Singular response of bistable systems driven by telegraph noise

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(Received 20 December 2011; published 6 March 2012)

We show that weak periodic driving can exponentially strongly change the rate of escape from a potential well of a system driven by telegraph noise. The analysis refers to an overdamped system, where escape requires that the noise amplitude θ exceed a critical value θ_c. For θ close to θ_c, the exponent of the escape rate displays a nonanalytic dependence on the amplitude of an additional low-frequency modulation. This leads to giant nonlinearity of the response of a bistable system to periodic modulation. Also studied is the linear response to periodic modulation far from θ_c. We analyze the scaling of the logarithm of the escape rate with the distance to the saddle-node and pitchfork bifurcation points. The analytical results are in excellent agreement with numerical simulations.

DOI: 10.1103/PhysRevE.85.031106 PACS number(s): 05.40.−a, 05.70.Ln, 85.25.Cp, 62.25.−g

I. INTRODUCTION

Noise in physical systems contains important information about the system dynamics, and much effort is put into studying it, examples being the studies of the full counting statistics in quantum optics and mesoscopic electron systems. Besides direct but often complicated measurements of the noise statistics, one can characterize noise by investigating fluctuations in noise-driven dynamical systems. Following the suggestions in Refs. [1,2], the non-Gaussian character of current noise in Josephson junctions was seen in the experiment [3,4]. The specific scaling of the rate of escape from a metastable state near a bifurcation point predicted for Poisson noise [5] has been seen in a micromechanical resonator [6].

Telegraph (dichotomous Markov) noise is one of the most interesting types of noise, as it comes from such diverse sources as defects in metals [7,8], two-level defects in Josephson junctions [9,10], and two-state fluctuators in a broad range of semiconductor devices [11], to mention but a few. It was used also to describe environmental fluctuations in biological systems [12]. It has been known since the late 1970s that such noise leads to singular features in the dynamics of noise-driven systems [13–15]. It was found [16,17] that systems driven by multiplicative telegraph noise that multiplies a periodic potential can display very strong nonlinearity of response to an additional dc bias. A number of interesting effects result also from the interplay between telegraph and white Gaussian noise [18–20].

In this paper, we study the singular response of telegraph-noise-driven systems to low-frequency modulation. We consider overdamped systems, i.e., inertial effects play no role in the dynamics. An important class of such systems are systems close to bifurcation points, as the dynamics near a bifurcation point is controlled by a soft mode [21]. We provide explicit results for the rates of escape from a metastable state for two types of bifurcations that have been attracting much interest recently, namely the saddle-node and pitchfork bifurcations. We also predict the singular response of telegraph-noise-driven systems to a weak periodic modulation.

Telegraph noise can be thought of as coming from a fluctuator with two states, $|\uparrow\rangle$ and $|\downarrow\rangle$, between which it switches at random. The force it exerts on the dynamical system depends on the state; we set it equal to θ and $-\theta$ in states $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. The features of the noise-induced fluctuations come from the fact that the noise cannot drive an overdamped system away from its stable state beyond the point where the restoring force exceeds the noise amplitude θ. This can lead to a singular behavior of the probability distribution, since the system can spend much time close to the point(s) where the restoring force and the noise are balanced [14,15]. Another consequence is that, if the dynamical system has a metastable state, escape from this state due to telegraph noise may only occur for a sufficiently large noise amplitude, where θ exceeds a critical value θ_c [22–25].

For small $|\theta - \theta_c|$, an already weak regular periodic force can strongly change the escape rate. If the force amplitude exceeds $|\theta - \theta_c|$, depending on the sign of $\theta - \theta_c$ the force can make escape possible or impossible for a part of the period. As a result, in a bistable system a low-frequency force can almost completely synchronize populations of the coexisting stable states. This means that the system is very sensitive to the force. As we show, the response of the system is strongly nonlinear and displays strong frequency dispersion.

II. SCALING OF THE RATE OF TELEGRAPH-NOISE-INDUCED ESCAPE

A. Model

The dynamics of an overdamped system with coordinate q driven by telegraph noise $f(t)$ from a two-state fluctuator is described by the equation

$$\dot{q} = -U'(q) + f(t). \tag{1}$$

The noise $f(t)$ takes on values $\pm\theta$ depending on the fluctuator state $|\uparrow\rangle$ or $|\downarrow\rangle$. The rates of interstate switching $|\uparrow\rangle \to |\downarrow\rangle$ and $|\downarrow\rangle \to |\uparrow\rangle$ are $\nu_{\uparrow\downarrow}$ and $\nu_{\downarrow\uparrow}$, respectively. In the absence of noise, the system has a stable state (stable states) at the minimum (minima) of the potential $U(q)$. The noise causes fluctuations about the stable states and can cause escape from a potential well.

In the general case of asymmetric noise, $\nu_{\uparrow\downarrow} \neq \nu_{\downarrow\uparrow}$, the noise exerts a nonzero average force on the system $\langle f \rangle$, which can be incorporated into the potential, $U(q) \to \tilde{U}(q)$,

$$\begin{aligned} \tilde{U}(q) &= U(q) + \theta q \nu_-/\nu, \\ \nu_- &= \nu_{\uparrow\downarrow} - \nu_{\downarrow\uparrow}, \quad \nu = \nu_{\uparrow\downarrow} + \nu_{\downarrow\uparrow}. \end{aligned} \tag{2}$$

Parameter ν is the overall rate of decay of noise correlations, whereas ν_-/ν characterizes noise asymmetry, i.e., the difference of the mean populations of the fluctuator states $|\downarrow\rangle$ and $|\uparrow\rangle$.

The average equilibrium position of the system $\tilde{q}_a$ is at the minimum of potential $\tilde{U}(q)$,

$$\tilde{U}'(\tilde{q}_a) = 0, \quad \tilde{U}''(\tilde{q}_a) \equiv U''(\tilde{q}_a) > 0. \tag{3}$$

We will assume that fluctuations about $\tilde{q}_a$ are small on average. This is the case if the noise amplitude θ is small. A small-amplitude telegraph noise is of limited interest; in particular, it cannot lead to escape of the system from a potential well as θ should exceed the potential gradient that drives the system toward the stable state. A more interesting case is one in which θ is not small, but the noise correlation time ν^{-1} is small compared to the characteristic relaxation time of the system t_r,

$$\nu t_r \gg 1, \quad t_r = 1/U''(\tilde{q}_a). \tag{4}$$

In this case, on average the noise changes sign fast. The system cannot follow such fast variations. The noise is then largely averaged to its mean value $\langle f(t) \rangle$ and fluctuations of the system become effectively weak. However, there also occur rare fluctuations where $f(t)$ remains constant for a time comparable to t_r. Such noise fluctuations lead to comparatively large system fluctuations.

We assume that both noise-switching rates are large: $\nu_{\uparrow\downarrow}, \nu_{\downarrow\uparrow} \gg t_r^{-1}$. Of central interest to us will be weakly to moderately asymmetric noise, where $\nu_{\uparrow\downarrow}$ and $\nu_{\downarrow\uparrow}$ are close to each other, and potentials $U(q)$ and $\tilde{U}(q)$ have a similar structure, e.g., both of them are either single- or double-well potentials.

B. Intrawell probability distribution and the escape rate

We will describe the system and the fluctuator using the probability densities $\rho_\uparrow(q)$ and $\rho_\downarrow(q)$ for the system to be at a given q and the fluctuator to be in state $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. They satisfy the equations [14]

$$\partial_t \rho_{\uparrow,\downarrow} = -\partial_q\{[-U'(q) \pm \theta]\rho_{\uparrow,\downarrow}\} \mp [\nu_{\uparrow\downarrow}\rho_\uparrow - \nu_{\downarrow\uparrow}\rho_\downarrow] \tag{5}$$

(the upper and lower signs refer to equations for $\rho_\uparrow$ and $\rho_\downarrow$, respectively). For $\nu t_r \gg 1$, function $\rho_\uparrow - \rho_\downarrow$ relaxes over time $\sim\nu^{-1}$, which is much shorter than the relaxation time t_r of the probability distribution of the system itself $\rho(q) = \rho_\uparrow(q) + \rho_\downarrow(q)$. On times large compared to ν^{-1}, function $\rho_\uparrow(q) - \rho_\downarrow(q)$ follows $\rho(q)$ adiabatically,

$$\rho_\uparrow - \rho_\downarrow \approx -\hat{L}(\theta\partial_q\rho + \nu_-\rho), \quad \hat{L} = (\nu - \partial_q U')^{-1}. \tag{6}$$

We note that operator $\hat{L}$ cannot be expanded in $\nu^{-1}U'\partial_q$, because the distribution ρ can be steep.

Using Eq. (6), we obtain from Eq. (5) a Markovian equation for adiabatic evolution of ρ,

$$\partial_t \rho = \partial_q[U'\rho + \theta\hat{L}(\theta\partial_q\rho + \nu_-\rho)]. \tag{7}$$

For a system prepared close to the minimum of the potential well, over time $\gtrsim 1/U''(\tilde{q}_a)$ the distribution inside the well becomes stationary, $\rho(q) \to \rho_{\rm st}(q)$. From Eq. (7),

$$\rho_{\rm st}(q) = \frac{\nu U''(\tilde{q}_a)\left(\sigma_a^2/2\pi\right)^{1/2}}{\theta^2 - U'^2(q)} \exp[-\Psi(q,\tilde{q}_a)], \tag{8}$$

where

$$\begin{aligned} \Psi(q_1,q_2) &= \nu \int_{q_2}^{q_1} dq' \tilde{U}'(q')/[\theta^2 - U'^2(q')], \\ \sigma_a^2 &= 4\nu_{\uparrow\downarrow}\nu_{\downarrow\uparrow}\theta^2/[\nu^3 U''(\tilde{q}_a)]. \end{aligned} \tag{9}$$

Equation (8) coincides with the result of Ref. [14] obtained in a different way. For $\nu t_r \gg 1$ distribution $\rho_{\rm st}$ has a maximum at $\tilde{q}_a$ and is Gaussian near the maximum, $\rho_{\rm st} \propto \exp[-(q - \tilde{q}_a)^2/2\sigma_a^2]$; parameter σ_a^2 in Eq. (9) is the variance of the distribution. We assumed in Eq. (8) that $\theta > |U'(q)|$ in the range $|q - \tilde{q}_a| \gtrsim \sigma_a$; in what follows, we assume that $\theta > |U'(q)|$ in a broad range $|q - \tilde{q}_a| \gg \sigma_a$.

Of primary interest for this paper is the situation where the potential $U(q)$ in the absence of noise and the average potential $\tilde{U}(q)$ have a metastable potential well, i.e., a well of finite depth from which the system can escape due to the noise. In this case, along with the minimum at $\tilde{q}_a$, the potential $\tilde{U}(q)$ has a local maximum at $\tilde{q}_{\mathcal{S}}$,

$$\tilde{U}'(\tilde{q}_{\mathcal{S}}) = 0, \quad \tilde{U}''(\tilde{q}_{\mathcal{S}}) \equiv U''(\tilde{q}_{\mathcal{S}}) < 0. \tag{10}$$

For weak on average telegraph noise, escape is a rare event and can be characterized by rate $W_{\rm e} \ll t_r^{-1}$. This rate was studied in a number of papers [22–25]. In the Appendix, we provide an alternative derivation which is based on solving Eq. (7) using the Kramers method [26]. Such a derivation does not involve the assumptions that underlie the mean first-passage time approach used for this problem or the assumption that particles are injected into a potential well at a constant rate. The method works equally well for symmetric ($\nu_- = 0$) and asymmetric telegraph noise and gives both the exponent and the prefactor in the switching rate,

$$W_{\rm e} = \frac{|U''(\tilde{q}_a)U''(\tilde{q}_{\mathcal{S}})|^{1/2}}{2\pi} \exp(-\Psi_{\rm e}), \quad \Psi_{\rm e} = \Psi(\tilde{q}_{\mathcal{S}},\tilde{q}_a). \tag{11}$$

Equation (11) applies when the noise amplitude is sufficiently large, $\theta > \max_{q\in(\tilde{q}_a,\tilde{q}_{\mathcal{S}})_r} |U'(q)|$ [here, notation $(x,y)_r$ is used to indicate the range between x and y, $(x,y)_r = (\min(x,y),\max(x,y))$]. The condition of the escape rate being small, $W_{\rm e} \ll U''(\tilde{q}_a)$, requires that $\Psi_{\rm e} \gg 1$. The latter condition is closely related to the condition that the distance between the extrema of $\tilde{U}(q)$ exceeds the width of the distribution peak, $|\tilde{q}_{\mathcal{S}} - \tilde{q}_a| \gg \sigma_a$, which is related, in turn, to the condition $\nu t_r \gg 1$. Indeed, if we use for an estimate $\theta \gtrsim U''(\tilde{q}_a)|\tilde{q}_a - \tilde{q}_{\mathcal{S}}|$, from Eq. (9) for $\nu_{\uparrow\downarrow} \sim \nu_{\downarrow\uparrow} \sim \nu$ we obtain $(\tilde{q}_a - \tilde{q}_{\mathcal{S}})^2/\sigma_a^2 \lesssim \nu t_r, \Psi_{\rm e}$. Therefore, condition $\nu t_r \gg 1$ is necessary for the escape rate to be small and for escape events to obey Poisson statistics.

C. Escape exponent near bifurcation points

The model of overdamped motion (1) is of particular interest for systems close to bifurcation points. In such systems, one of the motions is slow [21]; other degrees of freedom follow it adiabatically. We will be interested in the saddle-node and pitchfork bifurcations, in the vicinity of which fluctuations were recently studied in a number of experiments on nano- and micromechanical systems and Josephson-junction-based systems [27–31]. For these bifurcation points, the potential $U(q)$ in Eq. (1) has the form of $U_{\rm sn}(q)$ and $U_{\rm pf}(q)$, respectively, where

$$U_{\rm sn}(q) = \eta q - \tfrac{1}{3}q^3, \quad U_{\rm pf}(q) = -\tfrac{1}{2}\eta q^2 + \tfrac{1}{4}q^4. \tag{12}$$

Parameter η characterizes the distance to the bifurcation point in the parameter space. For $\eta > 0$, the system near the saddle-node bifurcation has a stable state $q_a = -\eta^{1/2}$ and a saddle point $q_{\mathcal{S}} = \eta^{1/2}$, whereas near the pitchfork bifurcation the system has two stable states $q_a^{(1,2)} = \pm\eta^{1/2}$ separated by the saddle point $q_{\mathcal{S}} = 0$. For a symmetric potential of the form of $U_{\rm pf}(q)$, the switching rate was studied in Refs. [24,25,32].

In contrast to systems driven by Gaussian noise or Poisson noise [5,33], for telegraph noise the escape exponent does not display simple scaling with η. From Eq. (11) in the interesting case of symmetric noise, $\nu_{\uparrow\downarrow} = \nu_{\downarrow\uparrow}$, we have for the escape exponents $\Psi_{\rm e}^{(\rm sn)}$ and $\Psi_{\rm e}^{(\rm pf)}$ for the saddle-node and pitchfork bifurcations, respectively,

$$\begin{aligned}\Psi_{\rm e}^{(\rm sn)} &= \nu\eta^{-1/2}R_{\rm sn}(\theta/\eta),\\ \Psi_{\rm e}^{(\rm pf)} &= \nu\eta^{-1}R_{\rm pf}(\theta/\eta^{3/2}).\end{aligned} \tag{13}$$

The scaling functions $R_{\rm sn}(z)$ and $R_{\rm pf}(z)$ are shown in Fig. 1. For $z \gg 1$, we have $R_{\rm sn}(z) \approx 4z^{-2}/3$ and $R_{\rm pf}(z) \approx z^{-2}/4$. Functions $R_{\rm sn}(z)$ and $R_{\rm pf}(z)$ diverge for $z \to 1$ and $z \to 2/3^{3/2}$, respectively. These values of z correspond to the critical value of the noise amplitude necessary for escape $\theta_c = \max_{q\in(q_a,q_{\mathcal{S}})_r} |U'(q)|$, which is determined by the condition that the noise can overcome the force that drives the system toward $\tilde{q}_a$. The behavior of the switching rate near θ_c has been studied [24,32] for the symmetric double-well potential of the form $U_{\rm pf}(q)$.

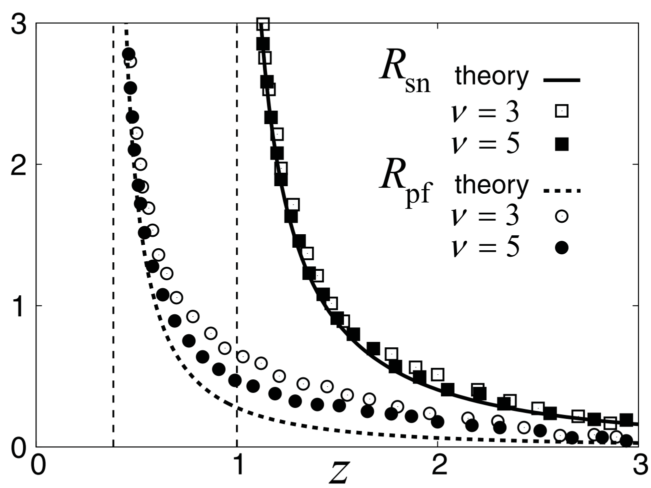

FIG. 1. The scaling functions $R_{\rm sn}(z)$ and $R_{\rm pf}(z)$ for the escape exponent near the saddle-node and pitchfork bifurcation points, Eq. (13). The vertical lines show the values of z where the corresponding functions diverge. The squares and circles are the results of simulations of the escape rate, with R obtained from Eqs. (11) and (13) with $\eta = 1$. The agreement between theory and simulations improves with increasing ν, which corresponds to the increasing exponent of the escape rate $\Psi_{\rm e}$, as seen from Eq. (13).

In the general case from Eqs. (9) and (11) for small $\delta\theta = \theta - \theta_c > 0$,

$$\Psi_{\rm e} \approx \pi\nu|\tilde{U}'(q_{\rm max})/\theta_c|[2|U'''(q_{\rm max})|\delta\theta]^{-1/2}, \tag{14}$$

where $q_{\rm max} \in (\tilde{q}_a,\tilde{q}_{\mathcal{S}})_r$ is the position of the local maximum of $|U'(q)|$. We note that, numerically, Eq. (14) sometimes applies only for very small $\delta\theta$; for example, for the potential $U_{\rm pf}(q)$ with $\eta = 1$, the difference between the numerical and asymptotic values of $\Psi_{\rm e}$ reaches $\sim$15% already for $\delta\theta = 0.01$.

If the telegraph noise has nonzero mean, the analysis of the vicinity of the bifurcation points has to be modified. For the saddle-node bifurcation, the average value of the bifurcation parameter η is shifted, whereas in the case of a pitchfork bifurcation the noise lifts the $q \to -q$ symmetry of the averaged system dynamics. The very concept of reducing multidimensional motion to the dynamics of a soft mode near a bifurcation point implies that, with nonzero $\langle f(t)\rangle$, the system should still be close to the bifurcation point, which means that $\theta|\nu_-|/\nu$ should be of the same order or less than η for the saddle-node bifurcation and than $\eta^{3/2}$ for the pitchfork bifurcation (which also becomes a saddle-node bifurcation after the degeneracy is lifted). If these conditions apply, the extension of the analysis of the escape exponent is straightforward, but the results will no longer have the simple form (13).

III. LINEAR RESPONSE TO A PERIODIC PERTURBATION

The analysis of the response of a telegraph-noise-driven system to a periodic perturbation has attracted significant attention in the context of stochastic resonance (see Refs. [34–36]), with an emphasis placed on systems in which the noise multiplies the potential. In contrast, we are considering systems with additive noise; in this section, we develop a linear response theory for telegraph-noise-driven systems.

Of utmost interest for the analysis of response is the case of weak noise, where the typical displacement from the equilibrium position $\Delta = \langle(q - \tilde{q}_a)^2\rangle^{1/2}$ is small, so that $\tilde{U}(q)$ remains parabolic for $|q - \tilde{q}_a| \lesssim \Delta$. This is the case for small noise amplitude θ or, if θ is not small, for high noise switching rate $\nu t_r \gg 1$; in the latter case, $\Delta = \sigma_a$. Linearizing Eq. (1) about $\tilde{q}_a$, we obtain

$$\Delta = (4\nu_{\uparrow\downarrow}\nu_{\downarrow\uparrow}\theta^2/\{\nu^2U''(\tilde{q}_a)[U''(\tilde{q}_a) + \nu]\})^{1/2}. \tag{15}$$

For a metastable potential well, the condition of weak noise means, in particular, that $\Delta \ll |\tilde{q}_{\mathcal{S}} - \tilde{q}_a|$.

A weak additive periodic force $A\cos\omega t$ leads to small-amplitude vibrations,

$$\langle\delta q(t)\rangle = \tfrac{1}{2}A\,{\rm Re}[\chi(\omega)\exp(-i\omega t)], \tag{16}$$

linearly superimposed on random motion. Here, $\chi(\omega)$ is the susceptibility. For weak noise and a single-well potential, $\chi(\omega) = [U''(\tilde{q}_a) - i\omega]^{-1}$ is independent of the noise. Note that a telegraph-noise-driven system is away from thermal equilibrium, and therefore there is no simple general relation between the susceptibility and the power spectrum of the system.

If the system is bistable, an important contribution to the susceptibility can come from interstate transitions. We will consider the case in which the potential $\tilde{U}(q)$ [and $U(q)$] has two wells, 1 and 2, located at $\tilde{q}_a^{(1)}$ and $\tilde{q}_a^{(2)}$, respectively; the local maximum of the potential between the wells is at $\tilde{q}_{\mathcal{S}}$. The dynamics in the presence of noise becomes trivial if the noise amplitude $\theta > \max|U'(q)|$ for $q \in (\tilde{q}_a^{(1)},\tilde{q}_{\mathcal{S}})_r$ but $\theta < \max|U'(q)|$ for $q \in (\tilde{q}_{\mathcal{S}},\tilde{q}_a^{(2)})_r$. In this case, the noise leads to transitions from well 1, but not from well 2. Then in the stationary regime, well 1 will be empty and the system will be monostable. For the opposite inequalities, well 2 will be empty.

We will consider a more interesting case in which $\theta > |U'(q)|$ in the whole region between the potential minima, $q \in (\tilde{q}_a^{(1)},\tilde{q}_a^{(2)})_r$, and $\nu \gg U''(\tilde{q}_a^{(1,2)})$. In this case, the rate W_{nm} of $n \to m$ interwell transitions $(n,m = 1,2)$ is given by Eq. (11) with $\tilde{q}_a$ replaced with $\tilde{q}_a^{(n)}$, i.e., $W_{nm} \propto \exp(-\Psi_{\rm e}^{(n)})$ with $\Psi_{\rm e}^{(n)} \equiv \Psi(\tilde{q}_{\mathcal{S}},\tilde{q}_a^{(n)})$. As a result of interwell transitions, there is formed a stationary distribution over the wells. The well populations w_1 and $w_2 = 1 - w_1$ can be found from the balance equation $\dot{w}_1 = -[W_{12} + W_{21}]w_1 + W_{21}$.

Periodic modulation changes the well populations. For weak modulation, this change can be comparatively large if the modulation frequency is small, $\omega \ll U''(\tilde{q}_a^{(1,2)})$. For such ω, one can think of the modulation as causing a slow change of the potential, $U(q) \to U(q) - Aq\cos\omega t$, which the system follows adiabatically. The change of $U(q)$ leads to a change of the instantaneous transition rates, and first of all their exponents,

$$\Psi_{\rm e}^{(n)} \to \Psi_{\rm e}^{(n)} + A\psi_n\cos(\omega t),$$
$$\psi_n = -\int_{\tilde{q}_a^{(n)}}^{\tilde{q}_{\mathcal{S}}} dq \frac{\delta\Psi\big(\tilde{q}_{\mathcal{S}},\tilde{q}_a^{(n)}\big)}{\delta U'(q)}, \tag{17}$$

where $\delta\Psi/\delta U'(q)$ is the functional derivative of the Ψ functional, Eq. (9), with respect to $U'(q)$. For large $\Psi(\tilde{q}_{\mathcal{S}},\tilde{q}_a^{(n)})$, parameter ψ_n is also large; it becomes particularly large when the difference between θ and the maximal value of $|U'(q)|$ in the interval $(\tilde{q}_{\mathcal{S}},\tilde{q}_a^{(n)})$ is small.

From Eq. (17), the transition rates become time-dependent due to the modulation. To first order in the modulation amplitude, $W_{nm} \to W_{nm}(t) \approx W_{nm}(1 - A\psi_n\cos\omega t)$. The corresponding change of the populations $w_{1,2}$ can be found as a periodic solution of the balance equation. The resulting contribution to the susceptibility $\chi_{\rm tr}(\omega)$ has a form reminiscent of the corresponding term in the theory of stochastic resonance for Gaussian-noise-driven systems (see Ref. [38] for a review),

$$\chi_{\rm tr}(\omega) = w_1w_2(\psi_1 - \psi_2)\big(\tilde{q}_a^{(1)} - \tilde{q}_a^{(2)}\big)\frac{W}{W - i\omega},$$
$$W = W_{12} + W_{21}, \quad w_1 = 1 - w_2 = W_{21}/W. \tag{18}$$

The total transition rate W in Eq. (18) is exponentially small for weak noise. Therefore, the susceptibility $\chi_{\rm tr}$ can be large only for low frequency ω. For a given ω, the transition-induced part $|\chi_{\rm tr}(\omega)|$ of the amplitude of the signal $\langle\delta q(t)\rangle$, Eq. (16), sharply depends on the noise amplitude θ and the noise switching rate ν. If, for the initially chosen noise parameters, $\omega \gg W$, then $|\chi_{\rm tr}(\omega)|$ increases exponentially with increasing θ and decreasing ν in a certain range of θ,ν, reminiscent of the Gaussian-noise-induced stochastic resonance. The signal-to-noise ratio also displays stochastic-resonance-type behavior. Another feature in common with the conventional stochastic resonance is that $|\chi_{\rm tr}|$ can be large only in the range where the stationary well populations w_1 and w_2 are close to each other.

An important distinction from stochastic resonance in Gaussian-noise-driven systems is that the transition rates W_{nm} display an extremely sharp dependence on the noise amplitude θ close to its critical value θ_c. This dependence is described by Eqs. (11) and (14). In this range of θ, the response becomes strongly nonlinear already for relatively small force amplitude A.

IV. CRITICAL RESPONSE

We now consider the nonlinear response of a telegraph-noise-driven system near the critical noise amplitude θ. We will analyze the most interesting case of a system with a symmetric double-well potential $U(q) = U(-q)$ and symmetric noise, $\nu_{\uparrow\downarrow} = \nu_{\downarrow\uparrow}$. Here, the critical values of θ are the same in both potential wells. In the absence of an extra force, for $\theta > \theta_c$ the system is equally distributed over the potential wells in the stationary regime. For $\theta < \theta_c$, the system stays in the well in which it was initially prepared.

A weak periodic force $A\cos\omega t$ can strongly change the distribution once the amplitude A becomes comparable with $\delta\theta = \theta - \theta_c$. In the adiabatic limit of slowly varying force, from Eq. (14) the instantaneous exponent of the $n \to m$ transition rate is

$$\Psi_{\rm e}^{(n)} \approx \varkappa\big[\delta\theta + A\cos(\omega t){\rm sgn}\big(U'\big(q_{\max}^{(n)}\big)\big)\big]^{-1/2},$$
$$\varkappa = \pi\nu\big|2U'''\big(q_{\max}^{(n)}\big)\big|^{-1/2} \quad (n = 1,2). \tag{19}$$

Here, $q_{\max}^{(n)}$ is the position of the maximum of $|U'(q)|$ between $q_a^{(n)} \equiv \tilde{q}_a^{(n)}$ and $q_{\mathcal{S}} \equiv \tilde{q}_{\mathcal{S}}$. We have taken into account that, for a symmetric potential, $|U'''(q_{\max}^{(n)})|$ is the same for both potential wells and therefore $\varkappa$ is independent of the well number n. Equation (19) is written assuming $\varkappa \gg (\delta\theta + A)^{1/2}$, so that $\Psi_{\rm e}^{(n)} \gg 1$. We also assumed $\delta\theta + A > 0$, so that the system can make interstate transitions in the presence of the force at least for a part of the modulation period.

Equation (19) determines the transition rate for $\delta\theta + A\cos(\omega t){\rm sgn}(U'(q_{\max}^{(n)})) > 0$, whereas for $\delta\theta + A\cos(\omega t){\rm sgn}(U'(q_{\max}^{(n)})) < 0$ there are no transitions from well n. A critical modulation amplitude for switching between attractors is known also for chaotic systems near crises [37].

From Eq. (19), the dependence of the transition rates $W_{nm} \propto \exp[-\Psi_{\rm e}^{(n)}]$ on the modulation amplitude A becomes strongly nonanalytic for $A \gtrsim |\delta\theta|$. The rates strongly depend on time. With overwhelming probability the system makes a transition from well n during a small portion of the modulation period where $W_{nm}(t)$ is maximal. In this time range,

$$\Psi_{\rm e}^{(n)} \approx \varkappa(\delta\theta + A)^{-1/2}\left[1 + \frac{A\omega^2(t - t_{nk})^2}{4(\delta\theta + A)}\right], \tag{20}$$

where $t_{nk} = 2\pi k/\omega$ for $U'(q_{\max}^{(n)}) > 0$ and $t_{nk} = \pi(2k+1)/\omega$ for $U'(q_{\max}^{(n)}) < 0$, with $k = 0,\pm 1,\pm 2,\ldots$.

From Eq. (20), $W_{nm} \propto \exp(-\Psi_{\rm e}^{(n)})$ as a function of time has a Gaussian peak at t_{nk} with typical width

$$t_{\rm e} = \omega^{-1}(4\pi/A\varkappa)^{1/2}(\delta\theta + A)^{3/4} \quad (\omega t_{\rm e} \ll 1). \tag{21}$$

It is important that this width is small, $\omega^2 t_{\rm e}^2 \ll (\delta\theta + A)/A$, so that expansion (20) applies for $|t - t_{nk}| \lesssim t_{\rm e}$ even for negative $\delta\theta$ provided $\varkappa \gg (\delta\theta + A)^{1/2}$.

For a symmetric potential $U(q)$, the potential gradients in different wells have opposite signs, $U'(q_{\max}^{(1)}) = -U'(q_{\max}^{(2)})$. Therefore, as seen from Eq. (19), for $A \gtrsim |\delta\theta|$ the rates of transitions $1 \to 2$ and $2 \to 1$ are exponentially different except for a part of the modulation period where ωt is close to $(2k+1)\pi/2$ with integer k; moreover, if $\delta\theta < 0$, each of the rates vanishes for a part of the period and only one of them can be nonzero at a time.

The evolution of the nth well population w_n during the time where $W_{nm} \gg W_{mn}$ is described by the equation $\dot w_n(t) = -W_{nm}w_n(t)$. In this time range, w_n changes from its maximal value $w_>$ to the minimal value $w_<$. At the same time, w_{3-n} changes from $w_<$ to $w_>$, where we use the fact that, for a symmetric potential, the maximal and minimal populations are the same for both wells. Therefore, $w_< \approx 1 - w_> \approx w_> \exp(-\int W_{nm}dt)$, where the integral is taken over a time interval centered at t_{nk} and largely exceeding $t_{\rm e}$, but still small compared to the modulation period. From Eqs. (11), (20), and (21),

$$w_> \approx (1 + e^{-W_{\max}t_{\rm e}})^{-1}, \quad w_< = w_> e^{-W_{\max}t_{\rm e}}, \tag{22}$$

where $W_{\max}$ is the maximal over time value of the transition rate,

$$W_{\max} = \frac{|U''(\tilde q_a)U''(\tilde q_S)|^{1/2}}{2\pi}\exp[-\varkappa(\delta\theta + A)^{-1/2}]. \tag{23}$$

It follows from Eqs. (21)–(23) that, for low modulation frequency, an already weak modulation can lead to an exponentially strong change of the populations. This happens where $W_{\max}t_{\rm e} \gg 1$. Practically, the two populations periodically change in this case between 0 and 1, in counterphase. The change occurs over a small portion of the period $\sim t_{\rm e}$, and therefore the populations as functions of time look almost like square waves. The signal in the system $\langle q(t)\rangle$ is almost a square wave too, varying between $q_a^{(1)}$ and $q_a^{(2)}$.

The onset of a square-wave signal in response to sinusoidal modulation is seen in Fig. 2. The simulations show appreciable fluctuations about the stable states; however, for the chosen parameters the interstate transitions look almost instantaneous. As expected from Eq. (22), the signal amplitude decreases with increasing ω.

A square wave signal occurs also in the conventional stochastic resonance in the presence of Gaussian noise provided the driving force amplitude exceeds the noise intensity in appropriate units; see Ref. [38]. In contrast to that situation, here the strong nonlinearity of the response comes from the singular dependence of the escape rate on the noise parameter. The dependence of the switching rate (23) on the modulation amplitude is nonanalytic for $\delta\theta \to 0$.

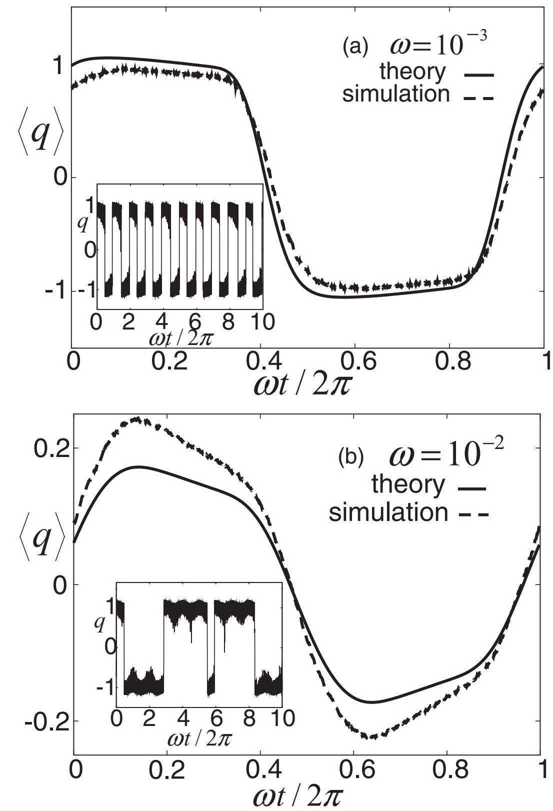

FIG. 2. Strong square-wave-like signal induced by a comparatively weak sinusoidal force $A\cos\omega t$. The results refer to the potential $U(q) = -q^2/2 + q^4/4$ and a symmetric telegraph noise, $\nu_{\uparrow\downarrow} = \nu_{\downarrow\uparrow} = \nu/2$. The noise amplitude $\theta = 0.384$ is close to the maximal slope of $U(q)$ between the minima, $\theta_c = |U'(q_{\max}^{(1,2)})| \approx 0.385$. The noise switching rate is $\nu = 2$ and the signal amplitude is $A = 0.1$. The panels refer to $\omega = 10^{-3}$ and 10^{-2}. They show how the shape of the response changes and the amplitude decreases with increasing frequency.

V. CONCLUSIONS

We have studied the scaling of the rate of escape from a metastable state and the response to periodic perturbation of systems driven by telegraph noise. As a part of the analysis, we extended to such systems the Kramers theory of activated escape [26]. A major distinction of telegraph noise from Gaussian noise is that it takes on only two values, whereas Gaussian noise can take any value, albeit with different probabilities. As a result, for telegraph noise the logarithm of the escape rate does not scale as a simple power of the distance to a bifurcation point, but instead displays a more complicated behavior. It is described for the saddle-node and pitchfork bifurcations, which are of interest for the experiment, in particular for experiments on Josephson junctions and nanoelectromechanical systems [6,27–31]. The obtained scaling functions are compared with simulations.

We have analyzed the linear response of a telegraph-noise-driven system to a periodic force. For bistable systems, the susceptibility displays a characteristic structure at frequencies of the order of the rate of interstate switching. This structure is pronounced in the range where the populations of the states are close to each other.

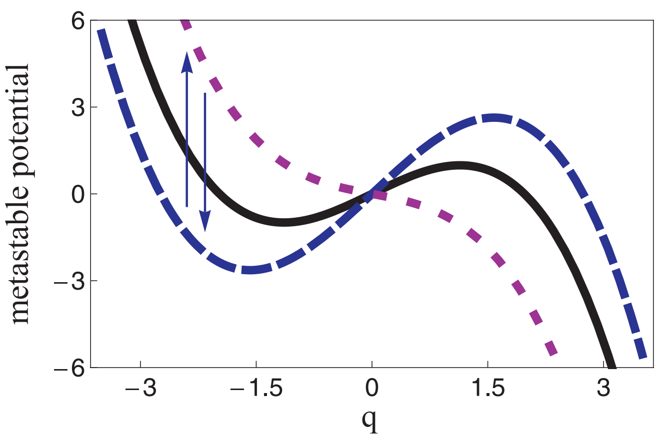

FIG. 3. (Color online) A sketch of a metastable potential well. The average potential $\tilde{U}(q) = U(q) + \theta q \nu_-/\nu$ is shown by the solid line. The long- and short-dashed lines show the potentials $U(q) + \theta q$ and $U(q) - \theta q$; they correspond to the instantaneous values of the noise $\mp\theta$. The potential is switching between these two values. Escape from the potential well occurs once the system goes sufficiently far beyond the local maximum of $\tilde{U}(q)$ (point $\tilde{q}_S$), from where the probability of returning to the interior of the well is very small. The plots refer to $U(q) = q - q^3/3, \theta = 1.5, \nu_-/\nu = 0.2$.

We found that bistable systems can also display an extremely strong nonlinearity of the response to an already weak low-frequency periodic force. It occurs in the range where the amplitude of the telegraph noise is close to the critical value below which the noise cannot cause transitions between the stable states of the system. The exponent of the transition rate depends nonanalytically on the force amplitude, whereas the populations of the stable states vary in time almost as a square wave in response to sinusoidal force.

ACKNOWLEDGMENTS

We are grateful to C. R. Doering for useful comments.

APPENDIX: KRAMERS THEORY OF ESCAPE RATE FOR TELEGRAPH-NOISE-DRIVEN SYSTEMS

The rate of escape from a potential well due to telegraph noise can be calculated by relating it, following Kramers [26], to current j away from the well. We will consider the geometry shown in Fig. 3, where the potential wells of both $U(q)$ and $\tilde{U}(q)$ are located in the region $q < \tilde{q}_S$. The escape rate is

$$W_e = j = -\partial_t \int_{-\infty}^{q} dq_1 \rho(q_1,t). \tag{A1}$$

Here, point q is chosen on the tail of the distribution, $q - \tilde{q}_a \gg \sigma_a$. In this range, the current is independent of q since the major contribution to the integral over q_1 in Eq. (A1) comes from the region $|q_1 - \tilde{q}_a| \lesssim \sigma_a$ where ρ is large.

The quasistationary solution of Eq. (7) on the tail, for a given current j, is

$$\rho(q;j) \approx -j \int_{q_f}^{q} dq_1 \frac{\nu - U''(q_1)}{\theta^2 - U'^2(q)} e^{-\Psi(q,q_1)}, \tag{A2}$$

where the function Ψ is given by Eq. (9). We set the lower limit of the integral q_f outside the potential well far from $\tilde{q}_S$, $q_f - \tilde{q}_S \gg \sigma_S$. Here, parameter $\sigma_S = 1/|\partial_q^2 \Psi(q,q_f)|_{\tilde{q}_S}^{1/2}$ determines the curvature of the distribution at point $\tilde{q}_S$; it is given explicitly by Eq. (A3) below. The probability of a fluctuation that would bring the system from q_f back into the potential well is negligibly small; one can think that at q_f there is placed an absorbing boundary. The escape rate is independent of q_f; see below.

For a single-well potential where $|U'(q)|$ increases behind $\tilde{q}_S$, cf. Fig. 3, and reaches the value θ, it is convenient to choose q_f in such a way that $q_f < q_\theta$, where q_θ is given by the equation $-U'(q_\theta) = \theta$. Since $U''(q_\theta) < 0$ and $\tilde{U}'(q_\theta) < 0$, the distribution $\rho(q;j) \to 0$ for $q \to q_f$. If $\tilde{U}(q) \to -\infty$ behind $\tilde{q}_S$ but $|U'(q)| < \theta$, one can set $q_f \to \infty$. In the case of a double-well potential $\tilde{U}(q)$, point q_f should be between $\tilde{q}_S$ and the second minimum of $\tilde{U}(q)$; Eq. (A2) does not apply close to this minimum.

For q inside the well and $\tilde{q}_S - q \gg \sigma_S$, the exponent $-\Psi(q,q_1)$ in Eq. (A2) is maximal as a function of q_1 for $q_1 = \tilde{q}_S$, where $\tilde{U}(q)$ has a local maximum. We can then calculate $\rho(q;j)$ by the steepest-descent method. Taking into account that $\nu \gg |U''(\tilde{q}_S)|$, we obtain

$$\rho(q;j) \approx \frac{\nu j \left(2\pi\sigma_S^2\right)^{1/2}}{\theta^2 - U'^2(q)} \exp[-\Psi(q,\tilde{q}_S)],$$
$$\sigma_S^2 = \frac{4\nu_{\uparrow\downarrow}\nu_{\downarrow\uparrow}\theta^2}{\nu^3 |U''(\tilde{q}_S)|}. \tag{A3}$$

The condition that the probability distributions (A3) and (8) match in a broad range of q gives the value of j and Eq. (11) for the escape rate $W_e = j$.

[1] J. Tobiska and Y. V. Nazarov, Phys. Rev. Lett. **93**, 106801 (2004).
[2] J. P. Pekola, Phys. Rev. Lett. **93**, 206601 (2004).
[3] A. V. Timofeev, M. Meschke, J. T. Peltonen, T. T. Heikkila, and J. P. Pekola, Phys. Rev. Lett. **98**, 207001 (2007).
[4] Q. Le Masne, H. Pothier, N. O. Birge, C. Urbina, and D. Esteve, Phys. Rev. Lett. **102**, 067002 (2009).
[5] L. Billings, I. B. Schwartz, M. McCrary, A. N. Korotkov, and M. I. Dykman, Phys. Rev. Lett. **104**, 140601 (2010); M. I. Dykman, Phys. Rev. E **81**, 051124 (2010).
[6] J. Zou, S. Buvaev, M. Dykman, and H. B. Chan, e-print arXiv:1102.1151 (cond-mat.mes-hall).
[7] S. Nowick and B. S. Berry, *Anelastic Relaxation in Crystalline Solids* (Academic, New York, 1972).
[8] N. Birge and B. Golding, *Tunneling of Defects in Metals*, in *Mesoscopic Quantum Phenomena* (Nova, New York, 2002).
[9] R. W. Simmonds, K. M. Lang, D. A. Hite, S. Nam, D. P. Pappas, and J. M. Martinis, Phys. Rev. Lett. **93**, 077003 (2004).
[10] Y. Shalibo, Y. Rofe, D. Shwa, F. Zeides, M. Neeley, J. M. Martinis, and N. Katz, Phys. Rev. Lett. **105**, 177001 (2010).
[11] D. Ielmini, F. Nardi, and C. Cagli, Appl. Phys. Lett. **96**, 053503 (2010).

[12] M. Thattai and A. Van Oudenaarden, Genetics **167**, 523 (2004).
[13] V. I. Klyatskin, Radiophys. Quantum Electron. **20**, 382 (1977).
[14] K. Kitahara, W. Horsthemke, and R. Lefever, Phys. Lett. A **70**, 377 (1979).
[15] W. Horsthemke and R. Lefever, *Noise-Induced Transitions. Theory and Applications in Physics, Chemistry, and Biology* (Springer-Verlag, Berlin, 1984).
[16] S. Ginzburg and M. Pustovoit, Phys. Lett. A **291**, 77 (2001).
[17] I. Bena, C. Van den Broeck, R. Kawai, and K. Lindenberg, Phys. Rev. E **66**, 045603 (2002); **68**, 041111 (2003).
[18] C. R. Doering and J. C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992).
[19] C. R. Doering, W. Horsthemke, and J. Riordan, Phys. Rev. Lett. **72**, 2984 (1994).
[20] V. Berdichevsky and M. Gitterman, Phys. Rev. E **56**, 6340 (1997).
[21] J. Guckenheimer and P. Holmes, *Nonlinear Oscillators, Dynamical Systems and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1997).
[22] C. Van den Broeck and P. Hänggi, Phys. Rev. A **30**, 2730 (1984).
[23] J. M. Sancho, Phys. Rev. A **31**, 3523 (1985).
[24] I. L'Heureux and R. Kapral, J. Chem. Phys. **88**, 7468 (1988).
[25] J. M. Porrà, J. Masoliver, and K. Lindenberg, Phys. Rev. A **44**, 4866 (1991).
[26] H. Kramers, Physica (Utrecht) **7**, 284 (1940).
[27] J. S. Aldridge and A. N. Cleland, Phys. Rev. Lett. **94**, 156403 (2005).
[28] R. Vijay, M. H. Devoret, and I. Siddiqi, Rev. Sci. Instrum. **80**, 111101 (2009).
[29] C. Stambaugh and H. B. Chan, Phys. Rev. B **73**, 172302 (2006); H. B. Chan and C. Stambaugh, Phys. Rev. Lett. **99**, 060601 (2007).
[30] A. Lupaşcu, S. Saito, T. Picot, P. C. De Groot, C. J. P. M. Harmans, and J. E. Mooij, Nat. Phys. **3**, 119 (2007).
[31] C. M. Wilson, T. Duty, M. Sandberg, F. Persson, V. Shumeiko, and P. Delsing, Phys. Rev. Lett. **105**, 233907 (2010).
[32] K. Ouchi, T. Horita, and H. Fujisaka, Phys. Rev. E **74**, 031106 (2006).
[33] M. I. Dykman and M. A. Krivoglaz, Physica A **104**, 480 (1980).
[34] V. Berdichevsky and M. Gitterman, Europhys. Lett. **36**, 161 (1996).
[35] J.-H. Li and Y.-X. Han, Phys. Rev. E **74**, 051115 (2006).
[36] M. Gong and F. Guo, J. Stat. Phys. **142**, 386 (2011).
[37] S. Sinha and B. K. Chakrabarti, Phys. Rev. E **58**, 8009 (1998).
[38] M. I. Dykman, D. G. Luchinsky, R. Mannella, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, Nuovo Cimento D **17**, 661 (1995).