Critical Exponent Crossovers in Escape near a Bifurcation Point

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In periodically driven systems, near a bifurcation (critical) point the period-averaged escape rate $\overline{W}$ scales with the field amplitude A as $|\ln\overline{W}| \propto (A_c - A)^{\xi}$, where A_c is a critical amplitude. We find three scaling regions. With increasing field frequency or decreasing $|A_c - A|$, the critical exponent ξ changes from $\xi = 3/2$ for a stationary system to a dynamical value $\xi = 2$ and then again to $\xi = 3/2$. Monte Carlo simulations agree with the scaling theory.

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In many physical systems the barriers for thermally activated transitions are high, leading to low transition rates. An activation barrier can be reduced and the rate increased if the system is brought close to a critical (bifurcation) point where a metastable state disappears [1]. This idea is used, for example, in studies of thermal magnetization reversal in nanomagnets close to the coercive magnetic field or a critical spin-polarized current [2–4]. Because of their exponential sensitivity, transition probabilities provide an important means of characterizing the dynamics of a system, as has been exploited also in studies of Josephson junctions [5,6].

Experiments on nanomagnets and Josephson junctions are often performed by ramping the control parameter (magnetic field or current) and measuring the time distribution of escape events [1,5]. In interpreting the data it is usually assumed that, for sufficiently slow ramp rates, the system remains quasistationary. Then the activation energy of a transition, or barrier height, usually scales with the control parameter η, measured from its critical value, as $\eta^{3/2}$ [7,8].

In this Letter a theory of activated transitions is developed for periodically driven systems. In such systems the notion of a stable state is well defined irrespective of the modulation rate, and the applicability of the quasistationary, or adiabatic approximation can be carefully studied. It turns out that, near a critical point, this approximation breaks down even for slow driving. This is a result of the strong dependence of the relaxation time t_r on the distance to the bifurcation point η, e.g., along the modulation amplitude axis.

As a consequence of the slowing down of one of the motions [9] close to the bifurcation point, the transition rate displays system-independent features: the activation energy of a transition scales as η^{ξ}. We find three scaling regions in the parameter space. As the modulation frequency ω_F increases, the critical exponent ξ changes from 3/2 to 2 and then again to 3/2. The emergence of $\xi = 2$ scaling is a result of a new time scale, which signifies the breakdown of adiabaticity long before the standard condition $\omega_F t_r \ll 1$ is violated. We present numerical calculations and Monte Carlo simulations for a specific system, which agree well with the scaling theory.

Activated transitions in modulated systems have been investigated in various contexts [10–19]; stochastic resonance and diffusion in periodically driven ratchets are recent examples [20–22]. In this Letter we study the previously unexplored region of modulation amplitudes close to critical and reveal the universality that emerges.

We will consider a periodically modulated overdamped one-dimensional system. The results will apply also to multidimensional systems [23], because near the critical point of interest (saddle-node bifurcation) a system has only one "soft" mode. We will describe the dynamics by a Langevin equation,

$$\dot{q} = K(q; A, t) + f(t), \qquad K(q; A, t + \tau_F) = K(q; A, t). \tag{1}$$

Here, A is the characteristic modulation amplitude, and $\tau_F = 2\pi/\omega_F$. The function $f(t)$ is Gaussian noise. The precise shape of its power spectrum will not be important. The noise intensity D will be the smallest parameter of the theory (for thermal fluctuations $D = k_B T$). Then the rate of noise-induced escape $W \ll t_r^{-1}, \omega_F$.

We will assume that the modulation amplitude A is close to a critical value A_c where a stable periodic state of the system $q_a(t)$ merges with an unstable periodic state, $q_b(t)$. In the limit of slow modulation, these states can be visualized as the instantaneous minimum and barrier top of the oscillating potential $U(q, t)$ that defines the force K in Eq. (1), $K = -\partial U/\partial q$, see Fig. 1.

Escape from the stable state $q_a(t)$ occurs as a result of an appropriate large fluctuation of $f(t)$. This fluctuation drives the system from $q_a(t)$ over the boundary $q_b(t)$ of the basin of attraction. The motion of the system during escape is random. However, different trajectories have different probabilities. The system is most likely to move along a particular trajectory called the optimal path $q_{\rm opt}(t)$ [24]. It corresponds to the most probable noise realization $f_{\rm opt}(t)$ [17].

For a stationary Gaussian noise $f(t)$, the probability distribution of noise trajectories is given by the functional $\mathcal{P}[f(t)] = \exp(-\mathcal{R}[f]/D)$,

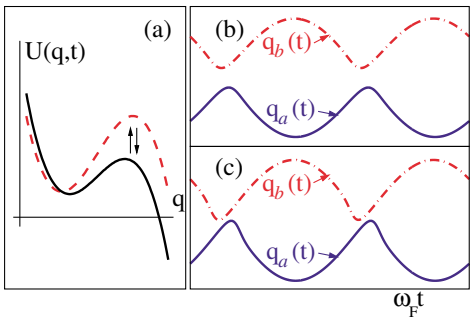

FIG. 1 (color online). (a) An oscillating potential barrier. In the limit of slow modulation, the stable and unstable periodic states q_a and q_b are the instantaneous positions of the potential minimum and barrier top, respectively. (b) For slow modulation, when the driving amplitude A is close to $A_c^{\rm ad}$, the states $q_{a,b}(t)$ come close to each other once per period. (c) As A further approaches A_c, the states $q_{a,b}(t)$ become skewed compared to the adiabatic picture, to avoid crossing.

$$\mathcal{R}[f] = \frac{1}{2}\iint dt dt'\, f(t)\hat{\mathcal{F}}(t-t')f(t'). \qquad (2)$$

Here, the kernel $\hat{\mathcal{F}}(t-t')/D$ is the inverse of the noise correlation function $\varphi(t-t') = \langle f(t)f(t')\rangle$. In particular, for white-noise $\varphi(t) = 2D\delta(t)$ and $\hat{\mathcal{F}}(t) = \delta(t)/2$.

The optimal paths $q_{\rm opt}(t)$, $f_{\rm opt}(t)$ are obtained by minimizing the functional $\mathcal{R}$ with the constraint that $q(t)$ and $f(t)$ are interrelated via Eq. (1). The path $q_{\rm opt}(t)$ starts at the stable state for $t \to -\infty$ and ends at the unstable periodic state for $t \to \infty$ [17].

The escape rate of a driven system W depends periodically on time. From Eq. (2), the maximal as well as the period-averaged escape rate $\overline{W}$ are $\propto \exp(-R/D)$. The activation energy of escape is $R = \mathcal{R}[f_{\rm opt}]$.

We will calculate R near the bifurcation point starting with the case of slow driving, when the modulation period $\tau_F \gg t_{\rm r}$. Here, a natural framework is provided by the adiabatic approximation in which the system follows modulation without delay.

The adiabatic stable and unstable states of the system $q_{a,b}^{\rm ad}(t)$ are obtained from Eq. (1) by neglecting $\dot{q}$ and the noise term

$$K(q_{a,b}^{\rm ad}; A, t) = 0. \qquad (3)$$

The derivative $-\partial K/\partial q$ calculated for $q_a^{\rm ad}(t)$ gives the reciprocal instantaneous relaxation time $1/t_{\rm r}^{\rm ad}$. Adiabaticity implies that $t_{\rm r}^{\rm ad} \ll \tau_F$.

The adiabatic critical value of the modulation amplitude $A = A_c^{\rm ad}$ is determined by the condition that the states $q_a^{\rm ad}(t)$ and $q_b^{\rm ad}(t)$ touch each other. Such "adiabatic" bifurcation generally happens once per period. We set $t = n\tau_F$ at this time ($n = 0, \pm 1, \ldots$). We also set $q_a^{\rm ad}(0) = q_b^{\rm ad}(0) = 0$ for $A = A_c^{\rm ad}$.

At the adiabatic bifurcation point the potential barrier of $U(q, t)$ in Fig. 1 disappears. Therefore escape is most likely to happen over a small portion of the modulation period centered at $t = n\tau_F$.

When the states $q_a^{\rm ad}$ and $q_b^{\rm ad}$ approach each other, we have $\partial K/\partial q \to 0$ and $t_{\rm r}^{\rm ad} \to \infty$. This means that the adiabatic approximation breaks down near the bifurcation point. Retardation effects can be taken into account by expanding K near $A_c^{\rm ad}$, $t = 0$ (or $t = n\tau_F$) and keeping the lowest order terms in q, $\delta A^{\rm ad} = A - A_c^{\rm ad}$, and t

$$K \approx \alpha q^2 + \beta\,\delta A^{\rm ad} - \alpha\gamma^2(\omega_F t)^2. \qquad (4)$$

Here, $\alpha = \frac{1}{2}\partial^2 K/\partial q^2$, $\beta = \partial K/\partial A$, and $\gamma = [-(2\alpha\omega_F^2)^{-1}\partial^2 K/\partial t^2]^{1/2}$ (the derivatives are evaluated for $A = A_c^{\rm ad}$, $t = 0$; γ is independent of ω_F, $\gamma > 0$).

Equation (4) is written for the case of an additive periodic force, $K = K_0(q) + F(t)$ [25,26]. Then, for $A = A_c^{\rm ad}$, the states $q_{a,b}^{\rm ad}(t)$ touch each other when $F(t)$ is maximal or minimal over t, and the expansion of K in t starts with terms $\propto t^2$. For K of a general form, the dynamics can still be described by Eqs. (1) and (4) after a change of variables [23].

The adiabatic periodic states (3) exist for $\alpha\beta\,\delta A^{\rm ad} < 0$. For concreteness we set $\alpha > 0$. Then $q_{a,b}^{\rm ad} = \mp 1/2\alpha t_{\rm r}^{\rm ad}$, and the adiabatic relaxation time is

$$t_{\rm r}^{\rm ad} = \frac{1}{2}[(\alpha\gamma\omega_F t)^2 - \alpha\beta\,\delta A^{\rm ad}]^{-1/2}. \qquad (5)$$

The applicability of the adiabatic approximation requires not only that $\omega_F t_{\rm r}^{\rm ad} \ll 1$, but also $|\partial t_{\rm r}^{\rm ad}/\partial t| \ll 1$, otherwise the system cannot follow the modulation. From Eq. (5), near the bifurcation point the time dependence of $t_{\rm r}^{\rm ad}$ is pronounced, so that $\max|\partial t_{\rm r}^{\rm ad}/\partial t| = 3^{-3/2}\gamma\omega_F/|\beta\,\delta A^{\rm ad}| \gg \omega_F t_{\rm r}^{\rm ad}$. Therefore the inequality $|\partial t_{\rm r}^{\rm ad}/\partial t| \ll 1$ is much stronger than $\omega_F t_{\rm r}^{\rm ad} \ll 1$. It holds if

$$t_{\rm r}^{\rm ad} \ll t_l, \qquad t_l = (\alpha\gamma\omega_F)^{-1/2}, \qquad (6)$$

i.e., $\omega_F \ll |\beta\,\delta A^{\rm ad}|/\gamma$. The time t_l sets a new dynamical time scale that restricts the adiabatic approximation.

As A approaches $A_c^{\rm ad}$, the criterion (6) is violated. The periodic stable and unstable states $q_{a,b}(t)$ are pressed against each other. Since they cannot cross, they become distorted, as shown in Fig. 1(c). Ultimately they merge, but along a line rather than at a point. From Eqs. (1) and (4), this line is $q_{a,b}(t) = \gamma\omega_F t$. This defines the nonadiabatic bifurcation for slow driving. The critical amplitude is

$$A_c^{\rm sl} = A_c^{\rm ad} + \gamma\omega_F/\beta. \qquad (7)$$

We now calculate the escape rate for both the adiabatic and nonadiabatic regions. With reduced variables the equations of motion (1) and (4) take the form

$$\dot{Q} = G(Q, \eta, \tau) + \tilde{f}(\tau), \qquad G = Q^2 - \tau^2 + 1 - \eta, \qquad (8)$$

where $Q = \alpha t_l q$, $\tau = t/t_l$, and $\dot{Q} \equiv dQ/d\tau$. The control

parameter

$$\eta = \beta(\gamma\omega_F)^{-1}(A_c^{\rm sl} - A) \tag{9}$$

is the reduced distance to the slow-driving bifurcation point. The adiabatic approximation applies for $\eta \gg 1$. The region $\eta \lesssim 1$ is nonadiabatic.

The function $\tilde{f}(\tau) = (\gamma\omega_F)^{-1} f(t)$ in Eq. (8) describes noise. If the correlation time of $f(t)$ is much less than $t_l, t_{\rm r}^{\rm ad}$, then $\tilde{f}(\tau)$ is effectively δ correlated, $\langle \tilde{f}(\tau)\tilde{f}(0)\rangle = 2\tilde{D}\delta(\tau)$, with $\tilde{D} = |\alpha/4|^{1/2}(\gamma\omega_F)^{-3/2}\int dt\, \varphi(t)$. From Eq. (2), the probability distribution of $\tilde{f}(\tau)$ is $\propto \exp[-\tilde{\mathcal{R}}(\tilde{f})/\tilde{D}]$, with $\tilde{\mathcal{R}} = \int d\tau\, \tilde{f}^2(\tau)/4$.

To find the activation energy of escape R we have to minimize $\tilde{\mathcal{R}}$ with respect to trajectories leading to escape. It can be conveniently done by replacing $\tilde{f}(\tau)$ with $\dot{Q} - G$. This leads to a variational problem for the reduced activation energy $\tilde{R} = \min\tilde{\mathcal{R}} = (\tilde{D}/D)R$,

$$\tilde{R} = \min \int_{-\infty}^{\infty} d\tau\, L(\dot{Q}, Q, \tau), \qquad L = \frac{1}{4}(\dot{Q} - G)^2, \tag{10}$$

with the boundary conditions $Q(\tau) \to Q_a(\tau)$ for $\tau \to -\infty$ and $Q(\tau) \to Q_b(\tau)$ for $\tau \to \infty$, where $Q_{a(b)} = \alpha t_l q_{a(b)}$. The escape rate is $\overline{W} \propto \exp(-\tilde{R}/\tilde{D})$.

Equation (10) gives $\tilde{R}$ as a function of the reduced distance to the bifurcation point η. It has the form of a mechanical action and can be found from the Hamilton equations

$$\dot{Q} = 2P + G, \quad \dot{P} = -P\,\partial_Q G \quad (P \equiv \partial_{\dot{Q}} L). \tag{11}$$

The activation energy $\tilde{R}(\eta)$ obtained by solving Eqs. (11) numerically is shown in Fig. 2. In addition to the adiabatic scaling $\eta^{3/2}$ for large η, it displays a crossover to a new nonadiabatic η^2 behavior in the region $\ln\eta \approx 1$.

We now analyze the asymptotic behavior of $\tilde{R}$. In the adiabatic range, $\eta \gg 1$, escape is most likely to occur when the states $Q_{a,b}(\tau)$ are closest, i.e., for $\tau = 0$ cf. Eq. (5). If we disregard the term τ^2 in G, from Eq. (11) we find $Q_{\rm opt}(\tau) = (\eta - 1)^{1/2}\tanh[(\eta - 1)^{1/2}(\tau - \tau_0)]$ with arbitrary τ_0. The adiabatic activation energy is $\tilde{R} \approx (4/3)(\eta - 1)^{3/2} \propto (A_c^{\rm ad} - A)^{3/2}$. This can be easily understood by noticing that, for $\tau = 0$ we have $G = -\partial_Q U$, with the potential $U = -Q^3/3 + (\eta - 1)Q$. Then $\tilde{R}$ is just the height of the potential barrier $U(Q_b) - U(Q_a)$ [1].

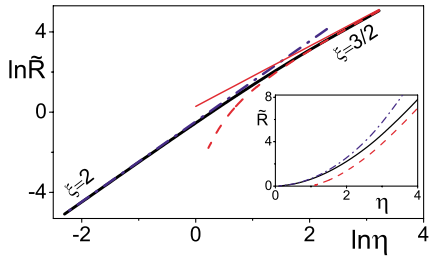

FIG. 2 (color online). Scaling crossover of the escape activation energy $\tilde{R}$ vs the distance to the bifurcation point $\eta \propto A_c^{\rm sl} - A$ (9) for slow modulation. The escape rate $\overline{W} \propto \exp(-\tilde{R}/\tilde{D})$. The thick solid lines show the numerical solution of Eqs. (10) and (11). They display the occurrence of two scaling regions $\tilde{R} \propto \eta^{\xi}$. The dash-dot lines show the nonadiabatic scaling (14), with $\tilde{R} \propto \eta^2$. The dashed lines show the adiabatic result $\tilde{R} = (4/3)(\eta - 1)^{3/2}$; its asymptote $\tilde{R} \propto \eta^{3/2}$ is shown by the thin solid line. The analytical and numerical results agree in the regions $\eta \ll 1$ and $\eta \gg 1$.

The term τ^2 in G leads to a nonadiabatic correction. To first order we obtain [23]

$$\tilde{R} \approx (4/3)(\eta - 1)^{3/2} + (\pi^2/6)(\eta - 1)^{-1/2}. \tag{12}$$

Equation (12) and the numerical results are in good agreement (better than 7%) for $\eta > 3$.

Our primary interest is to calculate the activation energy close to the bifurcation point where $\eta \ll 1$. From Eq. (8), in the whole range $\tau < |\ln\eta|^{1/2}$

$$Q_a(\tau) = -Q_b(-\tau) \approx \tau - \eta \int_{-\infty}^{\tau} d\tau_1\, e^{\tau^2 - \tau_1^2}. \tag{13}$$

For such $Q_{a,b}$, even though the problem (10) and (11) is of an instanton type, Eqs. (11) can be linearized in $Q(\tau) - \tau, P$, with solution

$$Q_{\rm opt}(\tau) = \tau - \eta \int_0^{\tau} d\tau'[1 - \sqrt{2}e^{-\tau'^2}]e^{\tau^2 - \tau'^2},$$
$$\tilde{R} = (\pi/8)^{1/2}\eta^2, \qquad \eta \ll 1. \tag{14}$$

Equation (14) shows our central result that, for slow driving, close to the bifurcation point the activation energy scales as $\tilde{R} \propto \eta^2$. Figure 2 shows the agreement of Eq. (14) with the numerical calculations in the region $\eta \lesssim 1$.

The activation energy displays yet another scaling in the opposite limit where $\omega_F t_{\rm r} \gg 1$ close to the bifurcation point. Because of critical slowing down, this region is reached for A close to A_c even if $\omega_F t_{\rm r} \lesssim 1$ far from A_c (however, it is exponentially narrow for small ω_F).

For $\omega_F t_{\rm r} \gg 1$ and small $\delta A = A - A_c$, the stable and unstable state $q_{a,b}(t)$ are close to each other all the time. The slow motion transverse to the periodic cycles $q_{a,b}$, with duration $\sim t_{\rm r}$, averages out oscillations at frequency ω_F. It is described by an equation with time-independent coefficients, as in the case of a stationary system near a bifurcation point [8],

$$\dot{Q}_0 = \alpha_0 Q_0^2 + \beta_0\,\delta A + f_0(t). \tag{15}$$

Equation (15) can be derived [23] by expanding the function $K(q; A, t)$ in Eq. (1) to second order in $q - q_c(t)$ and first order in δA, where $q_c(t)$ is the critical cycle into which the states $q_{a,b}(t)$ coalesce for $A = A_c$. The coefficients α_0 and β_0 are the period-averaged values of the appropriately weighted derivatives of K.

The function $f_0(t) \propto f(t)$ describes noise. It is effectively δ correlated on the slow time scale $t_{\rm r}$. Its intensity D_0 can be obtained by period averaging the appropriately weighted correlator $\phi(t)$ of the noise $f(t)$.

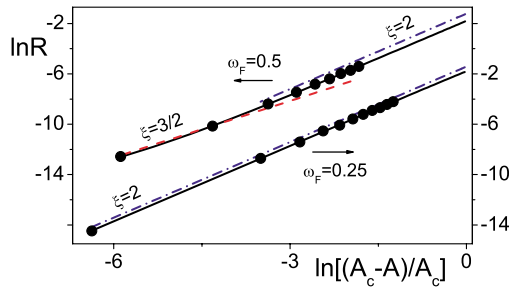

FIG. 3 (color online). The results showing the nonadiabatic $\xi = 2$ scaling of the activation energy, $R \propto (A_c - A)^{\xi}$, over a wide range of modulation amplitude A and the crossover to the $\xi = 3/2$ scaling at larger modulation frequency. The data refer to a driven Brownian particle in a time-dependent potential $U(q, t) = -q^3/3 + q^2/2 - Aq\cos\omega_F t$, with $\omega_F = 0.25$ and $\omega_F = 0.5$; the relaxation time in the absence of modulation is $t_r = 1$; the escape rate $\overline{W} \propto \exp(-R/D)$. The solid line shows the results of a numerical minimization of the functional $\mathcal{R}$ (2). The dots show the results of Monte Carlo simulations. The dash-dot line shows the $\xi = 2$ scaling (14) with appropriate parameters. The dashed line shows the high-frequency $\xi = 3/2$ scaling.

From Eq. (15), the activation energy of escape is $R = (D/D_0)R_0$, with $R_0 = (4/3)|\alpha_0|(-\beta_0\,\delta A/\alpha_0)^{3/2}$. This means that, in the limit of large $\omega_F t_r$, the activation energy scales with the distance to the bifurcation point as $(\delta A)^{3/2}$.

To verify the predicted scaling behavior, we studied the escape rate for a model system, a modulated Brownian particle with $K = q^2 - q + A\cos\omega_F t$. Over a broad range of ω_F and A we obtained the activation energy R by numerically minimizing the functional $\mathcal{R}$, Eq. (2), and by Monte Carlo simulations [23]. In the simulations R was found for each A and ω_F from $\sim 10^5$ escape events observed for 2–4 values of the noise intensity D, keeping $R/D > 6$. The calculations and simulations are in excellent agreement, see Fig. 3. They confirm that R indeed displays scaling crossovers near a bifurcation point. For $\omega_F \lesssim 0.1$ the behavior of R agrees with the results shown in Fig. 2. For relatively slow driving with $\omega_F = 0.25$, the region of nonadiabatic scaling $R \propto (A_c - A)^{\xi}$ with $\xi = 2$ significantly expands, whereas the adiabatic scaling practically disappears. For higher ω_F the $\xi = 3/2$ scaling becomes visible, first very close to the bifurcation point, but ultimately (for $\omega_F \gtrsim 1$) it becomes the only scaling.

In conclusion, we have identified the regions near a bifurcation point where the activation energy of escape displays scaling behavior as a function of the amplitude of periodic modulation. Because of emergent nonadiabaticity associated with a new time scale, even for slow modulation, $\omega_F t_r \ll 1$, there necessarily occurs a crossover from the adiabatic exponent $\xi = 3/2$ to $\xi = 2$. With increasing modulation frequency, the crossover $\xi = 2$ to $\xi = 3/2$ takes place. We expect that these scalings can be observed in systems such as modulated Josephson junctions, nanomagnets, and optically trapped Brownian particles.

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