} = K$ with K linearized in $q - q_i(t)$. The evolution of $q(t) - q_i(t)$ is given by the factors

$$\kappa_i(t,t') = \exp\left[\int_{t'}^{t} d\tau \mu_i(\tau)\right] \qquad (i = a, b), \tag{2}$$

where $\mu_i(t) = \mu_i(t+\tau_F) \equiv [\partial_q K(q,t)]_{q_i(t)}$. Over the period τ_F the distance $q(t) - q_i(t)$ decreases (for $i = a$) or increases (for $i = b$) by the Floquet multiplier $M_i = \kappa_i(t+\tau_F, t) \equiv \exp(\bar{\mu}_i \tau_F)$, where $\bar{\mu}_i$ is the period-average value of $\mu_i(t)$, with $\bar{\mu}_a < 0$, $\bar{\mu}_b > 0$.

For weak noise the expansions of K can be used to find $\rho(q,t)$ near $q_{a,b}(t)$. Near the metastable state q_a, the distribution is Gaussian [14], $\rho(q,t) \propto \exp\{-[q - q_a(t)]^2/2D\sigma_a^2(t)\}$. The reduced time-periodic variance is given by the equation

$$\sigma_i^2(t) = 2|M_i^{-2} - 1|^{-1} \int_0^{\tau_F} dt_1 \kappa_i^{-2}(t + t_1, t) \qquad (3)$$

with $i = a$ (in the absence of modulation $\sigma_a^2 = 1/|\mu_a|$).

The general form of the periodic distribution near the unstable state $q_b(t)$ (the boundary-layer distribution) can be found from Eq. (1) using the Laplace transform, similar to the weak-driving limit [10]. With K linear in $q - q_b$, the equation for the Laplace transform of $\rho(q, t)$ is of the first order, giving

$$\begin{aligned} \rho(q, t) &= \int_0^\infty dp e^{-pQ/D} \tilde{\rho}(p, t), \qquad Q = q - q_b(t), \\ \tilde{\rho}(p, t) &= \mathcal{E} D^{-1/2} \exp\{-[s(\phi) + p^2 \sigma_b^2(t)/2]/D\}. \end{aligned} \qquad (4)$$

In Eq. (4), $\mathcal{E}$ is a constant, $s(\phi)$ is an arbitrary zero-mean periodic function, $s(\phi + 2\pi) = s(\phi)$, and $\phi \equiv \phi(p, t)$,

$$\phi(p, t) = \Omega_F \ln[p\kappa_b(t, t')/\bar{\mu}_b l_D]. \qquad (5)$$

Here, $\Omega_F = \omega_F/\bar{\mu}_b \equiv 2\pi/\ln M_b$ is the reduced field frequency, $l_D = (2D/\bar{\mu}_b)^{1/2}$ is the typical diffusion length, and t' determines the initial value of ϕ; from Eq. (5), $\phi(p, t + \tau_F) = \phi(p, t) + 2\pi$. In Eq. (4) we assumed that the basin of attraction to q_a lies for $q < q_b(t)$, and $|Q| \ll \Delta q \equiv \min_t[q_b(t) - q_a(t)]$.

The form (4) is advantageous as it immediately gives the current $j(q, t)$ from the occupied region $(-\infty, q]$. Well behind the basin boundary, where $Q = q - q_b(t) \gg l_D$, diffusion can be disregarded; the current becomes convective and gives the instantaneous escape rate, $j(q, t) \approx \mu_b(t)\rho(q, t)Q$ at a given Q. Disregarding the term $\propto p^2/D$ in $\tilde{\rho}$ for $Q \gg l_D$, we obtain from Eq. (4)

$$j(q, t) = \mu_b(t)\mathcal{E} D^{1/2} \int_0^\infty dx e^{-x} \exp[-s(\phi_d)/D]. \qquad (6)$$

Here, $\phi_d = \Omega_F \ln[x\kappa_b(t_d, t')]$, and $t_d \equiv t_d(Q, t)$ is given by the equation $\kappa_b(t_d, t) = l_D/2Q$. In the whole harmonic range j depends on the observation point Q only in terms of the delay time t_d, which shows how long it took the system to roll down to the point Q, $\partial t_d/\partial Q = -1/\mu_b(t_d)Q$. We note that $\mu_b(t)$ can be negative for a part of the period, leading to reversals of the instantaneous current.

The escape rate $\overline{W}$ is given by the period-averaged $j(q, t)$ and is independent of q. From Eq. (6)

$$\overline{W} = \frac{\bar{\mu}_b}{2\pi} \mathcal{E} D^{1/2} \int_0^{2\pi} d\phi \exp[-s(\phi)/D]. \qquad (7)$$

Equations (6) and (7) provide a complete solution of the Kramers problem of escape of a modulated system and reduce it to finding the function s. They are similar in form to the expressions for the instantaneous and average escape rates for comparatively weak modulation, $|s| \ll R$, where s was obtained explicitly [10].

Unless the modulation is very weak or has a high frequency, for small noise intensity $\max s \sim |\min s| \gg D$. In this case the major contribution to the integrals in Eqs. (6) and (7) comes from the range where s is close to its minimum s_m reached for some $\phi = \phi_\mathrm{m}$. Then the escape rate $j(q, t)$ sharply peaks as a function of time once per period when $\phi_d(t) = \phi_\mathrm{m}$. This means that escape events are *strongly synchronized*. As we show, both $j(q, t)$ and $\overline{W}$ are determined just by the curvature of $s(\phi)$ near ϕ_m.

To find $j(q, t)$ we match Eq. (4) to the distribution $\rho(q, t)$ close to $q_b(t)$ but well inside the attraction basin, $-Q \gg l_D$. For small D this distribution can be found, for example, by solving the FPE (1) in the eikonal approximation, $\rho(q, t) = \exp[-S(q, t)/D]$. To zeroth order in D, the equation for $S = S_0$ has the form of the Hamilton-Jacobi equation $\partial_t S_0 = -H$ for an auxiliary nondissipative system with Hamiltonian [15]

$$H(q, p; t) = p^2 + pK(q, t), \qquad p = \partial_q S_0. \qquad (8)$$

The Hamiltonian trajectories of interest $q(t), p(t)$ start in the vicinity of the metastable state. The initial conditions follow from the Gaussian form of $\rho(q, t)$ near $q_a(t)$, with $S_0 = [q - q_a(t)]^2/2\sigma_a^2(t)$.

To logarithmic accuracy, the escape rate is determined by the probability to reach the basin boundary $q_b(t)$, i.e., by the action $S_0(q_b(t), t)$ [8]. The Hamiltonian trajectory $q_\mathrm{opt}(t), p_\mathrm{opt}(t)$, which minimizes $S_0(q_b(t), t)$, approaches $q_b(t)$ asymptotically as $t \to \infty$. It is periodically repeated in time with period τ_F; $q_\mathrm{opt}(t)$ is the most probable escape path (MPEP) of the original system.

Close to $q_b(t)$, the Hamiltonian equations for $q(t), p(t)$ can be linearized and solved. On the MPEP

$$\begin{aligned} p_\mathrm{opt}(t) &= -Q_\mathrm{opt}(t)/\sigma_b^2(t) = \kappa_b^{-1}(t, t')p_\mathrm{opt}(t'), \\ S_0(q_\mathrm{opt}(t), t) &= R - Q_\mathrm{opt}^2(t)/2\sigma_b^2(t), \end{aligned} \qquad (9)$$

where $Q_\mathrm{opt}(t) = q_\mathrm{opt}(t) - q_b(t)$. The quantity $R = S_0(q_\mathrm{opt}(t), t)_{t\to\infty}$ is the activation energy of escape.

The surface $S_0(q, t)$ is flat for small $Q - Q_\mathrm{opt}$ due to nonintegrability of the dynamics with Hamiltonian (8) [13]. It touches the surface $S_b(q, t) = R - Q^2/2\sigma_b^2(t)$ on the MPEP, $Q = Q_\mathrm{opt}(t)$. Away from the MPEP $S_0(q, t) > S_b(q, t)$, and therefore the function $\rho_b(q, t) = \rho(q, t) \times \exp[S_b(q, t)/D]$ is maximal on the MPEP.

We match on the MPEP ρ_b found in the eikonal approximation to the maximum of ρ_b found from Eq. (4) near the basin boundary. For $|s_\mathrm{m}| \gg D$ and $-Q \gg l_D$, the integral over p in Eq. (4) can be evaluated by the steepest descent method. The integrand is maximal if $p = -Q/\sigma_b^2(t)$ and s is minimal for this p; i.e., $\phi(p, t) = \phi_\mathrm{m}$ and $s = s_\mathrm{m}$. These conditions can be met on the whole MPEP at once, because $\phi(p_\mathrm{opt}(t), t) = \mathrm{const}$. Then from Eq. (4)

$$\begin{aligned} \rho(q, t) &= \mathcal{E}_b(t) \exp[-S_b(q, t)/D], \\ \mathcal{E}_b(t) &= \tilde{\mathcal{E}} D^{-1/2}[\sigma_b^2(t) + \Omega_F^2 s_\mathrm{m}'' p_\mathrm{opt}^{-2}(t)]^{-1/2}, \end{aligned} \qquad (10)$$

where $\tilde{\mathcal{E}} = \mathcal{E}(2\pi D)^{1/2}\exp[(R - s_{\rm m})/D]$, and $s''_{\rm m} \equiv [d^2s/d\phi^2]_{\phi_{\rm m}}$. From Eqs. (9) and (10), not only the exponents, but also their slopes coincide along the MPEP for the boundary-layer and eikonal-approximation distributions.

The function $\mathcal{E}_b(t)$ should match on the MPEP the prefactor of the eikonal-approximation distribution $\rho = \exp(-S/D)$, which is given by the term $S_1 \propto D$ in S. On the MPEP, $z = \exp(2S_1/D)$ obeys the equation [16]

$$d^2z/dt^2 - 2d(z\partial_q K)/dt + 2zp\partial_q^2 K = 0, \qquad (11)$$

where $q = q_{\rm opt}(t)$, $p = p_{\rm opt}(t)$. The initial condition to this equation follows from $\rho(q,t) = z^{-1/2}\exp(-S_0/D)$ being Gaussian near $q_a(t)$, which gives $z(t) \to 2\pi D\sigma_a^2(t)$ for $t \to -\infty$. Close to $q_b(t)$, from Eq. (11) $z(t) = D[z_1\sigma_b^2(t) + z_2 p_{\rm opt}^{-2}(t)]$, where $z_{1,2}$ are constants [11]; the term $\propto z_1$ was disregarded in the analysis [11]. Remarkably, $z^{-1/2}(t)$ is of the same functional form near $q_b(t)$ as $\mathcal{E}_b(t)$. Thus the prefactors in $\rho(q,t)$ as given by the eikonal and the boundary-layer approximations also match each other.

Explicit expressions for the escape rate in the regime of strong synchronization can be obtained for comparatively weak or slow modulation, where $s''_{\rm m} \sim |s_{\rm m}| \gg D$ but

$$\Omega_F^2 s''_{\rm m} \ll R. \qquad (12)$$

The results for $D \ll |s_{\rm m}| \ll R$ should coincide with the results of Ref. [10], which were obtained in a different way. We have verified this by finding $s''_{\rm m}$ from Eq. (11) by perturbation theory in the modulation amplitude A.

Condition (12) can be met for large A, where $s''_{\rm m} \sim R$, provided the modulation frequency is small, $\omega_F t_{\rm r} \sim \Omega_F \ll 1$ (adiabatic modulation). Here, the MPEP is given by the equation $\dot{q}_{\rm opt} = -K(q_{\rm opt}, t_{\rm m})$, with $t_{\rm m}$ found from the condition of the minimum of the adiabatic barrier height $\Delta U(t) = U(q_b(t),t) - U(q_a(t),t)$. The activation energy $R = \Delta U_{\rm m} \equiv \Delta U(t_{\rm m})$.

The value of $s''_{\rm m}$ can be obtained from $z(t)$ or by matching the adiabatic intrawell distribution $\propto \exp[-U(q,t)/D]$ and the boundary-layer distribution (4) in the region $|Q| \gg l_D$ and $\Omega_F^2 s''_{\rm m} \ll \mu_b(t_{\rm m})Q^2$ for $|t - t_{\rm m}| \ll \tau_F$. Both approaches give $\Omega_F^2\mu_b^2 s''_{\rm m} = \Delta\ddot{U}_{\rm m}$, where μ_b and $\Delta\ddot{U}_{\rm m} \equiv \partial_t^2\Delta U$ are calculated for $t = t_{\rm m}$.

The form of $j(q,t)$ depends on the parameter $\Omega_F^2 s''_{\rm m}/D$. When it is small, the term $\propto p_{\rm opt}^{-2}$ in $\mathcal{E}_b(t)$ [Eq. (10)] and $z(t)$ is also small away from the diffusion region around q_b. Then $z = 2\pi D\sigma_a^2(t_{\rm m})$. The pulses of $j(q,t)$ are Gaussian,

$$j(q,t) = \frac{|\mu_a\mu_b|^{1/2}}{2\pi}e^{-R/D}\sum_k e^{-(t-t_k)^2\Delta\ddot{U}_{\rm m}/2D} \qquad (13)$$

$[\mu_{a,b} \equiv \mu_{a,b}(t_{\rm m})]$. They are centered at $t_k = t_{\rm m} + k\tau_F$, with $k = 0, \pm1, \ldots$ [we disregard the delay $\sim\mu_b^{-1}\ln(Q/l_D)$ in t_k]. Equation (13) corresponds to the fully adiabatic picture, where the escape rate is given by the instantaneous barrier height $\Delta U(t)$.

The current has a different form for $\Omega_F^2 s''_{\rm m}/D \gg 1$. Because $p_{\rm opt}^{-2}(t) \propto \kappa_b^2(t,t')$ exponentially increases in time near q_b, the term $\propto p_{\rm opt}^{-2}$ in $\mathcal{E}_b$ and z becomes dominating before the MPEP reaches the diffusion region $|Q| \sim l_D$. Then Eqs. (6) and (10) give

$$j(q,t) = \frac{\mu_b(t)\tilde{\mathcal{E}}D^{1/2}}{\Omega_F\sqrt{s''_{\rm m}}}e^{-R/D}\sum_{k=-\infty}^{\infty}x_k e^{-x_k}, \qquad (14)$$
$$x_k = x_0\exp(2\pi k/\Omega_F), \qquad x_0 = p_{\rm opt}(t)Q/D.$$

Note that here $p_{\rm opt}(t)$ can be smaller than $l_D/\sigma_b^2(t)$.

Equation (14) describes the escape rate in the whole region $\Omega_F^2 s''_{\rm m} \gg D$; it does not require the adiabatic approximation. Its form is totally different from that of the diffusion current $-D\partial_Q\rho$ on the basin boundary $Q = 0$ as given by Eqs. (4) and (10). The ratio $\tilde{\mathcal{E}}/\sqrt{s''_{\rm m}} = \Omega_F z_2^{-1/2}$ can be obtained by solving Eq. (11).

For $\Omega_F \ll 1$ the current (14) is a series of distinct strongly asymmetric peaks, with $x_k \approx \exp[-(t - k\tau_F - t_{\rm m})\mu_b(t_{\rm m})]$ near the maximum. The transition between the pulse shapes (13) and (14) occurs for $\Omega_F^2 s''_{\rm m}/D \sim 1$. It is described by Eq. (6) with $\mathcal{E} = (2\pi)^{-1}D^{-1/2}|\mu_a/\mu_b|^{1/2}\exp[-(R - s_{\rm m})/D]$. For $\Omega_F \ll 1$, the shape of current pulses in the whole range (12) is the same as for weak modulation [17], but the parameters depend on A, ω_F differently. With increasing Ω_F the peaks of j (14) are smeared out and the escape synchronization is weakened. For $\Omega_F \gg 1$ it disappears.

The escape current (14) is completely different from the current on the basin boundary [11]. The regime $\Omega_F^2 s''_{\rm m}/D \ll 1$, where the current has the form (13), cannot be studied in the approximation [11] at all.

In the range $s''_{\rm m} \sim |s_{\rm m}| \gg D$, the period-averaged escape rate (7) is

$$\overline{W} = \nu\exp(-R/D), \qquad \nu = \bar{\mu}_b\tilde{\mathcal{E}}D^{1/2}/2\pi\sqrt{s''_{\rm m}}. \quad (15)$$

The prefactor ν can be expressed in terms of z_2, giving the result [11] even where the theory [11] does not apply.

The asymptotic technique developed in this Letter allows obtaining the prefactor ν in several limiting cases. For comparatively weak modulation, $D \ll |s_{\rm m}| \ll R$, Eqs. (11) and (15) give the same result as in Ref. [10]. Since the theory [10] covers the whole range $|s_{\rm m}| \ll R$, a transition from the limit of no modulation to the case of arbitrarily strong modulation is now fully described.

In the whole range where the adiabatic approximation applies, $\Omega_F \ll 1$, we obtain

$$\nu = (2\pi)^{-3/2}|\mu_a\mu_b|^{1/2}D^{1/2}\omega_F(\Delta\ddot{U}_{\rm m})^{-1/2}, \qquad (16)$$

where $\mu_{a,b}$ are calculated for $t = t_{\rm m}$. Interestingly, ν (16) is independent of the modulation frequency.

Equation (16) is simplified for the modulation amplitude close to the bifurcational value A_c where the metastable and unstable states merge. For small $\delta A = A_c - A$ and

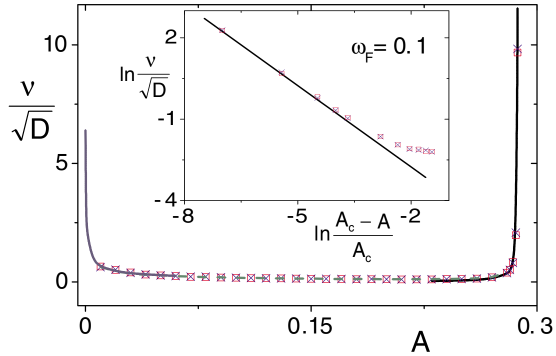

FIG. 1 (color online). The prefactor ν in the average escape rate $\overline{W}$ (15). The results refer to a Brownian particle with $K(q,t) = q^2 - 1/4 + A\cos(\omega_F t)$, $\omega_F = 0.1$ and describe escape in the regime of strong synchronization, where $\nu \propto D^{1/2}$. The solid line for small A shows the scaling $\nu \propto A^{-1/2}$ [10]. The solid line for small $A_c - A$ ($A_c \simeq 0.29$) shows the scaling (17). The dashed line shows the result of the numerical solution of Eq. (11). The squares and crosses show the results of Monte Carlo simulations for $R/D = 5$ and $R/D = 6$, respectively.

$\omega_F|t - t_m|$ the adiabatic barrier is $\Delta U(t) \propto [\delta A + a_c\omega_F^2(t - t_m)^2]^{3/2}$ (here $a_c \sim A_c$), and $|\mu_{a,b}| \propto (\delta A)^{1/2}$; cf. Refs. [18–23]. Then, from Eq. (16), the prefactor in the adiabatic limit scales as $\nu \propto (\delta A)^{1/4}$.

The slowing down of the system motion makes the adiabatic approximation inapplicable in the region $\delta A/A_c \lesssim \Omega_F$. In contrast to the adiabatic scaling $R \propto (\delta A)^{3/2}$, the activation energy scales here as $R \propto (\delta A)^2$ [22]. Using the results [22], we obtain from Eq. (11)

$$\nu = \nu_0 D^{1/2}(\delta A)^{-1}\omega_F^{5/4}, \tag{17}$$

where $\nu_0 = (64\pi^7\omega_F)^{-1/4}|\partial_t^2 K\partial_q^2 K|^{1/8}/|\partial_A K|$. Here all derivatives are evaluated for q, t, and the amplitude $A = A_c^{\mathrm{ad}} \approx A_c$ where the minimum and maximum over q of the potential $U(q,t)$ merge (once per period).

From Eq. (17), the prefactor $\nu \propto (\delta A)^{-1}$ sharply increases as the modulation amplitude approaches A_c. This is qualitatively different from the decrease of ν in the adiabatic approximation. The scaling $\nu \propto (\delta A)^{-1}$ agrees with the numerical solution of Eqs. (11) and (15) for a model system shown in Fig. 1. The calculations in a broad range of A are also confirmed by Monte Carlo simulations.

For high frequencies, $\Omega_F \gg 1$, escape is not synchronized by the modulation. The prefactor in the escape rate is $\nu = |\bar{\mu}_a\bar{\mu}_b|^{1/2}/2\pi$; it is independent of the noise intensity D. Near the bifurcation point it scales as in stationary systems, where $\nu \propto (\delta A)^{1/2}$ and $R \propto (\delta A)^{3/2}$ [18,19]. Very close to the bifurcation point modulation is necessarily fast, because $|\bar{\mu}_{a,b}| \to 0$ for $A \to A_c$. Therefore the prefactor always goes to zero for $A \to A_c$. However, for small ω_F the corresponding region of δA is exponentially narrow [22].

In conclusion, we have obtained a general solution of the problem of noise-induced escape in periodically modulated overdamped systems. With increasing modulation frequency, the pulses of escape current change from Gaussian to strongly asymmetric; for large ω_F current modulation is smeared out. The prefactor ν in the period-averaged escape rate is a strongly nonmonotonic function of the modulation amplitude A for low frequencies. It first drops with increasing A to $\nu \propto (D/A)^{1/2}$ [10], then varies with A smoothly [11,12], and then sharply increases, $\nu \propto D^{1/2}/(A_c - A)$, near the bifurcation amplitude A_c. We found three scaling regimes near A_c, where $\nu \propto (A_c - A)^{\zeta}$ with $\zeta = 1/4$, -1, or $1/2$. The widths of the corresponding scaling ranges strongly depend on the modulation frequency.

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